Section 10.1: Sequences

Definition: A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers,

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots.$$

The order is important here, for example 2, 4, 6, 8, ... is *not* the same as 4, 2, 6, 8, ...

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_1 = \dfrac{1}{1+1} = \dfrac{1}{2},\ a_2 = \dfrac{2}{2+1} = \dfrac{2}{3},\ a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}, \dots$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ $\quad a_1 = \dfrac{(-1)^1(1+1)}{3^1} = \dfrac{-2}{3},\ a_2 = \dfrac{(-1)^2(2+1)}{3^2} = \dfrac{1}{3},\ a_3 = \dfrac{(-1)^3(3+1)}{3^3} = \dfrac{-4}{27}, \dots$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

$$\begin{aligned} f_3 &= f_2 + f_1 = 1 + 1 = 2, \\ f_4 &= f_3 + f_2 = 2 + 1 = 3, \\ f_5 &= f_4 + f_3 = 3 + 2 = 5, \\ f_6 &= f_5 + f_4 = 5 + 3 = 8, \quad \dots \end{aligned}$$

Definition: (**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$\text{for all } n \geq N \qquad |a_n - L| < \varepsilon.$$

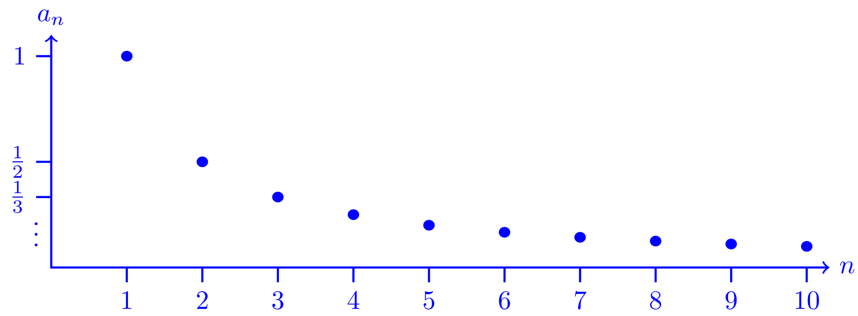
If no such number L exists, we say that $\{a_n\}$ **diverges**.

Definition: (**Friendly Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if

$$\lim_{n\to\infty} a_n = L.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?



From the plot above it looks as if the sequence is tending towards 0. It seems that plotting sequences looks a lot like plotting a function. In fact, we can use our knowledge of functions to infer things about sequences.

Theorem: **(Continuous Function Theorem)** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ whenever n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

We know that $f(x) = \frac{1}{x}$ satisfies $f(n) = a_n$ for every positive integer n, so then

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

In truth, the limit of this sequence is clear without invoking the power of this theorem. But, the theorem is still a great tool that we can use for more complicated sequences.

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences: (i.e. the limit rules you already know for functions)

If $a_n \longrightarrow L$, $b_n \longrightarrow M$, then:

1. Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = L + M$,
2. Constant Rule: $\lim_{n\to\infty} c = c$ for any $c \in \mathbb{R}$,
3. Product Rule: $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$,
4. Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, if $M \neq 0$
5. Power Rule: $\lim_{n\to\infty} a_n^p = L^p$, if $p > 0$, $a_n > 0$

Squeeze Theorem: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$$

2. $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

Note that $f(x) := \frac{\ln(x)}{x}$ satisfies $f(n) = a_n$ for each positive integer n. So,

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = \boxed{0}$$

3. $\left\{\frac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

Since $-1 \leq \cos(n) \leq 1$ for all $n \in \mathbb{N}$, we have $-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ and since

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

we have $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$, by the Squeeze Theorem.

4. $\left\{\frac{(-1)^n}{n}\right\}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \boxed{0}$$

Examples of Divergent Sequences:

1. $\{(-1)^n\}_{n=1}^{\infty}$

2. $\{(-1)^n n\}_{n=1}^{\infty}$

3. $\{\sin(n)\}_{n=1}^{\infty}$

Definition: The product of the first n positive integers,

$$n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is denoted by $n!$ (read n **factorial**.)

Convention: $0! = 1$

Example 1: Find the limit of the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Observe,

$$a_1 = \frac{1!}{1^1} = \frac{1}{1} \leq \frac{1}{1}$$

$$a_2 = \frac{2!}{2^2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{2} \cdot \frac{1}{2} \leq \frac{1}{2}$$

$$a_3 = \frac{3!}{3^3} = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} = \underbrace{\frac{3}{3} \cdot \frac{2}{3}}_{\leq 1} \cdot \frac{1}{3} \leq \frac{1}{3}$$

$$\vdots$$

$$a_n = \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n}}_{\leq 1} \cdot \frac{1}{n} \leq \frac{1}{n}$$

So we have $0 \leq a_n \leq \frac{1}{n}$, so by the Squeeze Theorem $\lim_{n \to \infty} a_n = 0$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

- If $r > 1$, $\lim_{n \to \infty} r^n = \infty$
- If $r = 1$, $\lim_{n \to \infty} r^n = 1$
- If $0 < r < 1$, $\lim_{n \to \infty} r^n = 0$
- If $r = 0$, $\lim_{n \to \infty} r^n = 0$
- If $-1 < r < 0$, $\lim_{n \to \infty} r^n = 0$
- If $r \leq -1$, $\{r^n\}_{n=1}^{\infty}$ diverges

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______bounded from above______ if there exists a number M such that $a_n \leq M$ for all n.

