MATH 142: Calculus II

Joseph C Foster
University of South Carolina
Summer 19

Last updated: May 15, 2019

Section 8.1: Using Basic Integration Formulas

A Review: The basic integration formulas summarise the forms of indefinite integrals for may of the functions we have studied so far, and the substitution method helps us use the table below to evaluate more complicated functions involving these basic ones. So far, we have seen how to apply the formulas directly and how to make certain u-substitutions. Sometimes we can rewrite an integral to match it to a standard form. More often however, we will need more advanced techniques for solving integrals. First, let's look at some examples of our known methods.

Basic integration formulas			
1. $\int k\,dx = kx + C$	(any number k)	12. $\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$(n \neq -1)$	13. $\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$	
3. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$		14. $\int \sec(x)\,dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$	
4. $\int e^x\,dx = e^x + C$		15. $\int \csc(x)\,dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$	
5. $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$	$(a > 0,\ a \neq 1)$	16. $\int \sinh(x)\,dx = \cosh(x) + C$	
6. $\int \sin(x)\,dx = -\cos(x) + C$		17. $\int \cosh(x)\,dx = \sinh(x) + C$	
7. $\int \cos(x)\,dx = \sin(x) + C$		18. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
8. $\int \sec^2(x)\,dx = \tan(x) + C$		19. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
9. $\int \csc^2(x)\,dx = -\cot(x) + C$		20. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$	$(a > 0)$
10. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$		21. $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
11. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$		22. $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$	$(x > a > 0)$

Example 1 - Substitution: Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

$u = x^2 - 3x + 1$

$du = 2x - 3\,dx$

$u = (3)^2 - 3(3) + 1 = 1$

$u = (5)^2 - 3(5) + 1 = 11$

$$\begin{aligned}\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx &= \int_1^{11} \frac{1}{\sqrt{u}}\,du \\ &= \int_1^{11} u^{-1/2}\,du \\ &= 2u^{1/2}\Big|_1^{11} \\ &= 2\sqrt{11} - 2\sqrt{1} \\ &= \boxed{2\left(\sqrt{11}-1\right)}\end{aligned}$$

Example 2 - Complete the Square: Find

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx.$$

$$\begin{aligned}8x - x^2 &= -(x^2 - 8x) \\ &= -((x-4)^2 - 4^2) \\ &= 4^2 - (x-4)^2\end{aligned}$$

$u = x - 4$

$du = dx$

$$\begin{aligned}\int \frac{1}{\sqrt{8x-x^2}}\,dx &= \int \frac{1}{\sqrt{4^2-(x-4)^2}}\,dx \\ &= \int \frac{1}{\sqrt{4^2-(u)^2}}\,du \\ &= \sin^{-1}\left(\frac{u}{4}\right) + C \\ &= \boxed{\sin^{-1}\left(\frac{x-4}{4}\right) + C}\end{aligned}$$

Example 3 - Trig Identities: Calculate

$$\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx.$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$\begin{aligned}\int \cos(x)\sin(2x) + \sin(x)\cos(2x)\,dx &= \int \sin(x+2x)\,dx \\ &= \int \sin(3x)\,dx \\ &= \int \frac{1}{3}\sin(u)\,du \\ &= -\frac{1}{3}\cos(u) + C \\ &= \boxed{-\frac{1}{3}\cos(3x) + C}\end{aligned}$$

Example 4 - Trig Identities: Find

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\, dx.$$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)} \cdot \frac{1+\sin(x)}{1+\sin(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\sin(x)}{1-\sin^2(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(x)} + \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}\, dx \\
&= \int_0^{\frac{\pi}{4}} \sec^2(x) + \sec(x)\tan(x)\, dx \\
&= \tan(x) + \sec(x)\Big|_0^{\frac{\pi}{4}} \\
&= \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) - (\tan(0) + \sec(0)) \\
&= 1 + \sqrt{2} - (0+1) \\
&= \boxed{\sqrt{2}}
\end{aligned}$$

Example 5 - Clever Substitution Evaluate

$$\int \frac{1}{(1+\sqrt{x})^3}\, dx.$$

$$\begin{aligned}
u &= 1+\sqrt{x} \\
du &= \frac{1}{2\sqrt{x}}\, dx \\
2\sqrt{x}\, du &= dx \\
2(u-1)\, du &= dx
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{(1+\sqrt{x})^3}\, dx &= \int \frac{2(u-1)}{u^3}\, du \\
&= \int \frac{2}{u^2} - \frac{2}{u^3}\, du \\
&= \int 2u^{-2} - 2u^{-3}\, du \\
&= -2u^{-1} + u^{-2} + C \\
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \boxed{-\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C}
\end{aligned}$$

Example 6 - Properties of Trig Integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos(x)\, dx.$$

$$f(x) = x^3 \implies f(-x) = (-x)^3 = -x^3 = -f(x)$$
$$\implies x^3 \text{ is an odd function}$$

$$g(x) = \cos(x) \implies f(-x) = \cos(-x) = \cos(x) = f(x)$$
$$\implies x^3 \text{ is an even function}$$

Putting these two facts together we see that $x^3\cos(x)$ is an odd function and is symmetric over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus (by Theorem 8, Section 5.6)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos(x)\, dx = \boxed{0}$$

Section 8.2: Techniques of Integration

A New Technique: **Integration by parts** is a technique used to simplify integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when one of the functions ($f(x)$ or $g(x)$) can be differentiated repeatedly and the other function can be integrated repeatedly without difficulty. The following are two such integrals:

$$\int x\cos(x)\,dx \text{ and } \int x^2e^x\,dx.$$

Notice $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly (they are even eventually zero) and $g(x) = \cos(x)$ and $g(x) = e^x$ can be integrated repeatedly without difficulty.

