Section 10.3: The Integral Test

Tests for Convergence: The most basic question we can ask about a series is whether or not it converges. In the next few sections we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series we need to investigate the error involved when using a partial sum to approximate its total sum.

Non-decreasing Partial Sums: Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $S_{n+1} = S_n + a_{n+1}$, so

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \dots$$

Since the partial sums form a non-decreasing sequence, the Monotone Convergence Theorem give us the following result:

Corollary Of MCT: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Example 1: Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

n^{th} term test:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow \; n^{\text{th}} \text{ term test is inconclusive.}$$

Note however,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1 + \frac{1}{2}}_{\frac{3}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{1}{2}$. If $n = 2^k$, the sum S_n is greater than $\frac{k}{2}$, so S_n is not bounded from above. So **the Harmonic Series diverges**. Another way of seeing this is

$$S_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} > \frac{k}{2} \overset{k\to\infty}{\longrightarrow} \infty,$$

so then $S_n \longrightarrow \infty$ and the series diverges.

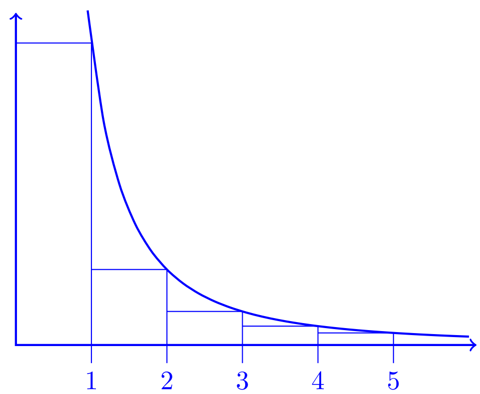
We now introduce the Integral Test with a series that is related to the harmonic series, but whose n^{th} term is $1/n^2$ instead of $1/n$.

Example 2: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We will compare the series to $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$.

$$\begin{aligned} S_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + \cdots + f(n) \\ &< f(1) + \int_1^{n} \frac{1}{x^2}\,dx \\ &< f(1) + \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= 1 + 1 \\ &= 2 \end{aligned}$$



Since the partial sums are bounded above by 2, the sum *converges*.

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a ___positive___, ___continuous___, ___decreasing___ function of x. Then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ and the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ both converge or diverge.

Example 3: Show that the *p*-**series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

If $p > 1$ then $f(x) = \dfrac{1}{x^p}$ is a positive, continuous, decreasing function of x. Since $\displaystyle\int_1^{\infty} f(x)\,dx = \frac{1}{p-1}$, the series converges by the Integral Test. Note that the sum of this series is <u>not</u> generally $\dfrac{1}{p-1}$. If $p \leq 0$, the sum diverges by the n^{th} term test. If $0 < p < 1$ then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^p}\,dx = \frac{1}{p-1}\left(\lim_{b\to\infty} b^{1-p} - 1\right) = \infty.$$

Example 4: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

$f(x) = xe^{-x^2}$ is positive, continuous, decreasing and $f(n) = a_n$ for all n. Further,

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^{b} xe^{-x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\left[-e^{-b^2} - (-e^{-1})\right] = \frac{1}{2e}.$$

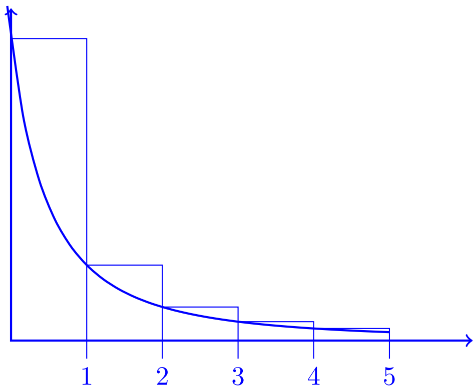
Since the integral converges, the series also converges.

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Suppose a series $\sum a_n$ is shown to be convergent by the integral test and we want to estimate the size of the remainder R_n measuring the difference between the total sum S and its n^{th} partial sum S_n.

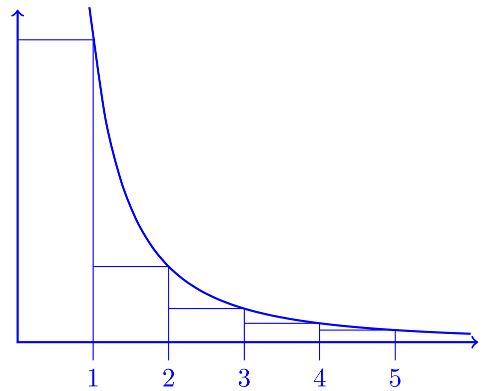
$$R_n = S - S_n = a_{n+1} + a_{n+1} + a_{n+1} + \cdots$$

Lower Bound: Shift the integral test function left 1 unit.



$$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$$

Upper Bound: The integral test function.



$$R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Example 5: Estimate the sum, S, of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.

$$\int_n^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_n^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty} \left[-\frac{1}{b} + \frac{1}{n}\right] = \frac{1}{n} \quad \Longrightarrow \quad S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}$$

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdot + \frac{1}{100} \approx 1.54977 \quad \Longrightarrow \quad 1.64068 \leq S \leq 1.64977$$

It seems reasonable that taking the midpoint of this interval would give a good estimate, so

$$S \approx 1.6452.$$

It turns out that using fancy advanced calculus (Fourier Analysis) we actually know that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493.$$