Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	1
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ________________________. The number a_n is the ____________________ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the _________________________________ of the series, the number S_n being the __________________________.

If the sequence of partial sums converges to a limit L, we say that the series _____________________ and that the

________________ is L. In this case we write

If the sequence of partial sums of the series does not converge, we say that the series ____________________.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The _________________ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The n^{th} Term Test for Divergence: The series $\sum_{n=1}^{\infty} a_n$ *diverges* if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\sum_{n=1}^{\infty} (a_n + b_n)$ 2) Constant Multiple Rule : $\sum_{n=1}^{\infty} ca_n$

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.