Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\dfrac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

First we need to calculate the value of t at the point $(\sqrt{2}, 1)$. Since $\tan(x)$ is a one-to-one function on the parameter interval we see that

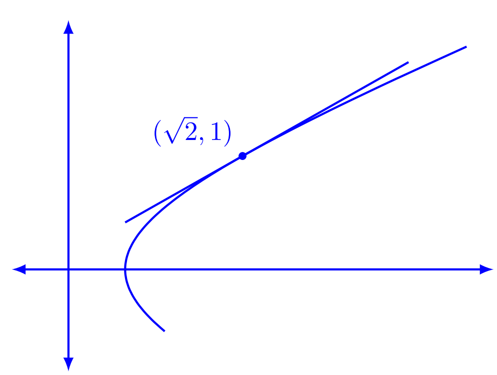
$$t = \tan^{-1}(1) = \frac{\pi}{4}$$

Using this we calculate the slope of the tangent line.

$$m = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = \frac{dy/dt}{dx/dt}\Big|_{t=\frac{\pi}{4}} = \frac{\sec^2(t)}{\sec(t)\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sec(t)}{\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Thus the equation of the tangent line at the point $(\sqrt{2}, 1)$ is

$$\boxed{y = \sqrt{2}(x - \sqrt{2}) + 1}$$



Example 2: Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

$$\frac{dx}{dt} = 1 - 2t \qquad \frac{dy}{dt} = 1 - 3t^2$$

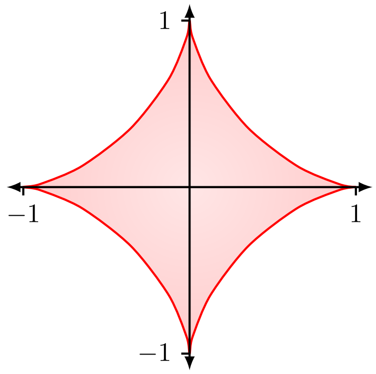
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dt}\right) &= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)\\ &= \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}\\ &= \frac{-6t+12t^2+2-6t^2}{(1-2t)^2}\\ &= \frac{2-6t+6t^2}{(1-2t)^2}\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt} = \boxed{\frac{2-6t+6t^2}{(1-2t)^3}}$$

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$



The shape we are dealing with is symmetric, so the area we are interested in is four times the area beneath the curve in the first quadrant, corresponding to $0 \le t \le \frac{\pi}{2}$. We will apply the Fundamental Theorem of Calculus using substitution to express the curve y ad the differential dx in terms of t.

$$x = \cos^3(t)$$
$$dx = -3\cos^2(t)\sin(t)\,dt$$

$$u = \sin(2t)$$
$$du = 2\cos(2t)\,dt$$

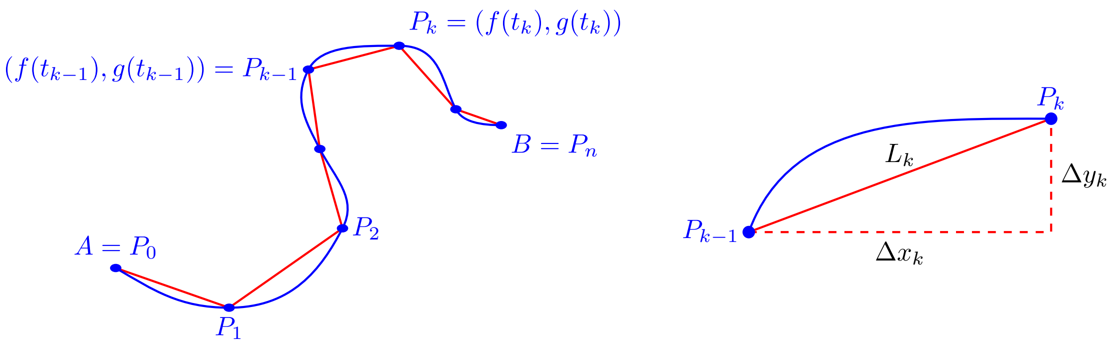
$$\begin{aligned}A = 4\int_0^1 y(x)\,dx &= 4\int_{\frac{\pi}{2}}^0 \sin^3(t)\left(-3\cos^2(t)\sin(t)\right)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))^2(1+\cos(2t))\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))\left(1-\cos^2(2t)\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin^2(2t)-\cos(2t)\sin^2(2t)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4t)}{2}\,dt - \frac{3}{2}\int_{t=0}^{t=\frac{\pi}{2}} \frac{u^2}{2}\,du\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)\right]_0^{\frac{\pi}{2}} - \frac{3}{4}\left[\frac{u^3}{3}\right]_{t=0}^{t=\frac{\pi}{2}}\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)-\frac{1}{3}\sin^3(2t)\right]_0^{\frac{\pi}{2}}\\ &= \boxed{\frac{3\pi}{8}}\end{aligned}$$

Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions $f(t)$ and $g(t)$ are continuously differentiable on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a smooth curve.



The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \le t \le 2\pi.$$

$$\frac{dx}{dt} = -r\sin(t)$$

$$\frac{dy}{dt} = r\cos(t)$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\left(\sin^2(t) + \cos^2(t)\right)}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2}\, dt \\ &= \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = \boxed{2\pi r} \end{aligned}$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$

As in Example 3, the perimeter of the astroid is 4 times the length of the curve in the first quadrant.

$$\frac{dx}{dt} = 3\cos^2(t)\sin(t)$$

$$\frac{dy}{dt} = -3\cos^2(t)\sin(t)$$

$$u = \sin(t)$$

$$du = \cos(t)\, dt$$

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t)\, dt \\ &= 12\int_{t=0}^{t=\frac{\pi}{2}} u\, du \\ &= 12\left[\frac{u^2}{2}\right]_{t=0}^{t=\frac{\pi}{2}} \\ &= 12\left[\frac{\sin^2(t)}{2}\right]_0^{\frac{\pi}{2}} \\ &= 12\left[\frac{1}{2} - 0\right] \\ &= \boxed{6} \end{aligned}$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.

$$\frac{dx}{dt} = -\sin(t)$$

$$\frac{dy}{dt} = \cos(t)$$

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi \int_0^{2\pi} (2 + \sin(t)) \sqrt{(-\sin(t))^2 + (\cos(t))}\, dt \\ &= 2\pi \int_0^{2\pi} 2 + \sin(t)\, dt \\ &= 2\pi \left[2t - \cos(t)\right]_0^{2\pi} \\ &= 2\pi \left[(4\pi - 1) - (0 - 1)\right] \\ &= \boxed{8\pi^2} \end{aligned}$$