The number M is an ______upper bound______ for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the

______least upper bound (supremum)______ of $\{a_n\}_{n=1}\infty$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______bounded from below______ if there exists a number m such that $a_n \geq m$ for all n.

The number m is a ______lower bound______ for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the

______greatest lower bound (infimum)______ of $\{a_n\}_{n=1}\infty$.

(c) **Completeness Axiom**: If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound b. Similarly for least upper bound.

(d) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is ______bounded______.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an ______unbounded______ sequence.

(e) Every ______convergent______ sequence is ______bounded______ but **not** every ______bounded sequence______

______converges______. (consider ______$a_n = (-1)^n$______).

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______non-decreasing______ if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ______non-increasing______ if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ______monotonic______ if it is either non-decreasing or non-increasing.

The Monotone Convergence Theorem: Every bounded, monotonic sequence converges.

Note: The Monotone Convergence Theorem ONLY tells us that the limit exists, NOT the value of the limit. It also tells us that a non-decreasing sequence converges when it is bounded from above, but diverges to infinity otherwise.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

$$a_1 = 2, \qquad a_2 = \frac{1}{2}(2+6) = 4, \qquad a_3 = \frac{1}{2}(4+6) = 5, \qquad \frac{11}{2}, \qquad \frac{23}{4}, \ldots.$$

It seems that the sequence is increasing. Lets prove this by *induction*. Suppose that $a_{k-1} > a_k$ for some $k > 2$. If we can show $a_{k+1} > a_k$ then we are done. Indeed,

$$a_{k-1} < a_k \Longrightarrow a_{k-1} + 6 < a_k + 6 \Longrightarrow a_k = \frac{1}{2}(a_{k-1} + 6) < \frac{1}{2}(a_k + 6) = a_{k+1}.$$

Thus $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence. If we show that the sequence is bounded we can use the Monotone Convergence Theorem. We know that it is bounded below by 2, since we just showed it was an increasing sequence. Note too that, at least for the ones we checked, $a_k < 6$. So,

$$a_{k+1} = \frac{1}{2}\left(a_k + 6\right) < \frac{1}{2}(6+6) = 6.$$

So we have $\{a_n\}_{n=1}^{\infty}$ is bounded above by 6. So, by the Monotone Convergence Theorem $\{a_n\}_{n=1}^{\infty}$ converges.

To find the limit, let $L := \lim_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L+6) \Longrightarrow 2L = L + 6 \Longrightarrow L = 6.$$

So $\lim\limits_{n\to\infty} a_n = 6$.

Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	$1 = \frac{2^1 - 1}{2^{1-1}}$
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2} = \frac{2^2 - 1}{2^{2-1}}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4} = \frac{2^3 - 1}{2^{3-1}}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = \lim_{n\to\infty} \left(\frac{2^n}{2^{n-1}} - \frac{1}{2^{n-1}}\right) = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right) 2.$$

Since the sequence of partial sums converges, the *infinite series* converges. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ___infinite series___. The number a_n is the ___n^{th} term___ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the ___sequence of partial sums___ of the series, the number S_n being the ___n^{th} partial sum___.

If the sequence of partial sums converges to a limit L, we say that the series ___converges___ and that the

___sum___ is L. In this case we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sums of the series does not converge, we say that the series ___diverges___.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

in which a and r are fixed real numbers and $a \neq 0$. The ___ratio___ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

$$S_n = a_a(1) + a(1)^2 + a(1)^3 + \cdots + a(1)^{n-1} = na$$

and the series diverges since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \pm\infty$ (depending on the sign of a).

If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

$$S_1 = a, \quad S_2 = a + a(-1) = 0, \quad a + a(-1) = a(-1)^2 = a, \quad \ldots$$

If $|r| \neq 1$, then we use the following "trick":

$$\begin{aligned} & S_n = a + ar + ar^2 + \cdots + sr^{n-1} \\ \Longrightarrow \quad & rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \\ \Longrightarrow \quad & S_n - rS_n = a - ar^n \\ \Longrightarrow \quad & S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}. \end{aligned}$$

If $|r| < 1$ then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$, so $S_n \longrightarrow \dfrac{a}{1-r}$. If $|r| > 1$ then $|r^n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and the series diverges.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 5}{4^{n-1}} = \sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}.$$

So this series is a geometric series with $a = 5$ and $r = -\dfrac{1}{4}$. Since $|r| < 1$ the series converges and so,

$$\sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1} = \frac{5}{1 - \left(-\frac{1}{4}\right)} = \boxed{4}$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

$$\begin{aligned} 5.232323\ldots &= 5 + \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \cdots \\ &= 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\ &= 5 + \frac{23}{100}\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^{n-1} \qquad\qquad a = 1, \quad r = \frac{1}{100} \\ &= 5 + \frac{23}{100}\left(\frac{1}{1 - \frac{1}{100}}\right) \\ &= 5 + \frac{23}{100} \cdot \frac{100}{99} \\ &= \boxed{\frac{518}{99}} \end{aligned}$$

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

If we take the partial sum decomposition,

$$\sum_{n=1} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

then its easy to see that the partial sums are,

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \overset{n\to\infty}{\longrightarrow} 1.$$

Since the sequence of partial sums converges, the series converges and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \boxed{1}$

Theorem: If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Suppose $\{S_n\}_{n=1}^{\infty}$ converges to L. Then note that $\{S_{n+1}\}_{n=1}^{\infty}$ also converges to L. So then,

$$0 = L - L = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n+1} - S_n) = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

The n^{th} Term Test for Divergence: The series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* if $\displaystyle\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

1. $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges since $\displaystyle\lim_{n\to\infty} n^2 = \infty$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\displaystyle\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B,$ 2) Constant Multiple Rule : $\displaystyle\sum_{n=1}^{\infty} ca_n = cA,$ for any $c \in \mathbb{R}$.

