Section 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim\limits_{n \to \infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the requirements with $N = 1$ and therefore converges.

Instead of verifying $b_n \geq b_{n+1}$, we can follow the steps we did in the integral test to verify the sequence is decreasing. Define a differentiable function $f(x)$ satisfying $f(n) = b_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is non-increasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $b_n \geq b_{n+1}$ for all N.

Example 2: Consider the sequence where $b_n == \dfrac{10n}{n^2 + 16}$. Define $f(x) = \dfrac{10x}{x^2 + 16}$. Then $f'(x) = \dfrac{10(16 - x^2)}{(x^2 + 16)} \geq 0$ when $x \geq 4$. It follows that $b_n \geq b_{n+1}$ for $n \geq 4$.

The Alternating Series Test Estimation Theorem: If the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies the conditions of the AST, then for $n \geq N$,

$$S_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n$$

approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term.

Furthermore, the sum L lies between any two successive partial sums S_n and S_{n+1}, and the remainder, $L - S_n$, has the same sign as the first unused term.

Example 3: Let's apply the Estimation Theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

n	Sum	S_n	$L - S_n$
0	1	1	$-\frac{1}{3}$
1	$1-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$
2	$1-\frac{1}{2}+\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{12}$
3	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{24}$
4	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$	$\frac{11}{16}$	$-\frac{1}{48}$
5	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$	$\frac{21}{32}$	$\frac{1}{96}$
6	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}$	$\frac{43}{64}$	$-\frac{1}{192}$
7	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$	$\frac{85}{128}$	$\frac{1}{384}$
8	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}$	$\frac{171}{256}$	$-\frac{1}{768}$

Example 4 - Conditional Convergence: We have seen that in absolute value, the Alternating Harmonic Series diverges. The presence of infinitely many negative terms is essential to its convergence. We say the Alternating Harmonic Series if **conditionally convergent**. We can extend this idea to the alternating p-series.

If p is a positive constant, the sequence $\frac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p} = 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots, \qquad p>0$$

converges.

- If $p > 1$, the series converges absolutely.
- If $0 < p \le 1$, the series converges conditionally.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \ldots, b_n \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Example 5: We know $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to some number L.

By the Estimation Theorem, we know $L \neq 0$ (our partial sums never "hop" over 0). So,

$$\begin{aligned}
2L &= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\right)\\
&= 2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\cdots\\
&= (2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\cdots\\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\
&= L
\end{aligned}$$

(group all the terms with odd denominators together, leaving the even denominator terms alone)

So $2L = L\ldots$ so $L = 0$? But $L \neq 0\ldots$ oops. Thus we cannot rearrange the sum in a conditionally convergent sequence.