Section 10.1: Sequences

Definition: A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The order is important here, for example 2, 4, 6, 8, ... is *not* the same as 4, 2, 6, 8, ...

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

Definition: (**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$\text{for all } n \geq N \qquad |a_n - L| < \varepsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences:

If $a_n \longrightarrow L$, $b_n \longrightarrow M$, then:

1. Sum Rule: $\lim\limits_{n\to\infty} (a_n + b_n) =$,
2. Constant Rule: $\lim\limits_{n\to\infty} c =$,
3. Product Rule: $\lim\limits_{n\to\infty} a_n \cdot b_n =$,
4. Quotient Rule: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} =$
5. Power Rule: $\lim\limits_{n\to\infty} a_n^p =$

Squeeze Theorem: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim\limits_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

2. $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

3. $\left\{\frac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

4. $\left\{\frac{(-1)^n}{n}\right\}$

Examples of Divergent Sequences:

1. $\{(-1)^n\}_{n=1}^{\infty}$

2. $\{(-1)^n n\}_{n=1}^{\infty}$

3. $\{\sin(n)\}_{n=1}^{\infty}$

Definition: The product of the first n positive integers,

$$n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is denoted by $n!$ (read n **factorial**.)

Example 1: Find the limit of the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is __________________________ if there exists a number M such that $a_n \leq M$ for all n.

The number M is an ____________________ for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the

__________________________________ of $\{a_n\}_{n=1}\infty$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is __________________________ if there exists a number m such that $a_n \geq m$ for all n.

The number m is a ____________________ for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the

__________________________________ of $\{a_n\}_{n=1}\infty$.

(c) **Completeness Axiom**: If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound b. Similarly for least upper bound.

(d) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is __________________.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an ____________________ sequence.

(e) Every ____________________ sequence is __________________ but **not** every _________________________

__________________. (consider ____________________).

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______________________ if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ______________________ if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ____________________ if it is either non-decreasing or non-increasing.

The Monotone Convergence Theorem: Every bounded, monotonic sequence converges.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	1
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an __________________________. The number a_n is the ______________________ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the ____________________________________ of the series, the number S_n being the ___________________________.

If the sequence of partial sums converges to a limit L, we say that the series _______________________ and that the

__________________ is L. In this case we write

If the sequence of partial sums of the series does not converge, we say that the series ______________________.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The ___________________ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The n^{th} Term Test for Divergence: The series $\sum_{n=1}^{\infty} a_n$ *diverges* if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\sum_{n=1}^{\infty} (a_n + b_n)$

2) Constant Multiple Rule : $\sum_{n=1}^{\infty} ca_n$

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.

Section 10.3: The Integral Test

Tests for Convergence: The most basic question we can ask about a series is whether or not it converges. In the next few sections we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series we need to investigate the error involved when using a partial sum to approximate its total sum.

Non-decreasing Partial Sums: Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $S_{n+1} = S_n + a_{n+1}$, so

Since the partial sums form a non-decreasing sequence, the Monotone Convergence Theorem give us the following result:

Corollary Of MCT: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Example 1: Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We now introduce the Integral Test with a series that is related to the harmonic series, but whose n^{th} term is $1/n^2$ instead of $1/n$.

Example 2: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a ____________, ____________, ____________ function of x. Then the series $\sum_{n=\_\_\_}^{\infty} a_n$ and the integral $\int_{\_\_\_}^{\infty} f(x)\,dx$ both converge or diverge.

Example 3: Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Example 4: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Example 5: Estimate the sum, S, of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.

Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \leq a_n \leq \frac{1}{n}$ for all n, we can conclude ____________________.

(b) If $\frac{1}{n} \leq a_n$ for all n, we can conclude ____________________.

(c) If $0 \leq a_n \leq \frac{1}{n^2}$ for all n, we can conclude ____________________.

(d) If $\frac{1}{n^2} \leq a_n$ for all n, we can conclude ____________________.

(e) If $\frac{1}{n^2} \leq a_n \leq \frac{1}{n}$ for all n, we can conclude ____________________.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ ________________________, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ ________________________, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

(b) How do the sizes of the terms a_n and b_n compare?

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

(c) How do the sizes of the terms a_n and b_n compare?

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

Problem 5: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 - 1}}{n^5 + 3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

Problem 6: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3 + n}}$ converges or diverges.

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2 - 1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} b_n$ converges or diverges?

(b) Show that $\lim_{n \to \infty} \frac{b_n}{a_n} = 1$.

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \leq 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ____________________ or both ____________________.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n + 1}{n(n+1)(n+2)}$ converges or diverges.

Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.

Section 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim\limits_{n\to\infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the requirements with $N = 1$ and therefore converges.

Instead of verifying $b_n \geq b_{n+1}$, we can follow the steps we did in the integral test to verify the sequence is decreasing. Define a differentiable function $f(x)$ satisfying $f(n) = b_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is non-increasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $b_n \geq b_{n+1}$ for all N.

Example 2: Consider the sequence where $b_n == \dfrac{10n}{n^2+16}$. Define $f(x) = \dfrac{10x}{x^2+16}$. Then $f'(x) = \dfrac{10(16-x^2)}{(x^2+16)} \geq 0$ when $x \geq 4$. It follows that $b_n \geq b_{n+1}$ for $n \geq 4$.