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Caution! $\sum (a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ diverge!.

$$\begin{cases} \sum_{n=1}^{\infty} 1 \text{ diverges} \\ \sum_{n=1}^{\infty} (-1) \text{ diverges} \\ \sum_{n=1}^{\infty} (1 + (-1)) = 0 \end{cases}$$

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.

Section 10.3: The Integral Test

Tests for Convergence: The most basic question we can ask about a series is whether or not it converges. In the next few sections we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series we need to investigate the error involved when using a partial sum to approximate its total sum.

Non-decreasing Partial Sums: Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $S_{n+1} = S_n + a_{n+1}$, so

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \ldots$$

Since the partial sums form a non-decreasing sequence, the Monotone Convergence Theorem give us the following result:

Corollary Of MCT: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Example 1: Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

n^{th} term test:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow \ n^{\text{th}} \text{ term test is inconclusive.}$$

Note however,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1 + \frac{1}{2}}_{\frac{3}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{1}{2}$. If $n = 2^k$, the sum S_n is greater than $\frac{k}{2}$, so S_n is not bounded from above. So **the Harmonic Series diverges**. Another way of seeing this is

$$S_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} > \frac{k}{2} \overset{k\to\infty}{\longrightarrow} \infty,$$

so then $S_n \longrightarrow \infty$ and the series diverges.

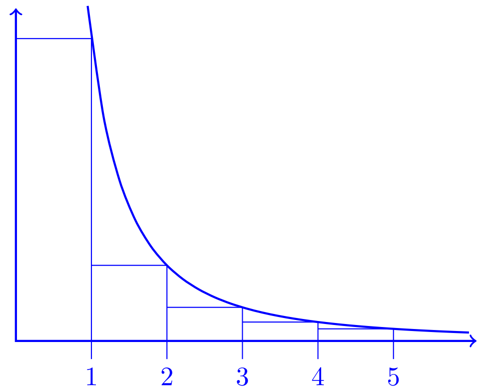
We now introduce the Integral Test with a series that is related to the harmonic series, but whose n^{th} term is $1/n^2$ instead of $1/n$.

Example 2: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We will compare the series to $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\begin{aligned} S_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2}\,dx \\ &< f(1) + \int_1^\infty \frac{1}{x^2}\,dx \\ &= 1 + 1 \\ &= 2 \end{aligned}$$



Since the partial sums are bounded above by 2, the sum *converges*.

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a ___positive___, ___continuous___, ___decreasing___ function of x. Then the series $\sum_{n=\underline{N}}^{\infty} a_n$ and the integral $\int_{\underline{N}}^{\infty} f(x)\,dx$ both converge or diverge.

Example 3: Show that the p**-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

If $p > 1$ then $f(x) = \frac{1}{x^p}$ is a positive, continuous, decreasing function of x. Since $\int_1^\infty f(x)\,dx = \frac{1}{p-1}$, the series converges by the Integral Test. Note that the sum of this series is <u>not</u> generally $\frac{1}{p-1}$. If $p \leq 0$, the sum diverges by the n^{th} term test. If $0 < p < 1$ then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \frac{1}{p-1}\left(\lim_{b\to\infty} b^{1-p} - 1\right) = \infty.$$

Example 4: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

$f(x) = xe^{-x^2}$ is positive, continuous, decreasing and $f(n) = a_n$ for all n. Further,

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\left[-e^{-b^2} - (-e^{-1})\right] = \frac{1}{2e}.$$

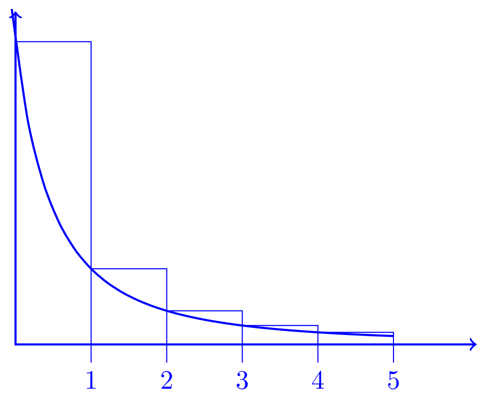
Since the integral converges, the series also converges.

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Suppose a series $\sum a_n$ is shown to be convergent by the integral test and we want to estimate the size of the remainder R_n measuring the difference between the total sum S and its n^{th} partial sum S_n.

