Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.