Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \leq a_n \leq \frac{1}{n}$ for all n, we can conclude ____nothing____.

(b) If $\frac{1}{n} \leq a_n$ for all n, we can conclude ____$\sum_{n=1}^{\infty} a_n$ diverges____.

(c) If $0 \leq a_n \leq \frac{1}{n^2}$ for all n, we can conclude ____$\sum_{n=1}^{\infty} a_n$ converges____.

(d) If $\frac{1}{n^2} \leq a_n$ for all n, we can conclude ____nothing____.

(e) If $\frac{1}{n^2} \leq a_n \leq \frac{1}{n}$ for all n, we can conclude ____nothing____.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

(a) (i) $\frac{1}{n^2} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (ii) $\frac{1}{n+1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

(d) (i) $\frac{1}{n^2} \leq \frac{1}{n^2-1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges. (ii) $\frac{1}{n^2} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(e) (i) $\frac{1}{n^2} \leq \frac{1}{n^2-1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges. (ii) $\frac{1}{n^2} \leq \frac{1}{n+1} \leq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ ____converges____, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ ____diverges____, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

Converges - its a Geometric Series with $r = \dfrac{1}{2}$.

(b) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{2^n + n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = b_n.$$

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

It converges!

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

$$b_n = \frac{1}{n^2}$$

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

Converges - its a $p-$series with $p = 2$

(c) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{n^2 + n + 1} \leq \frac{1}{n^2} = b_n$$

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

It converges!

Problem 5: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^5+3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

The dominant terms of a_n are $\frac{\sqrt{n^4}}{n^5} = \frac{n^2}{n^5} = \frac{1}{n^3}$.

- Choose $b_n = \frac{1}{n^3}$.
- $a_n = \frac{\sqrt{n^4-1}}{n^5+3} < \frac{\sqrt{n^4}}{n^5+3} = \frac{n^2}{n^5+3} < \frac{n^2}{n^5} = \frac{1}{n^3} = b_n$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 3 > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 6: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3+n}}$ converges or diverges.

- $\cos^2(n) \leq 1 \implies \frac{\cos^2(n)}{\sqrt{n^3+n}} \leq \frac{1}{\sqrt{n^3+n}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \implies$ choose $b_n = \frac{1}{n^{3/2}}$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = \frac{3}{2} > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2-1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2} \leq \frac{1}{n^2-1} \implies \text{So Direct Comparison is inconclusive.}$$

(b) Show that $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{n^2-1+1}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2-1}\right) = 1$$

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \leq 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2-1} \leq \frac{2}{n^2} \iff n^2 \leq 2(n^2-1) \iff n^2 \leq 2n^2 - 2 \iff 2 \leq n^2 \iff 1 < n.$$

Yes!

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \leq \sum_{n=2}^{\infty} \frac{2}{n^2} = 2\sum_{n=2}^{\infty} \frac{1}{n^2} \text{ converges since it is a } p\text{-series} \implies \sum_{n=1}^{\infty} b_n \text{ converges!}$$

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ____converge____ or both ____diverge____.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ converges or diverges.

$$\frac{n^3-2n}{n^4+3} > \frac{n^3}{n^4+3} \text{ which behaves like } \frac{1}{n}.$$

Let $b_n = \frac{1}{n}$ and use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n^3-2n}{n^4+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-2n}{n^4+3} \cdot n = \lim_{n\to\infty} \frac{n^4-2n^2}{n^4+3} = 1 > 0$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ also *diverges.*

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ converges or diverges.

$$0 < \frac{10n+1}{n(n+1)(n+2)} \approx \frac{10n}{n^3} = 10\frac{1}{n^2} \text{ so let } b_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{10n+1}{n^3+2n^2+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{10n^3+n^2}{n^3+2n^2+2n} = \lim_{n\to\infty} \frac{10+1\frac{1}{n}}{1+\frac{2}{n}+\frac{2}{n^2}} = 10 > 0.$$

So $\sum_{n=1}^{\infty} a_n$ behaves the same way $\sum_{n=1}^{\infty} b_n$ does. Thus by the limit comparison test, $\sum_{n=1}^{\infty} a_n$ *converges.*