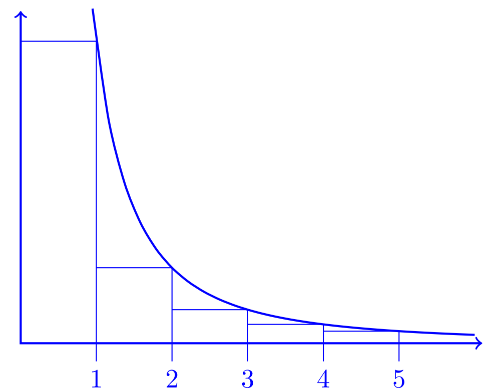
$$R_n = S - S_n = a_{n+1} + a_{n+1} + a_{n+1} + \cdots$$

Lower Bound: Shift the integral test function left 1 unit.



$$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$$

Upper Bound: The integral test function.



$$R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Example 5: Estimate the sum, S, of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.

$$\int_n^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_n^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty} \left[-\frac{1}{b} + \frac{1}{n}\right] = \frac{1}{n} \qquad \Longrightarrow \qquad S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}$$

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdot + \frac{1}{100} \approx 1.54977 \qquad \Longrightarrow \qquad 1.64068 \leq S \leq 1.64977$$

It seems reasonable that taking the midpoint of this interval would give a good estimate, so

$$S \approx 1.6452.$$

It turns out that using fancy advanced calculus (Fourier Analysis) we actually know that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493.$$

Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \leq a_n \leq \frac{1}{n}$ for all n, we can conclude nothing.

(b) If $\frac{1}{n} \leq a_n$ for all n, we can conclude $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If $0 \leq a_n \leq \frac{1}{n^2}$ for all n, we can conclude $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $\frac{1}{n^2} \leq a_n$ for all n, we can conclude nothing.

(e) If $\frac{1}{n^2} \leq a_n \leq \frac{1}{n}$ for all n, we can conclude nothing.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

(a) (i) $\frac{1}{n^2} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (ii) $\frac{1}{n+1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

(d) (i) $\frac{1}{n^2} \leq \frac{1}{n^2-1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges. (ii) $\frac{1}{n^2} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(e) (i) $\frac{1}{n^2} \leq \frac{1}{n^2-1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges. (ii) $\frac{1}{n^2} \leq \frac{1}{n+1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ _____converges_____, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ _____diverges_____, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

Converges - its a Geometric Series with $r = \dfrac{1}{2}$.

(b) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{2^n + n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = b_n.$$

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

It converges!

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

$$b_n = \frac{1}{n^2}$$

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

Converges - its a $p-$series with $p = 2$

(c) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{n^2 + n + 1} \leq \frac{1}{n^2} = b_n$$

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

It converges!

Problem 5: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^5+3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

The dominant terms of a_n are $\frac{\sqrt{n^4}}{n^5} = \frac{n^2}{n^5} = \frac{1}{n^3}$.

- Choose $b_n = \frac{1}{n^3}$.
- $a_n = \frac{\sqrt{n^4-1}}{n^5+3} < \frac{\sqrt{n^4}}{n^5+3} = \frac{n^2}{n^5+3} < \frac{n^2}{n^5} = \frac{1}{n^3} = b_n$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 3 > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 6: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3+n}}$ converges or diverges.

- $\cos^2(n) \leq 1 \implies \frac{\cos^2(n)}{\sqrt{n^3+n}} \leq \frac{1}{\sqrt{n^3+n}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \implies$ choose $b_n = \frac{1}{n^{3/2}}$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = \frac{3}{2} > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2-1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2} \leq \frac{1}{n^2-1} \implies \text{So Direct Comparison is inconclusive.}$$

(b) Show that $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{n^2-1+1}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2-1}\right) = 1$$

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \leq 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2-1} \leq \frac{2}{n^2} \iff n^2 \leq 2(n^2-1) \iff n^2 \leq 2n^2 - 2 \iff 2 \leq n^2 \iff 1 < n.$$

Yes!

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \leq \sum_{n=2}^{\infty} \frac{2}{n^2} = 2\sum_{n=2}^{\infty} \frac{1}{n^2} \text{ converges since it is a } p\text{-series} \implies \sum_{n=1}^{\infty} b_n \text{ converges!}$$

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ______converge______ or both ______diverge______.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ converges or diverges.

$$\frac{n^3-2n}{n^4+3} > \frac{n^3}{n^4+3} \text{ which behaves like } \frac{1}{n}.$$

Let $b_n = \frac{1}{n}$ and use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n^3-2n}{n^4+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-2n}{n^4+3} \cdot n = \lim_{n\to\infty} \frac{n^4-2n^2}{n^4+3} = 1 > 0$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ also *diverges.*

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ converges or diverges.

$$0 < \frac{10n+1}{n(n+1)(n+2)} \approx \frac{10n}{n^3} = 10\frac{1}{n^2} \text{ so let } b_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{10n+1}{n^3+2n^2+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{10n^3+n^2}{n^3+2n^2+2n} = \lim_{n\to\infty} \frac{10+1\frac{1}{n}}{1+\frac{2}{n}+\frac{2}{n^2}} = 10 > 0.$$

So $\sum_{n=1}^{\infty} a_n$ behaves the same way $\sum_{n=1}^{\infty} b_n$ does. Thus by the limit comparison test, $\sum_{n=1}^{\infty} a_n$ *converges.*

Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

This is a geometric series with $|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$, so it converges.

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

This is a geometric series with $|r = \left|-\frac{5}{4}\right| = \frac{5}{4} > 1$, so it diverges.

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$.

