Section 10.9: Convergence of Taylor Series

Taylors Theorem: In the last section, we asked when a Taylor Series for a function can be expected to that (generating) function. That question is answered by the following theorem:

If $f(x)$ and its first n derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ are continuous on the closed interval between a and b, and $f^{(n)}(x)$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Interesting Fact: Taylor's Theorem is a generalisation of the Mean Value Theorem!

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Stating Taylor's Theorem in this way says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where the function $R_n(x)$ is determined by the value of the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ at a point c that depends on both a and x, and that it lies somewhere between them.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the ____remainder____ ____of order n____ or the ____error term____ for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

$f(x)$ has derivatives of all orders on $(-\infty, \infty)$. Using the Taylor Polynomial generated by $f(x) = e^x$ at $a = 0$ and Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$ for some 0 between 0 and x. Recall that e^x is an increasing function, so;

$x > 0$: $0 < c < x \Longrightarrow e^0 < e^c < e^x \Longrightarrow 1 < e^c < e^x$

$x < 0$: $x < c < 0 \Longrightarrow e^x < e^c < e^0 \Longrightarrow e^x < e^c < 1$

$x = 0$: $e^x = 1$, $x^{n+1} = 0 \Longrightarrow R_n(x) = 0$

So,

$x > 0$: $|R_n(x)| = \left|\frac{e^c x^{n+1}}{(n+1)!}\right| \leq \frac{e^x x^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

$x \leq 0$: $|R_n(x)| = \left|\frac{e^c x^{n+1}}{(n+1)!}\right| \leq \frac{|x|^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

Thus $\lim_{n\to\infty} R_n(x) = 0$ for all x, so the series converges to e^x on $(-\infty, \infty)$. Thus,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

This gives us a new* definition for the number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

* Recall in Calc I we showed $e = \lim_{x\to 0^+} (1+x)^{1/x}$ using L'Hôpitals Rule.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 2: Show that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ converges to $\sin(x)$ for all x.

Recall that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$. Since for each n, $\left|f^{(2n+1)}(x)\right| = |\cos(x)| \leq 1$ and $\left|f^{(2n)}\right| = |\sin(x)| \leq 1$, let $M = 1$. Then,

$$|R_{2n+1}(x)| \leq 1 \cdot \frac{|x-0|^{2n+2}}{(2n+2)!} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus the Taylor Series converges to $f(x) = \sin(x)$. That is,

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}$$

Using Taylor Series: Since every Taylor series is a power series, the operations of adding, subtracting and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 3: Using known series, find the first few terms of the Taylor series for

$$\frac{1}{3}(2x + x\cos(x))$$

using power series operations.
We have,

$$\begin{aligned}
\frac{1}{3}(2x + x\cos(x)) &= \frac{2}{3}x + \frac{x}{3}\cos(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sin(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \\
&= \frac{2}{3}x + \sum_{n=0}^{\infty}\frac{(-1)^n}{3\cdot(2n)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{3\cdot 2!} + \frac{x^5}{3\cdot 4!} - \cdots \\
&= x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots
\end{aligned}$$

Example 4: For what values of x can we replace $\sin(x)$ by the polynomial $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?
We use the fact that the Taylor series for $\sin(x)$ is an alternating series for every non-zero value of x. By the Alternating Series Estimation Theorem (Section 10.6), the error in truncating

$$\sin(x) = x - \frac{x^3}{3!}\Bigg| + \frac{x^5}{5!} - \cdots$$

is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

So the error will be less than 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \Longleftrightarrow \quad |x|^5 < 360 \times 10^{-4} = 0.036 \quad \Longleftrightarrow \quad |x| < \sqrt[5]{0.036} \approx 0.514.$$

So, if $= -0.514 < x < 0.514$, the error obtained from using $x - \frac{x^3}{3!}$ to approximate $\sin(x)$ will be less than 10×10^{-4}.

Moreover, by the Alternating Series Estimation Theorem, we know the estimate $x - \frac{x^3}{3!}$ is an underestimate of $\sin(x)$ when x is positive, since $\frac{x^5}{120}$ would be positive, and an overestimate if x is negative.