Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

This is a geometric series with $|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$, so it converges.

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

This is a geometric series with $|r = \left|-\frac{5}{4}\right| = \frac{5}{4} > 1$, so it diverges.

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

$a_n = (-1)^{n+1}\frac{1}{n^2} \Longrightarrow |a_n| = \frac{1}{n^2}$:

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$,

so $\sum_{n=1}^{\infty} a_n$ converges absolutely

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3^{n+1}}\cdot\frac{3^n}{2^n+5}\right| \\ &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3\left(2^n+5\right)}\right| \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2^{n+1}+5}{2^n+5} \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}} \\ &= \frac{2}{3} < 1\end{aligned}$$

So, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ converges absolutely by the Ratio Test.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2(n+1))!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right| &= \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\cdot\frac{n!\cdot n!}{(2n)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}}{(n+1)\cdot\cancel{n!}\cdot(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{n!}\cdot\cancel{n!}}{\cancel{(2n)!}}\right| \\ &= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\ &= \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} \\ &= 4 > 1\end{aligned}$$

So, $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

The ratio test is super useful for factorials

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2}{2^n}\right|} &= \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{2} \\ &= \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2} \\ &= \frac{1^2}{2} \\ &= \frac{1}{2} < 1\end{aligned}$$

So, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges absolutely by the Root Test.

The ratio test is super useful for a^n

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln(\sqrt[n]{n})} = e^{\lim_{n\to\infty} \frac{\ln(n)}{n}} \overset{\text{L'H}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1$$