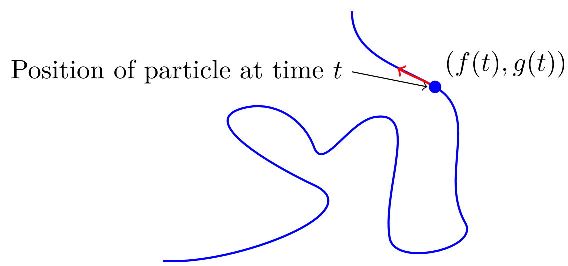
Section 11.1: Parametrisations of Plane Curves

Parametric Equations: Below we have the path of a moving particle on the xy-plane. We can sometimes describe such a path by a pair of equations, $x = f(t)$ and $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.



Definitions: If x and y are given as functions

$$x = f(t) \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ____________________.

The equations are ____________________ for the curve.

The variable t is the ____________________ for the curve and its domain I is the ____________________.

If I is a closed interval, $a \leq t \leq b$, the ____________________ of the curve is the point $(f(a), g(a))$ and the

____________________ of the curve is $(f(b), g(b))$.

Example 1: Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

Example 2: Identify geometrically the curve in Example 1 by eliminating the parameter t and obtaining an algebraic equation in x and y.

Example 3: Graph the parametric curves

(a) $x = \cos(t), \qquad y = \sin(t), \qquad 0 \leq t \leq 2\pi,$

(b) $x = a\cos(t), \quad y = a\sin(t), \quad 0 \leq t \leq 2\pi, \quad a \in \mathbb{R}.$

Example 4: The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

Example 5 - Natural Parametrisation: A parametrisation of the function $f(x) = x^2$ is given by

Example 6: Find a parametrisation for the line through the point (a, b) having slope m.

Example 7: Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

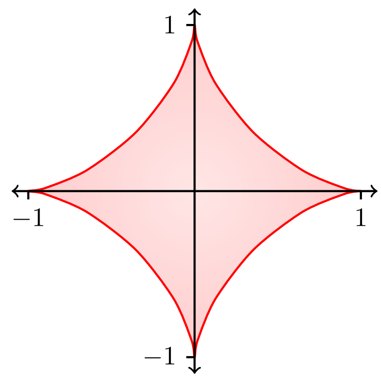
$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

Example 2: Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi.$$

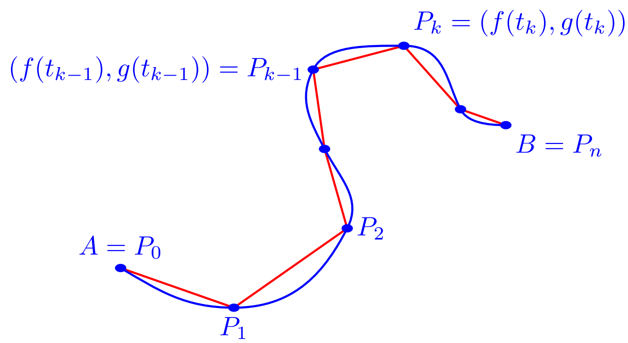


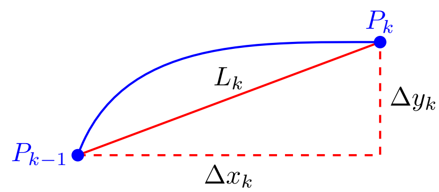
Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions $f(t)$ and $g(t)$ are ________________________________ on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a ______________________.





The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \leq t \leq 2\pi.$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi.$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

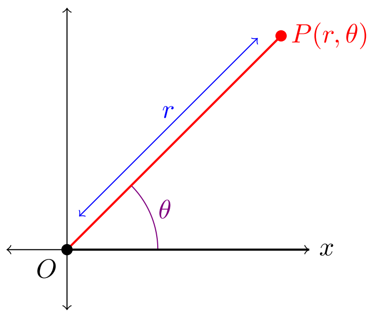
Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.

Section 11.3: Polar Coordinates

Definition: To define polar coordinates, we first fix an __________________ O (called the __________________) and an ______________________ from O (usually the positive x-axis). Then each point P can be located by assigning to it a ________________________________ (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray OP.



Just like trigonometry, θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is <u>not unique</u>.
In some cases, we allow r to be negative. For instance, the point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians anticlockwise from the initial ray and going forward 2 units, or we could turn $\pi/6$ radians clockwise and go backwards 2 units; corresponding to $P(-2, \pi/6)$.

Example 1: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

Polar Equations and Graphs: If we fix r at a constant value (not equal to zero), the point $P(r, \theta)$ will lie $|r|$ unites from the origin O. As θ varies over any interval of length 2π, P traces a what?

If we fix θ at a constant value and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces a what?

Example 2: A circle or line can have more than one polar equation.

Example 3: Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments and rays. Graph the sets of points whose polar coordinates satisfy the given conditions:

(a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$

(b) $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

(c) $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$

Relating Polar and Cartesian Coordinates: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following:

Example 4: Given the polar equation, find the Cartesian equivalent:

(a) $r\cos(\theta) = 2$

(b) $r^2\cos(\theta)\sin(\theta) = 4$

(c) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$

(d) $r = 1 + 2r\cos(\theta)$

(e) $r = 1 - \cos(\theta)$

Example 5: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.