$a_n = (-1)^{n+1}\frac{1}{n^2} \Longrightarrow |a_n| = \frac{1}{n^2}$:

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$,

so $\sum_{n=1}^{\infty} a_n$ converges absolutely

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3^{n+1}}\cdot\frac{3^n}{2^n+5}\right| \\ &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3\left(2^n+5\right)}\right| \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2^{n+1}+5}{2^n+5} \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}} \\ &= \frac{2}{3} < 1\end{aligned}$$

So, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ converges absolutely by the Ratio Test.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2(n+1))!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right| &= \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\cdot\frac{n!\cdot n!}{(2n)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}}{(n+1)\cdot\cancel{n!}\cdot(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{n!}\cdot\cancel{n!}}{\cancel{(2n)!}}\right| \\ &= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\ &= \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} \\ &= 4 > 1\end{aligned}$$

So, $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

The ratio test is super useful for factorials

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2}{2^n}\right|} &= \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{2} \\ &= \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2} \\ &= \frac{1^2}{2} \\ &= \frac{1}{2} < 1\end{aligned}$$

So, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges absolutely by the Root Test.

The ratio test is super useful for a^n

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln(\sqrt[n]{n})} = e^{\lim_{n\to\infty} \frac{\ln(n)}{n}} \overset{\text{L'H}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1$$

Section 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim_{n\to\infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the requirements with $N = 1$ and therefore converges.

Instead of verifying $b_n \geq b_{n+1}$, we can follow the steps we did in the integral test to verify the sequence is decreasing. Define a differentiable function $f(x)$ satisfying $f(n) = b_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is non-increasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $b_n \geq b_{n+1}$ for all N.

Example 2: Consider the sequence where $b_n == \frac{10n}{n^2+16}$. Define $f(x) = \frac{10x}{x^2+16}$. Then $f'(x) = \frac{10(16-x^2)}{(x^2+16)} \geq 0$ when $x \geq 4$. It follows that $b_n \geq b_{n+1}$ for $n \geq 4$.

The Alternating Series Test Estimation Theorem: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies the conditions of the AST, then for $n \geq N$,

$$S_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n$$

approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term.

Furthermore, the sum L lies between any two successive partial sums S_n and S_{n+1}, and the remainder, $L - S_n$, has the same sign as the first unused term.

Example 3: Let's apply the Estimation Theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

n	Sum	S_n	$L - S_n$
0	1	1	$-\frac{1}{3}$
1	$1-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$
2	$1-\frac{1}{2}+\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{12}$
3	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{24}$
4	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$	$\frac{11}{16}$	$-\frac{1}{48}$
5	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$	$\frac{21}{32}$	$\frac{1}{96}$
6	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}$	$\frac{43}{64}$	$-\frac{1}{192}$
7	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$	$\frac{85}{128}$	$\frac{1}{384}$
8	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}$	$\frac{171}{256}$	$-\frac{1}{768}$

Example 4 - Conditional Convergence: We have seen that in absolute value, the Alternating Harmonic Series diverges. The presence of infinitely many negative terms is essential to its convergence. We say the Alternating Harmonic Series if **conditionally convergent**. We can extend this idea to the alternating p-series.
If p is a positive constant, the sequence $\frac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p} = 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots, \qquad p>0$$

converges.

- If $p>1$, the series converges absolutely.
- If $0<p\leq 1$, the series converges conditionally.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \dots, b_n \dots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Example 5: We know $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to some number L.
By the Estimation Theorem, we know $L \neq 0$ (our partial sums never "hop" over 0). So,

$$\begin{aligned} 2L &= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\right) \\ &= 2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\cdots \\ &= (2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\cdots \\ &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots \\ &= L \end{aligned}$$

(group all the terms with odd denominators together, leaving the even denominator terms alone)

So $2L = L \dots$ so $L=0$? But $L \neq 0 \dots$ oops. Thus we cannot rearrange the sum in a conditionally convergent sequence.

Section 10.7: Power Series

Definition: A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

$$\begin{aligned}
S_n &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \\
\implies (1-x)S_n &= (1-x)\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) \\
&= \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) - \left(x + x^2 + x^3 + x^4 + x^5 \cdots + x^{n+1}\right) \\
&= 1 - x^{n+1} \\
\implies S_n &= \frac{1 - x^n}{1 - x}
\end{aligned}$$

So,

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - x^n}{1 - x} \text{ which converges if and only if } |x| < 1$$

Instead of focussing on finding a formula for the sum of a power series, we are now going to think of the partial sums of the series as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need only take a few terms of the series to get a good approximation. As we move toward $x = 1$ or $x = -1$, we need more terms.

One of the most important questions we can ask about a power series is "for what values of x will the series converge?" Since power series are functions, what we are really asking here is "what is the **domain** of the power series?"

Example 2: Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

$$\text{Centre: } a = 2, \qquad c_0 = 1,\ c_1 = -\frac{1}{2},\ c_2 = \frac{1}{4},\ \ldots,\ c_n = \left(-\frac{1}{2}\right)^n,$$

$$\text{Ratio: } r = \frac{c_{n+1}(x-2)^{n+1}}{c_n(x-2)^n} = \frac{c_1(x-2)}{c_0} = \frac{-\frac{1}{2}(x-2)}{1} = -\frac{x-2}{2}$$

The series converges when $|r| < 1$, that is,

$$\left|-\frac{x-2}{2}\right| < 1 \Longrightarrow \left|\frac{x-2}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow -2 < x-2 < 2 \Longrightarrow 0 < x < 4.$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n \frac{x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1}x^n}\right| = \left|\frac{nx}{x+1}\right| = |x|\frac{n}{n+1} \overset{n\to\infty}{\longrightarrow} |x|$$

The series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. It remains to see what happens at the endpoints; $x = -1$ and $x = 1$.

