Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

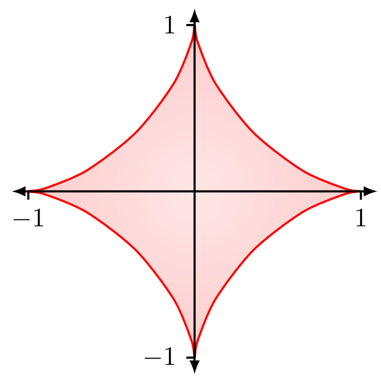
$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

Example 2: Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$

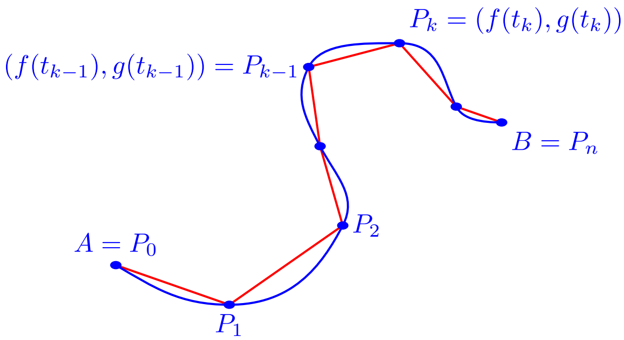


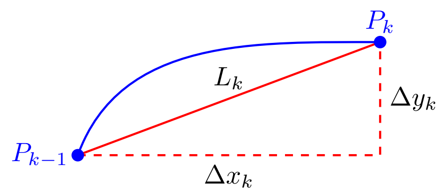
Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions $f(t)$ and $g(t)$ are ______________________________ on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a _____________________.





The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2} \Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2} \Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \leq t \leq 2\pi.$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi.$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.