The Alternating Series Test Estimation Theorem: If the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies the conditions of the AST, then for $n \geq N$,

$$S_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n$$

approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term.
Furthermore, the sum L lies between any two successive partial sums S_n and S_{n+1}, and the remainder, $L - S_n$, has the same sign as the first unused term.

Example 3: Let's apply the Estimation Theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

Example 4 - Conditional Convergence: We have seen that in absolute value, the Alternating Harmonic Series diverges. The presence of infinitely many negative terms is essential to its convergence. We say the Alternating Harmonic Series if **conditionally convergent**. We can extend this idea to the alternating p-series.
If p is a positive constant, the sequence $\frac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p} = 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots, \qquad p>0$$

converges.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \ldots, b_n \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Example 5: We know $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to some number L.

Section 10.7: Power Series

Definition: A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

Instead of focussing on finding a formula for the sum of a power series, we are now going to think of the partial sums of the series as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need only take a few terms of the series to get a good approximation. As we move toward $x = 1$ or $x = -1$, we need more terms.

One of the most important questions we can ask about a power series is "for what values of x will the series converge?" Since power series are functions, what we are really asking here is "what is the **domain** of the power series?"

Example 2: Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Fact: There is always at least one point for which a power series converges: the point $x = a$ at which the series is centred.

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Convergence Theorem and the previous examples lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways;

- If might converge on some interval of *radius* R.
- It might converge everywhere.
- It might converge only at $x = a$.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x $(R = \infty)$

3. The series converges only at $x = a$ and diverges elsewhere $(R = 0)$

R is called the **radius of convergence** of the power series, and the interval of radius R centred at $x = 1$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series.

How to test a Power Series for Convergence:

1. Use the Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test fo convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the n^{th} term does not approach zero for those values of x.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants. They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product.

The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B_n(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$. For example:

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

Term-by-Term Differentiation Theorem: If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on the interval $a - R < x < a + R$. This function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.
Note: When we differentiate we may have to start our index at one more than it was before. This is because we lose the constant term (if it exists) when we differentiate.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=0}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

which *diverges* for all x. This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find a series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

converges for $a - R < x < a + R$ for $R > 0$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$.

Example 6: Given $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$

Section 10.8: Taylor and Maclaurin Series

Series Representations: We've seen that geometric series can be used to generate a power series for functions having a special form, such as $f(x) = \dfrac{1}{1-x}$ or $g(x) = \dfrac{3}{x-2}$. Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial** of order n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as the linearisation of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $a = 0$.

Example 3: Find the Maclaurin Series generated by $f(x) = \sin(x)$.

Example 4: Find the Taylor Series generated by $f(x) = e^x$.

Section 10.9: Convergence of Taylor Series

Taylors Theorem: In the last section, we asked when a Taylor Series for a function can be expected to that (generating) function. That question is answered by the following theorem:

If $f(x)$ and its first n derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ are continuous on the closed interval between a and b, and $f^{(n)}(x)$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Interesting Fact: Taylor's Theorem is a generalisation of the Mean Value Theorem!

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Stating Taylor's Theorem in this way says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where the function $R_n(x)$ is determined by the value of the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ at a point c that depends on both a and x, and that it lies somewhere between them.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the ______________________

______________________or the ______________________ for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 2: Show that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ converges to $\sin(x)$ for all x.

Using Taylor Series: Since every Taylor series is a power series, the operations of adding, subtracting and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 3: Using known series, find the first few terms of the Taylor series for

$$\frac{1}{3}(2x + x\cos(x))$$

using power series operations.

Example 4: For what values of x can we replace $\sin(x)$ by the polynomial $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?

Section 10.10: Applications of Taylor Series

Evaluating Non-elementary Integrals: Taylor series can be used to express non-elementary integrals in terms of series. Integrals like the one in the next example arise in the study of the diffraction of light.

Example 1: Express

$$\int \sin(x^2)\,dx$$

as a power series.

Example 2: Estimate

$$\int_0^1 \sin(x^2)\,dx$$

with an error of less than 0.001.

If we extend this to 5 terms, we obtain

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} \approx 0.310268303.$$

This gives an error of about 1.08×10^{-9}. To guarantee this accuracy (using the error formula) for the Trapezium Rule, we would need to use about 8000 subintervals!

Euler's Identity: A complex number is a number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$. So then

$$i=\sqrt{-1} \qquad i^2=-1 \qquad i^3=-\sqrt{-1} \qquad i^4=1$$

If we substitute $x=i\theta$ into the Taylor series for e^x ans use the relations above, we obtain

This identity is actually amazing. You can use this identity to derive all of the angle sum formulas, so you never need to remember them all! Also we see that $e^{i\pi}=-1$, which we can rewrite to obtain

$$\boxed{e^{i\pi}+1=0}$$

which combines 5 of the most important constants in mathematics; e, π, i, 1 and 0.

Common Taylor Series

1. $\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$-1 < x \leq 1$
7. $\tan^{-1}(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$\lvert x\rvert \leq 1$