$x = -1$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty}\frac{-1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n} \Longrightarrow$ the series diverges at $x = -1$.

$x = 1$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} =$ the Alternating Harmonic Series $\Longrightarrow$ the series converges at $x = 1$.

So, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for $-1 < x \leq 1$ and diverges elsewhere.

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \left|\frac{x^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} \overset{n\to\infty}{=\!\!\longrightarrow} 0$$

Since the value of the limit is 0, no matter what real number we choose for x and $0 < 1$, the series converges absolutely for all values of x. $\quad (x \in \mathbb{R},\ \ -\infty < x < \infty,\ \ (-\infty, \infty))$.

Fact: There is always at least one point for which a power series converges: the point $x = a$ at which the series is centred.

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Convergence Theorem and the previous examples lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways;

- If might converge on some interval of *radius* R. an interval has radius R if its length is $2R$
- It might converge everywhere.
- It might converge only at $x = a$.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$)
3. The series converges only at $x = a$ and diverges elsewhere ($R = 0$)

R is called the **radius of convergence** of the power series, and the interval of radius R centred at $x = 1$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series.

How to test a Power Series for Convergence:

1. Use the Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test fo convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the n^{th} term does not approach zero for those values of x.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^{3/2}3^{n+1}} \cdot \frac{n^{3/2}3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn^{3/2}}{(n+1)^{3/2}3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{3/2} = \frac{|x|}{3}.$$

So the series converges absolutely when $\frac{|x|}{3} < 1 \Longrightarrow |x| < 3 \Longrightarrow -3 < x < 3$.

Check the endpoints:

$x = -3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ which is an alternating p-series with $p = \frac{3}{2}$, so it *converges.*

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2}$, so it *converges.*

Thus the interval of convergence is $[-3, 3]$ and the radius of convergence is $R = 3$.

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants. They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product.

The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B_n(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$. For example:

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

Term-by-Term Differentiation Theorem: If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n \left(x-a\right)^n$$

on the interval $a - R < x < a + R$. This function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n \left(x-a\right)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.
Note: When we differentiate we may have to start our index at one more than it was before. This is because we lose the constant term (if it exists) when we differentiate.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=0}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

which *diverges* for all x. This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find a series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

$$f'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1.$$

$$f''(x) = \frac{2}{(1-x)^3} = 0 + 0 + 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

converges for $a - R < x < a + R$ for $R > 0$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$.

Example 6: Given $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

$$\begin{aligned} \ln(1+x) = \int_0^x \frac{1}{1+t}\,dt &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x \\ ` &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad -1 < x < 1. \end{aligned}$$

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$

Differentiate

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Now we can integrate to find $f(x)$:

$$f(x) = \int f'(t)\,dt = \arctan(x) + C.$$

Since $f(0) = 0$, we have $0 = \arctan(0) + C = C$, so then

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\arctan(x)} \qquad -1 < x < 1$$

Section 10.8: Taylor and Maclaurin Series

Series Representations: We've seen that geometric series can be used to generate a power series for functions having a special form, such as $f(x) = \dfrac{1}{1-x}$ or $g(x) = \dfrac{3}{x-2}$. Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

$$f'(x) = 1 \cdot c_1 + 2 \cdot c_2(x-a) + 3 \cdot c_3(x-a)^2 + 4 \cdot c_4(x-a)^3 + \cdots + n \cdot c_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2 \cdot 1 \cdot c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \cdots + n \cdot (n-1) \cdot c_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 3 \cdot 2 \cdot 1 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + \cdots + n \cdot (n-1) \cdot (n-2) \cdot c_n(x-a)^{n-2} + \cdots$$

$$\vdots$$

Since $x = a$ is in the assumed interval of convergence, all of the above equations are valid when $x = a$:

$$f(a) = c_0, \qquad f'(a) = 1 \cdot c_1, \qquad f''(a) = 2 \cdot 1 \cdot c_2, \qquad f'''(a) = 3 \cdot 2 \cdot 1 \cdot c_3, \qquad f^{(n)}(a) = n! \cdot c_n$$

Solving for each c_k gives:

$$c_0 = f(a), \qquad c_1 = \frac{f'(a)}{1}, \qquad c_2 = \frac{f''(a)}{2 \cdot 1}, \qquad c_3 = \frac{f'''(a)}{3 \cdot 2 \cdot 1}, \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

Thus, if $f(x)$ has such a series representation, it must have the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

On the other hand, if we start with an arbitrary function $f(x)$ that is infinitely differentiable on an interval containing $x = a$ and use it to generate the series above, will the series then converge to $f(x)$ at each x in the interval of convergence? The answer is maybe.

