Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	$1 = \frac{2^1 - 1}{2^{1-1}}$
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2} = \frac{2^2 - 1}{2^{2-1}}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4} = \frac{2^3 - 1}{2^{3-1}}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = \lim_{n\to\infty} \left(\frac{2^n}{2^{n-1}} - \frac{1}{2^{n-1}} \right) = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}} \right) 2.$$

Since the sequence of partial sums converges, the *infinite series* converges. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ___infinite series___. The number a_n is the ___n^{th} term___ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the ___sequence of partial sums___ of the series, the number S_n being the ___n^{th} partial sum___.

If the sequence of partial sums converges to a limit L, we say that the series ___converges___ and that the ___sum___ is L. In this case we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sums of the series does not converge, we say that the series ___diverges___.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

in which a and r are fixed real numbers and $a \neq 0$. The ___ratio___ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

$$S_n = a_a(1) + a(1)^2 + a(1)^3 + \cdots + a(1)^{n-1} = na$$

and the series diverges since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \pm\infty$ (depending on the sign of a).

If $r=-1$, the series diverges since the n^{th} partial sums alternate between a and 0.

$$S_1 = a, \quad S_2 = a + a(-1) = 0, \quad a + a(-1) = a(-1)^2 = a, \quad \ldots$$

If $|r| \neq 1$, then we use the following "trick":

$$\begin{aligned} & S_n = a + ar + ar^2 + \cdots + sr^{n-1} \\ \Longrightarrow \quad & rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \\ \Longrightarrow \quad & S_n - rS_n = a - ar^n \\ \Longrightarrow \quad & S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}. \end{aligned}$$

If $|r| < 1$ then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$, so $S_n \longrightarrow \dfrac{a}{1-r}$. If $|r| > 1$ then $|r^n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and the series diverges.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 5}{4^{n-1}} = \sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}.$$

So this series is a geometric series with $a = 5$ and $r = -\dfrac{1}{4}$. Since $|r| < 1$ the series converges and so,

$$\sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1} = \frac{5}{1-(-\frac{1}{4})} = \boxed{4}$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

$$\begin{aligned} 5.232323\ldots &= 5 + \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \cdots \\ &= 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\ &= 5 + \frac{23}{100}\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^{n-1} \qquad\qquad a = 1, \quad r = \frac{1}{100} \\ &= 5 + \frac{23}{100}\left(\frac{1}{1-\frac{1}{100}}\right) \\ &= 5 + \frac{23}{100}\cdot\frac{100}{99} \\ &= \boxed{\frac{518}{99}} \end{aligned}$$

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

If we take the partial sum decomposition,

$$\sum_{n=1} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

then its easy to see that the partial sums are,

$$S_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \overset{n\to\infty}{\longrightarrow} 1.$$

Since the sequence of partial sums converges, the series converges and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \boxed{1}$

Theorem: If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Suppose $\{S_n\}_{n=1}^{\infty}$ converges to L. Then note that $\{S_{n+1}\}_{n=1}^{\infty}$ also converges to L. So then,

$$0 = L - L = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n+1} - S_n) = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

The n^{th} Term Test for Divergence: The series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* if $\displaystyle\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

1. $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges since $\displaystyle\lim_{n\to\infty} n^2 = \infty$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\displaystyle\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B,$ 2) Constant Multiple Rule : $\displaystyle\sum_{n=1}^{\infty} ca_n = cA,$ for any $c \in \mathbb{R}$.

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Caution! $\sum (a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ diverge!.

$$\begin{cases} \sum_{n=1}^{\infty} 1 \text{ diverges} \\ \sum_{n=1}^{\infty} (-1) \text{ diverges} \\ \sum_{n=1}^{\infty} (1 + (-1)) = 0 \end{cases}$$

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.