Section 10.1: Sequences

Definition: A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers,

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots.$$

The order is important here, for example 2, 4, 6, 8, ... is *not* the same as 4, 2, 6, 8, ...

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_1 = \dfrac{1}{1+1} = \dfrac{1}{2},\ a_2 = \dfrac{2}{2+1} = \dfrac{2}{3},\ a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}, \dots$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ $\qquad a_1 = \dfrac{(-1)^1(1+1)}{3^1} = \dfrac{-2}{3},\ a_2 = \dfrac{(-1)^2(2+1)}{3^2} = \dfrac{1}{3},\ a_3 = \dfrac{(-1)^3(3+1)}{3^3} = \dfrac{-4}{27}, \dots$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

$$\begin{aligned} f_3 &= f_2 + f_1 = 1 + 1 = 2, \\ f_4 &= f_3 + f_2 = 2 + 1 = 3, \\ f_5 &= f_4 + f_3 = 3 + 2 = 5, \\ f_6 &= f_5 + f_4 = 5 + 3 = 8, \quad \dots \end{aligned}$$

Definition: (**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$\text{for all } n \geq N \qquad |a_n - L| < \varepsilon.$$

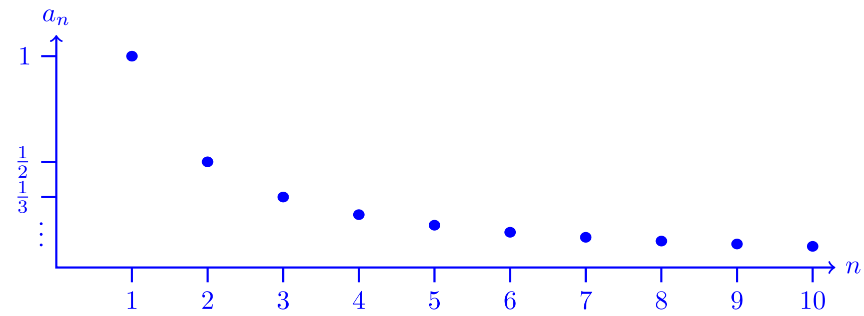
If no such number L exists, we say that $\{a_n\}$ **diverges**.

Definition: (**Friendly Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if

$$\lim_{n\to\infty} a_n = L.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?



From the plot above it looks as if the sequence is tending towards 0. It seems that plotting sequences looks a lot like plotting a function. In fact, we can use our knowledge of functions to infer things about sequences.

Theorem: **(Continuous Function Theorem)** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ whenever n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

We know that $f(x) = \frac{1}{x}$ satisfies $f(n) = a_n$ for every positive integer n, so then

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

In truth, the limit of this sequence is clear without invoking the power of this theorem. But, the theorem is still a great tool that we can use for more complicated sequences.

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences: (i.e. the limit rules you already know for functions)

If $a_n \longrightarrow L$, $b_n \longrightarrow M$, then:

1. Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = L + M$,
2. Constant Rule: $\lim_{n\to\infty} c = c$ for any $c \in \mathbb{R}$,
3. Product Rule: $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$,
4. Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, if $M \neq 0$
5. Power Rule: $\lim_{n\to\infty} a_n^p = L^p$, if $p > 0$, $a_n > 0$

Squeeze Theorem: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$$

2. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

Note that $f(x) := \dfrac{\ln(x)}{x}$ satisfies $f(n) = a_n$ for each positive integer n. So,

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = \boxed{0}$$

3. $\left\{\dfrac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

Since $-1 \leq \cos(n) \leq 1$ for all $n \in \mathbb{N}$, we have $-\dfrac{1}{n} \leq \dfrac{\cos(n)}{n} \leq \dfrac{1}{n}$ and since

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

we have $\lim_{n\to\infty} \dfrac{\cos(n)}{n} = 0$, by the Squeeze Theorem.

4. $\left\{\dfrac{(-1)^n}{n}\right\}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \boxed{0}$$

Examples of Divergent Sequences:

1. $\{(-1)^n\}_{n=1}^{\infty}$

2. $\{(-1)^n n\}_{n=1}^{\infty}$

3. $\{\sin(n)\}_{n=1}^{\infty}$

Definition: The product of the first n positive integers,

$$n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is denoted by $n!$ (read n **factorial**.

