Section 10.8: Taylor and Maclaurin Series

Series Representations: We've seen that geometric series can be used to generate a power series for functions having a special form, such as $f(x) = \dfrac{1}{1-x}$ or $g(x) = \dfrac{3}{x-2}$. Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

$$f'(x) = 1\cdot c_1 + 2\cdot c_2(x-a) + 3\cdot c_3(x-a)^2 + 4\cdot c_4(x-a)^3 + \cdots + n\cdot c_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2\cdot 1\cdot c_2 + 3\cdot 2\cdot c_3(x-a) + 4\cdot 3\cdot c_4(x-a)^2 + \cdots + n\cdot(n-1)\cdot c_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 3\cdot 2\cdot 1\cdot c_3 + 4\cdot 3\cdot 2\cdot c_4(x-a) + \cdots + n\cdot(n-1)\cdot(n-2)\cdot c_n(x-a)^{n-2} + \cdots$$

$$\vdots$$

Since $x = a$ is in the assumed interval of convergence, all of the above equations are valid when $x = a$:

$$f(a) = c_0, \qquad f'(a) = 1\cdot c_1, \qquad f''(a) = 2\cdot 1\cdot c_2, \qquad f'''(a) = 3\cdot 2\cdot 1\cdot c_3, \qquad f^{(n)}(a) = n!\cdot c_n$$

Solving for each c_k gives:

$$c_0 = f(a), \qquad c_1 = \frac{f'(a)}{1}, \qquad c_2 = \frac{f''(a)}{2\cdot 1}, \qquad c_3 = \frac{f'''(a)}{3\cdot 2\cdot 1}, \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

Thus, if $f(x)$ has such a series representation, it must have the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

On the other hand, if we start with an arbitrary function $f(x)$ that is infinitely differentiable on an interval containing $x = a$ and use it to generate the series above, will the series then converge to $f(x)$ at each x in the interval of convergence? The answer is maybe.

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \dfrac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\frac{1}{x}$	$\frac{1}{2}$
1	$(-1)\cdot\frac{1}{x^2}$	$(-1)\frac{1}{2^2}$
2	$(-1)^2\cdot\frac{2\cdot 1}{x^3}$	$(-1)^2\frac{2\cdot 1}{2^3}$
3	$(-1)^3\cdot\frac{3\cdot 2\cdot 1}{x^4}$	$(-1)^3\frac{3\cdot 2\cdot 1}{2^4}$
4	$(-1)^4\cdot\frac{4\cdot 3\cdot 2\cdot 1}{x^5}$	$(-1)^4\frac{4\cdot 3\cdot 2\cdot 1}{2^5}$
n	$(-1)^n\cdot\frac{n!}{x^{n+1}}$	$(-1)^n\frac{n!}{2^{n+1}}$

The key thing to do when looking for the general term is to not simplify everything. You should try and only group those terms that come from the "same place." For example, when $n = 2$ we could have cancelled a 2 from the numerator and denominator of $f''(2)$. But since the 2 in the numerator came from differentiating and the 2 on the denominator came from plugging in $x = a$, we leave them alone. Leaving factors alone this way will help you more easily see where each number in the factor is coming from and its relation to the value of n.

So, the Taylor Series generated by $f(x) = \dfrac{1}{x}$ centred at $a = 2$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n}$$

Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + \frac{(-1)^n(x-2)^n}{2^{n+1}}$$

is geometric with first term $\dfrac{1}{2}$ and ratio $r = -\dfrac{(x-2)}{2}$. So it converges (absolutely) for

$$\left|-\frac{(x-2)}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow 0 < x < 4.$$

Now we check the endpoints:

$x = 0$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(0-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} \Longrightarrow$ diverges. (Also clear since $f(x) = \frac{1}{x}$ is not defined at $x = 0$)

$x = 4$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(4-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \Longrightarrow$ diverges.

Thus the only values of x for which this Taylor Series converges are $\boxed{0 < x < 4}$.

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial** of order n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as the linearisation of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$2n$	$(-1)^n \cos(x)$	$(-1)^n$
$2n+1$	$(-1)^{n+1} \sin(x)$	$(-1)^n 0$

When terms are alternating between 0s and non-zero terms, take a look at the parity of the values of n for which they appear. That is, check if all the 0s occur when n is odd (or when n is even). Once you figure out which is which you can ignore all the zero terms by considering $2n$ or $2n+1$.

If you are dealing with trigonometric functions, it is likely that at some point there will be some repetition happening. For example here $f^{(4)}(x) = f(x)$. So then you might be able to see what is happening by only using the terms up until the repeat.

So the Taylor Series generated by $f(x) = \cos(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$$

To find the interval of convergence, we can use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)} \xrightarrow{n\to\infty} 0$$

So this Taylor Series converges for all $x \in \mathbb{R}$.

Finally, the Taylor polynomials are given by:

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 3: Find the Maclaurin Series generated by $f(x) = \sin(x)$.

Recall that $\cos(x)$ is an even function and we have just discovered in Example 2 that only even powers of x occur in its Maclaurin Series. One would expect then that since $f(x) = \sin(x)$ is an odd function that only odd powers of x will appear in its Maclaurin Series. Indeed this is actually the case. Doing the same calculations as in Example 2 will yield the desired result.

Here however we will just invoke the power of integration: Since $\int_0^x \cos(t)\,dt = \sin(x)$ and

$$\int_0^x \frac{(-1)^n}{(2n)!}t^{2n}\,dt = \frac{(-1)^n}{(2n)!} \cdot \frac{t^{2n+1}}{(2n+1)}\bigg|_0^x = \frac{(-1)^n}{(2n+1)!}t^{2n+1}\bigg|_0^x = \frac{(-1)^n}{(2n+1)!}x^{2n+1},$$

we have the Taylor Series generated by $f(x) = \sin(x)$ is

$$\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}t^{2n}\,dt = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}$$

Example 4: Find the Taylor Series generated by $f(x) = e^x$.

Note that $f^{(n)}(x) = f(x) = e^x$ for every positive integer n. So $f^{(n)}(0) = e^0 = 1$ for each n, so then the Taylor Series generated by $f(x) = e^x$ at $a = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$