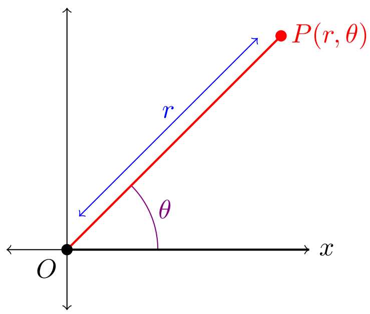
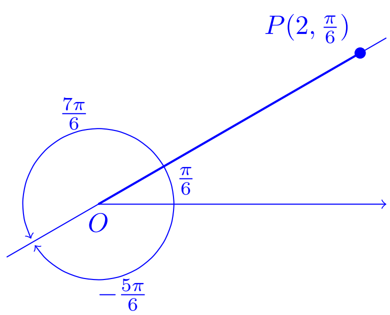
Section 11.3: Polar Coordinates

Definition: To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O (usually the positive x-axis). Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray OP.



Just like trigonometry, θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is not unique.
In some cases, we allow r to be negative. For instance, the point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians anticlockwise from the initial ray and going forward 2 units, or we could turn $\pi/6$ radians clockwise and go backwards 2 units; corresponding to $P(-2, \pi/6)$.

Example 1: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.



For $r = 2$,

$$\theta = \frac{\pi}{6},\ \frac{\pi}{6} \pm 2\pi,\ \frac{\pi}{6} \pm 4\pi,\ \frac{\pi}{6} \pm 6\pi,\ \ldots$$

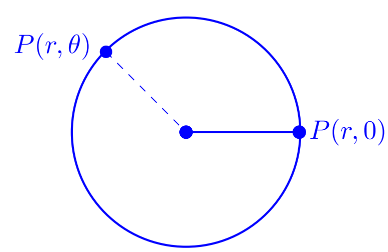
For $r = -2$,

$$\theta = -\frac{5\pi}{6},\ -\frac{5\pi}{6} \pm 2\pi,\ -\frac{5\pi}{6} \pm 4\pi,\ -\frac{5\pi}{6} \pm 6\pi,\ \ldots$$

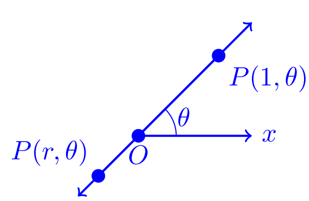
So,

$$\left\{ \left(2, \frac{\pi}{6} + 2n\pi\right), \left(-2, -\frac{5\pi}{6} + 2n\pi\right) \mid n \in \mathbb{Z} \right\}$$

Polar Equations and Graphs: If we fix r at a constant value (not equal to zero), the point $P(r, \theta)$ will lie $|r|$ unites from the origin O. As θ varies over any interval of length 2π, P traces a what? A circle!



If we fix θ at a constant value and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces a what? A line!



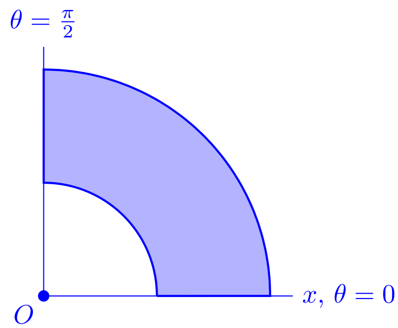
Example 2: A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for a circle of radius 1 centred at the origin.

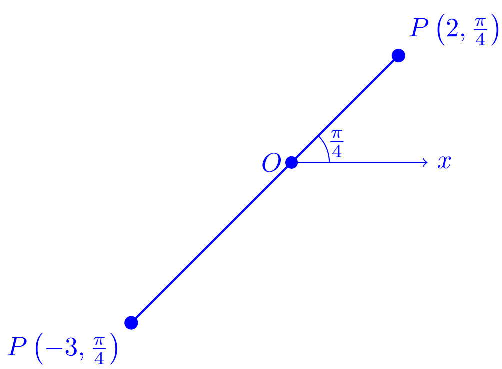
(b) $\theta = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, \ldots$ are all equations for the line passing through the Cartesian points $(0, 0)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 3: Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments and rays. Graph the sets of points whose polar coordinates satisfy the given conditions:

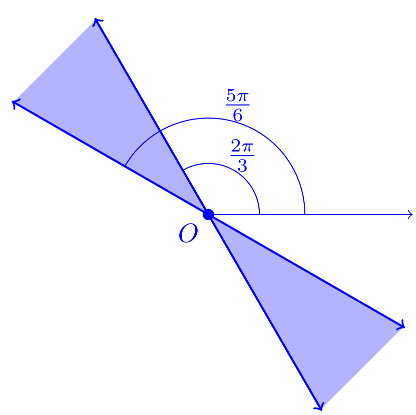
(a) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$



(b) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$



(c) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$



Relating Polar and Cartesian Coordinates: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r^2 = x^2 + y^2.$$

Example 4: Given the polar equation, find the Cartesian equivalent:

(a) $r\cos(\theta) = 2$

$$\boxed{x = 2}$$

(b) $r^2\cos(\theta)\sin(\theta) = 4$

$$r\cos(\theta) \cdot r\sin(\theta) = 4 \Longrightarrow \boxed{xy = 4}$$

(c) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$

$$\left(r\cos(\theta)\right)^2 - \left(r\sin(\theta)\right)^2 = 1 \Longrightarrow \boxed{x^2 - y^2 = 1}$$

(d) $r = 1 + 2r\cos(\theta)$

$$r^2 = \left(1 + 2r\cos(\theta)\right)^2 = 1 + 4r\cos(\theta) + 4\left(r\cos(\theta)\right)^2 \Longrightarrow \boxed{x^2 + y^2 = 1 + 4x + 4x^2}$$

(e) $r = 1 - \cos(\theta)$

$$\begin{aligned} r^2 = \left(1 - \cos(\theta)\right)r = r - r\cos(\theta) &\Longrightarrow r^2 + r\cos(\theta) = r \\ &\Longrightarrow \left(r^2 + r\cos(\theta)\right)^2 = r^2 \\ &\Longrightarrow \boxed{\left(x^2 + y^2 + x\right)^2 = x^2 + y^2} \end{aligned}$$

Example 5: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

$$x^2 + y^2 - 6y + 9 = 9 \Longrightarrow \left(x^2 + y^2\right) - 6y = 0 \Longrightarrow \boxed{r^2 - 6r\sin(\theta) = 0}$$