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \dfrac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\frac{1}{x}$	$\frac{1}{2}$
1	$(-1)\cdot\frac{1}{x^2}$	$(-1)\frac{1}{2^2}$
2	$(-1)^2\cdot\frac{2\cdot 1}{x^3}$	$(-1)^2\frac{2\cdot 1}{2^3}$
3	$(-1)^3\cdot\frac{3\cdot 2\cdot 1}{x^4}$	$(-1)^3\frac{3\cdot 2\cdot 1}{2^4}$
4	$(-1)^4\cdot\frac{4\cdot 3\cdot 2\cdot 1}{x^5}$	$(-1)^4\frac{4\cdot 3\cdot 2\cdot 1}{2^5}$
n	$(-1)^n\cdot\frac{n!}{x^{n+1}}$	$(-1)^n\frac{n!}{2^{n+1}}$

The key thing to do when looking for the general term is to not simplify everything. You should try and only group those terms that come from the "same place." For example, when $n = 2$ we could have cancelled a 2 from the numerator and denominator of $f''(2)$. But since the 2 in the numerator came from differentiating and the 2 on the denominator came from plugging in $x = a$, we leave them alone. Leaving factors alone this way will help you more easily see where each number in the factor is coming from and its relation to the value of n.

So, the Taylor Series generated by $f(x) = \dfrac{1}{x}$ centred at $a = 2$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n}$$

Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + \frac{(-1)^n(x-2)^n}{2^{n+1}}$$

is geometric with first term $\dfrac{1}{2}$ and ratio $r = -\dfrac{(x-2)}{2}$. So it converges (absolutely) for

$$\left|-\frac{(x-2)}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow 0 < x < 4.$$

Now we check the endpoints:

$x = 0$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(0-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} \Longrightarrow$ diverges. (Also clear since $f(x) = \frac{1}{x}$ is not defined at $x = 0$)

$x = 4$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(4-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \Longrightarrow$ diverges.

Thus the only values of x for which this Taylor Series converges are $\boxed{0 < x < 4}$.

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial** of order n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as the linearisation of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$2n$	$(-1)^n \cos(x)$	$(-1)^n$
$2n+1$	$(-1)^{n+1}\sin(x)$	$(-1)^n 0$

When terms are alternating between 0s and non-zero terms, take a look at the parity of the values of n for which they appear. That is, check if all the 0s occur when n is odd (or when n is even). Once you figure out which is which you can ignore all the zero terms by considering $2n$ or $2n+1$.

If you are dealing with trigonometric functions, it is likely that at some point there will be some repetition happening. For example here $f^{(4)}(x) = f(x)$. So then you might be able to see what is happening by only using the terms up until the repeat.

So the Taylor Series generated by $f(x) = \cos(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}}$$

To find the interval of convergence, we can use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)} \overset{n\to\infty}{\longrightarrow} 0$$

So this Taylor Series converges for all $x \in \mathbb{R}$.

Finally, the Taylor polynomials are given by:

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 3: Find the Maclaurin Series generated by $f(x) = \sin(x)$.

Recall that $\cos(x)$ is an <u>even function</u> and we have just discovered in Example 2 that only <u>even</u> powers of x occur in its Maclaurin Series. One would expect then that since $f(x) = \sin(x)$ is an <u>odd function</u> that only <u>odd</u> powers of x will appear in its Maclaurin Series. Indeed this is actually the case. Doing the same calculations as in Example 2 will yield the desired result.

Here however we will just invoke the power of integration: Since $\int_0^x \cos(t)\,dt = \sin(x)$ and

$$\int_0^x \frac{(-1)^n}{(2n)!} t^{2n}\,dt = \frac{(-1)^n}{(2n)!} \cdot \frac{t^{2n+1}}{(2n+1)}\bigg|_0^x = \frac{(-1)^n}{(2n+1)!} t^{2n+1}\bigg|_0^x = \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

we have the Taylor Series generated by $f(x) = \sin(x)$ is

$$\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}\,dt = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

Example 4: Find the Taylor Series generated by $f(x) = e^x$.

Note that $f^{(n)}(x) = f(x) = e^x$ for every positive integer n. So $f^{(n)}(0) = e^0 = 1$ for each n, so then the Taylor Series generated by $f(x) = e^x$ at $a = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Section 10.9: Convergence of Taylor Series

Taylors Theorem: In the last section, we asked when a Taylor Series for a function can be expected to that (generating) function. That question is answered by the following theorem:

If $f(x)$ and its first n derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ are continuous on the closed interval between a and b, and $f^{(n)}(x)$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Interesting Fact: Taylor's Theorem is a generalisation of the Mean Value Theorem!

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Stating Taylor's Theorem in this way says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where the function $R_n(x)$ is determined by the value of the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ at a point c that depends on both a and x, and that it lies somewhere between them.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the ______remainder______

______of order n______ or the ______error term______ for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

$f(x)$ has derivatives of all orders on $(-\infty, \infty)$. Using the Taylor Polynomial generated by $f(x) = e^x$ at $a = 0$ and Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$ for some 0 between 0 and x. Recall that e^x is an increasing function, so;

$x > 0$: $\quad 0 < c < x \Longrightarrow e^0 < e^c < e^x \Longrightarrow 1 < e^c < e^x$

$x < 0$: $\quad x < c < 0 \Longrightarrow e^x < e^c < e^0 \Longrightarrow e^x < e^c < 1$

$x = 0$: $\quad e^x = 1,\ x^{n+1} = 0 \Longrightarrow R_n(x) = 0$

So,

$x > 0$: $\quad |R_n(x)| = \left|\dfrac{e^c x^{n+1}}{(n+1)!}\right| \le \dfrac{e^x x^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

$x \le 0$: $\quad |R_n(x)| = \left|\dfrac{e^c x^{n+1}}{(n+1)!}\right| \le \dfrac{|x|^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

Thus $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, so the series converges to e^x on $(-\infty, \infty)$. Thus,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

This gives us a new* definition for the number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

* Recall in Calc I we showed $e = \lim\limits_{x\to 0^+} (1+x)^{1/x}$ using L'Hôpitals Rule.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 2: Show that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ converges to $\sin(x)$ for all x.

