Section 10.7: Power Series

Definition: A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

$$\begin{aligned}
S_n &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \\
\implies (1-x)S_n &= (1-x)\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) \\
&= \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) - \left(x + x^2 + x^3 + x^4 + x^5 \cdots + x^{n+1}\right) \\
&= 1 - x^{n+1} \\
\implies S_n &= \frac{1 - x^n}{1 - x}
\end{aligned}$$

So,

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} \text{ which converges if and only if } |x| < 1$$

Instead of focussing on finding a formula for the sum of a power series, we are now going to think of the partial sums of the series as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need only take a few terms of the series to get a good approximation. As we move toward $x = 1$ or $x = -1$, we need more terms.

One of the most important questions we can ask about a power series is "for what values of x will the series converge?" Since power series are functions, what we are really asking here is "what is the **domain** of the power series?"

Example 2: Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

$$\text{Centre: } a = 2, \qquad c_0 = 1,\ c_1 = -\frac{1}{2},\ c_2 = \frac{1}{4},\ \ldots,\ c_n = \left(-\frac{1}{2}\right)^n,$$

$$\text{Ratio: } r = \frac{c_{n+1}(x-2)^{n+1}}{c_n(x-2)^n} = \frac{c_1(x-2)}{c_0} = \frac{-\frac{1}{2}(x-2)}{1} = -\frac{x-2}{2}$$

The series converges when $|r| < 1$, that is,

$$\left|-\frac{x-2}{2}\right| < 1 \Longrightarrow \left|\frac{x-2}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow -2 < x-2 < 2 \Longrightarrow 0 < x < 4.$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n \frac{x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1}x^n}\right| = \left|\frac{nx}{x+1}\right| = |x|\frac{n}{n+1} \overset{n\to\infty}{\longrightarrow} |x|$$

The series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. It remains to see what happens at the endpoints; $x = -1$ and $x = 1$.

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} \Longrightarrow$ the series diverges at $x = -1$.

$x = 1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} =$ the Alternating Harmonic Series $\Longrightarrow$ the series converges at $x = 1$.

So, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges for $-1 < x \le 1$ and diverges elsewhere.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} \overset{n\to\infty}{=\!\!=\!\!\longrightarrow} 0$$

Since the value of the limit is 0, no matter what real number we choose for x and $0 < 1$, the series converges absolutely for all values of x. $\quad (x \in \mathbb{R},\ \ -\infty < x < \infty,\ \ (-\infty, \infty))$.

Fact: There is always at least one point for which a power series converges: the point $x = a$ at which the series is centred.

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Convergence Theorem and the previous examples lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways;

- If might converge on some interval of *radius* R. an interval has radius R if its length is $2R$
- It might converge everywhere.
- It might converge only at $x = a$.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x $(R = \infty)$

3. The series converges only at $x = a$ and diverges elsewhere $(R = 0)$

R is called the **radius of convergence** of the power series, and the interval of radius R centred at $x = 1$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series.

How to test a Power Series for Convergence:

1. Use the Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test fo convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the n^{th} term does not approach zero for those values of x.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^{3/2}3^{n+1}} \cdot \frac{n^{3/2}3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn^{3/2}}{(n+1)^{3/2}3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{3/2} = \frac{|x|}{3}.$$

So the series converges absolutely when $\frac{|x|}{3} < 1 \implies |x| < 3 \implies -3 < x < 3$.

Check the endpoints:

$x = -3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ which is an alternating p-series with $p = \frac{3}{2}$, so it *converges.*

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2}$, so it *converges.*

Thus the interval of convergence is $[-3, 3]$ and the radius of convergence is $R = 3$.

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants. They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product.

The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B_n(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$. For example:

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

Term-by-Term Differentiation Theorem: If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on the interval $a - R < x < a + R$. This function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.
Note: When we differentiate we may have to start our index at one more than it was before. This is because we lose the constant term (if it exists) when we differentiate.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=0}^{\infty} \frac{n! \cos(n!x)}{n^2}$$

which *diverges* for all x. This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find a series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

$$f'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1.$$

$$f''(x) = \frac{2}{(1-x)^3} = 0 + 0 + 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

converges for $a - R < x < a + R$ for $R > 0$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$.

Example 6: Given $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

$$\begin{aligned}\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x \\ ` &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad -1 < x < 1.\end{aligned}$$

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$

Differentiate

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Now we can integrate to find $f(x)$:

$$f(x) = \int f'(t)\,dt = \arctan(x) + C.$$

Since $f(0) = 0$, we have $0 = \arctan(0) + C = C$, so then

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\arctan(x)} \qquad -1 < x < 1$$