Convention: $0! = 1$

Example 1: Find the limit of the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Observe,

$$a_1 = \frac{1!}{1^1} = \frac{1}{1} \leq \frac{1}{1}$$

$$a_2 = \frac{2!}{2^2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{2} \cdot \frac{1}{2} \leq \frac{1}{2}$$

$$a_3 = \frac{3!}{3^3} = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} = \underbrace{\frac{3}{3} \cdot \frac{2}{3}}_{\leq 1} \cdot \frac{1}{3} \leq \frac{1}{3}$$

$$\vdots$$

$$a_n = \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n}}_{\leq 1} \cdot \frac{1}{n} \leq \frac{1}{n}$$

So we have $0 \leq a_n \leq \frac{1}{n}$, so by the Squeeze Theorem $\lim_{n \to \infty} a_n = 0$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

- If $r > 1$, $\lim_{n \to \infty} r^n = \infty$
- If $r = 1$, $\lim_{n \to \infty} r^n = 1$
- If $0 < r < 1$, $\lim_{n \to \infty} r^n = 0$
- If $r = 0$, $\lim_{n \to \infty} r^n = 0$
- If $-1 < r < 0$, $\lim_{n \to \infty} r^n = 0$
- If $r \leq -1$, $\{r^n\}_{n=1}^{\infty}$ diverges

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is ____bounded from above____ if there exists a number M such that $a_n \leq M$ for all n.

The number M is an ____upper bound____ for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the

____least upper bound (supremum)____ of $\{a_n\}_{n=1}\infty$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is ____bounded from below____ if there exists a number m such that $a_n \geq m$ for all n.

The number m is a ____lower bound____ for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the

____greatest lower bound (infimum)____ of $\{a_n\}_{n=1}\infty$.

(c) **Completeness Axiom**: If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound b. Similarly for least upper bound.

(d) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is ____bounded____.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an ____unbounded____ sequence.

(e) Every ____convergent____ sequence is ____bounded____ but **not** every ____bounded sequence____

____converges____. (consider ____$a_n = (-1)^n$____).

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ is ____non-decreasing____ if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ____non-increasing____ if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ____monotonic____ if it is either non-decreasing or non-increasing.

The Monotone Convergence Theorem: Every bounded, monotonic sequence converges.

Note: The Monotone Convergence Theorem ONLY tells us that the limit exists, NOT the value of the limit. It also tells us that a non-decreasing sequence converges when it is bounded from above, but diverges to infinity otherwise.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

$$a_1 = 2, \qquad a_2 = \frac{1}{2}(2+6) = 4, \qquad a_3 = \frac{1}{2}(4+6) = 5, \qquad \frac{11}{2}, \qquad \frac{23}{4}, \ldots.$$

It seems that the sequence is increasing. Lets prove this by *induction*. Suppose that $a_{k-1} > a_k$ for some $k > 2$. If we can show $a_{k+1} > a_k$ then we are done. Indeed,

$$a_{k-1} < a_k \Longrightarrow a_{k-1} + 6 < a_k + 6 \Longrightarrow a_k = \frac{1}{2}(a_{k-1} + 6) < \frac{1}{2}(a_k + 6) = a_{k+1}.$$

Thus $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence. If we show that the sequence is bounded we can use the Monotone Convergence Theorem. We know that it is bounded below by 2, since we just showed it was an increasing sequence. Note too that, at least for the ones we checked, $a_k < 6$. So,

$$a_{k+1} = \frac{1}{2}\left(a_k + 6\right) < \frac{1}{2}(6+6) = 6.$$

So we have $\{a_n\}_{n=1}^{\infty}$ is bounded above by 6. So, by the Monotone Convergence Theorem $\{a_n\}_{n=1}^{\infty}$ converges.

To find the limit, let $L := \lim_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L+6) \Longrightarrow 2L = L + 6 \Longrightarrow L = 6.$$

So $\lim\limits_{n\to\infty} a_n = 6$.