Recall that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$. Since for each n, $\left|f^{(2n+1)}(x)\right| = |\cos(x)| \le 1$ and $\left|f^{(2n)}\right| = |\sin(x)| \le 1$, let $M = 1$. Then,

$$|R_{2n+1}(x)| \le 1 \cdot \frac{|x-0|^{2n+2}}{(2n+2)!} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus the Taylor Series converges to $f(x) = \sin(x)$. That is,

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}$$

Using Taylor Series: Since every Taylor series is a power series, the operations of adding, subtracting and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 3: Using known series, find the first few terms of the Taylor series for

$$\frac{1}{3}(2x + x\cos(x))$$

using power series operations.
We have,

$$\begin{aligned}
\frac{1}{3}(2x + x\cos(x)) &= \frac{2}{3}x + \frac{x}{3}\cos(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sin(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \\
&= \frac{2}{3}x + \sum_{n=0}^{\infty}\frac{(-1)^n}{3\cdot(2n)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{3\cdot 2!} + \frac{x^5}{3\cdot 4!} - \cdots \\
&= x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots
\end{aligned}$$

Example 4: For what values of x can we replace $\sin(x)$ by the polynomial $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?
We use the fact that the Taylor series for $\sin(x)$ is an alternating series for every non-zero value of x. By the Alternating Series Estimation Theorem (Section 10.6), the error in truncating

$$\sin(x) = x - \frac{x^3}{3!}\Bigg| + \frac{x^5}{5!} - \cdots$$

is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

So the error will be less than 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \iff \quad |x|^5 < 360 \times 10^{-4} = 0.036 \quad \iff \quad |x| < \sqrt[5]{0.036} \approx 0.514.$$

So, if $= -0.514 < x < 0.514$, the error obtained from using $x - \frac{x^3}{3!}$ to approximate $\sin(x)$ will be less than 10×10^{-4}.

Moreover, by the Alternating Series Estimation Theorem, we know the estimate $x - \frac{x^3}{3!}$ is an underestimate of $\sin(x)$ when x is positive, since $\frac{x^5}{120}$ would be positive, and an overestimate if x is negative.

Section 10.10: Applications of Taylor Series

Evaluating Non-elementary Integrals: Taylor series can be used to express non-elementary integrals in terms of series. Integrals like the one in the next example arise in the study of the diffraction of light.

Example 1: Express

$$\int \sin(x^2)\,dx$$

as a power series.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \Longrightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{4n+2}$$

So,

$$\int \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{4n+2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}x^{4n+3}$$

Example 2: Estimate

$$\int_0^1 \sin(x^2)\,dx$$

with an error of less than 0.001.

Using the previous example we see

$$\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}x^{4n+3}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!} - [0] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}$$

We want to use the Alternating Series Estimation Theorem (section 10.6). So we want

$$\left|\frac{(-1)^{n+1}}{(4(n+1)+3)\cdot(2(n+1)+1)!}\right| < 0.001 \Longrightarrow \frac{1}{(4n+7)\cdot(2n+3)!} < 0.001$$
$$\Longrightarrow (4n+7)\cdot(2n+3)! > 1000$$

By trial and error we obtain $n = 1$ works. So then $\displaystyle\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} \approx \boxed{0.310}$.

If we extend this to 5 terms, we obtain

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} \approx 0.310268303.$$

This gives an error of about 1.08×10^{-9}. To guarantee this accuracy (using the error formula) for the Trapezium Rule, we would need to use about 8000 subintervals!

Euler's Identity: A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. So then

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -\sqrt{-1} \qquad i^4 = 1 \qquad i^{4n+k} = i^k \qquad i^{2n+k} = (-1)^n i^k.$$

If we substitute $x = i\theta$ into the Taylor series for e^x ans use the relations above, we obtain

$$\begin{aligned}
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{(i\theta)^{2n}}{(2n)!} + \frac{(i\theta)^{2n+1}}{(2n+1)!} \right) && \text{(split into even and odd terms)} \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n \theta^{2n}}{(2n)!} + \frac{(-1)^n i \theta^{2n+1}}{(2n+1)!} \right) && \text{(apply the indentities of } i\text{)} \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n)!} \theta^{2n} + i \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} \right) && \text{(rewrite for foreshadowing)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} && \text{(break up sum)} \\
&= \cos(\theta) + i \sin(\theta). && \text{(know things)}
\end{aligned}$$

$$\boxed{\text{Euler's Identity:} \quad e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

This identity is actually amazing. You can use this identity to derive all of the angle sum formulas, so you never need to remember them all! Also we see that $e^{i\pi} = -1$, which we can rewrite to obtain

$$\boxed{e^{i\pi} + 1 = 0}$$

which combines 5 of the most important constants in mathematics; e, π, i, 1 and 0.

Common Taylor Series

1. $\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\|x\|<1$
2. $\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty}(-1)^n x^n$	$\|x\|<1$
3. e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$\|x\|<\infty$
4. $\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$\|x\|<\infty$
5. $\cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$	$\|x\|<\infty$
6. $\ln(1+x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$	$-1<x\leq 1$
7. $\tan^{-1}(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$	$\|x\|\leq 1$