Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \le a_n \le \frac{1}{n}$ for all n, we can conclude ____________________.

(b) If $\frac{1}{n} \le a_n$ for all n, we can conclude ____________________.

(c) If $0 \le a_n \le \frac{1}{n^2}$ for all n, we can conclude ____________________.

(d) If $\frac{1}{n^2} \le a_n$ for all n, we can conclude ____________________.

(e) If $\frac{1}{n^2} \le a_n \le \frac{1}{n}$ for all n, we can conclude ____________________.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ ______________________, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ ______________________, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

(b) How do the sizes of the terms a_n and b_n compare?

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

(c) How do the sizes of the terms a_n and b_n compare?

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

Problem 5: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 - 1}}{n^5 + 3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

Problem 6: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3 + n}}$ converges or diverges.

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2 - 1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} b_n$ converges or diverges?

(b) Show that $\lim_{n \to \infty} \frac{b_n}{a_n} = 1$.

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \leq 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ____________________ or both ____________________.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n + 1}{n(n+1)(n+2)}$ converges or diverges.