An Application of the Product Rule: If $f(x)$ and $g(x)$ are differentiable functions of x, the product rule says that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides and rearranging gives us the **Integration by Parts** formula!

$$\begin{aligned} & \int \frac{d}{dx}\left[f(x)g(x)\right]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \\ \implies \quad & \int f(x)g'(x)\,dx = \int \frac{d}{dx}\left[f(x)g(x)\right]\,dx - \int f'(x)g(x)\,dx \\ \implies \quad & \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx} \end{aligned}$$

In differential form, let $u = f(x)$ and $v = g(x)$. Then,

Integration by Parts Formula:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int v\,du$, is easier to evaluate. The key to integration by parts is making the right choice for u and v. Sometimes we may need to try multiple options before we can apply the formula.

Example 1: Find

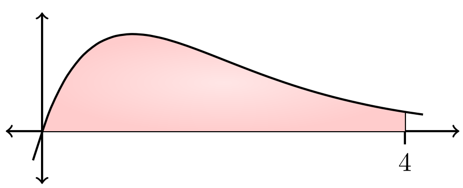
$$\int x\cos(x)\,dx.$$

We have to decide what to assign to u and what to assign to dv. Our goal is to make the integral *easier*. One thing to bear in mind is that whichever term we let equal u we need to differentiate - so if differentiating makes a part of the integrand simpler that's probably what we want! In this cases differentiating $\cos(x)$ gives $-\sin(x)$, which is no easier to deal with. But differentiating x gives 1 which *is* simpler. So we have,

$$u = x \qquad dv = \cos(x)\,dx$$
$$du = dx \qquad v = \sin(x)$$

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= \boxed{x\sin(x) + \cos(x) + C}\end{aligned}$$

Example 3 - Integration by Parts for Definite Integrals: Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



$$A = \int_0^4 xe^{-x}\,dx$$

$$u = x \qquad dv = e^{-x}\,dx$$
$$du = dx \qquad v = -e^{-x}$$

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big|_0^4 - \int_0^4 -e^{-x}\,dx \\ &= -xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}\,dx \\ &= \left(-4e^{-4} - 0\right) - e^{-x}\Big|_0^4 \\ &= -4e^{-4} - \left(e^{-4} - 1\right) \\ &= -4e^{-4} - \left(e^{-4} - 1\right) \\ &= \boxed{-5e^{-4} + 1}\end{aligned}$$

Example 3: Evaluate

$$\int x^2 e^x \, dx.$$

Here we go through the same thought process. If $u = e^x$ then $du = e^x \, dx$, which doesn't make the problem any easier (though it doesn't make it any harder either). But in this case $dv = x^2$ would give $v = \frac{1}{3}x^3$ which arguably is *not* simpler that x^2. So,

$$u = x^2 \qquad dv = e^x \, dx$$
$$du = 2x \, dx \qquad v = e^x$$

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

It's at this point we see that we still cannot integrate the integral on the write easily. This is okay. **Sometimes we may have to apply the integration by parts formula more than once!**

$$u = x \qquad dv = e^x \, dx$$
$$du = dx \qquad v = e^x$$

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= \boxed{\left(x^2 - 2x + 2\right) e^x + C} \end{aligned}$$

The previous technique works for any integral of the form $\int x^n e^{mx} \, dx$, where n is any *positive* integer and m is any integer. What if n was negative? Then this case we would set $u = e^x$.

Example 4 - Tabular Method: In Example 2 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 2 again.

Evaluate

$$\int x^2 e^x \, dx.$$

Let $f(x) = x^2$ and $g(x) = e^x$. Then,

Differentiate $f(x)$		Integrate $g(x)$
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Then the integral is,

$$\int x^2 e^x \, dx = +x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C = \boxed{\left(x^2 - 2x + 2\right) e^x + C}$$

We *have* actually used the integration by parts formula, but we have just made our lives easier by condensing the work into a neat table. This method is extremely useful when Integration by Parts needs to be used over and over again.

Example 5 - Recurring Integrals: Find the integral

$$\int e^x \sin(x)\,dx.$$

We need to apply Integration by Parts twice before we see something:

(1)

$$u = e^x \qquad dv = \sin(x)$$
$$du = e^x\,dx \qquad v = -\cos(x)$$

(2)

$$u = e^x \qquad dv = \cos(x)$$
$$du = e^x\,dx \qquad v = \sin(x)$$

$$\begin{aligned}\int e^x \sin(x)\,dx &= -e^x\cos(x) + \int e^x\cos(x)\,dx \\ &= -e^x\cos(x) + \left(e^x\sin(x) - \int e^x\sin(x)\,dx\right) \\ &= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx\end{aligned}$$

Notice that now the integral we are interested in, $\int e^x\sin(x)\,dx$, appears on both the left and right hand side of the equation. So, if we add this integral to both sides we get

$$\implies \quad 2\int e^x\sin(x)\,dx = e^x\left(-\cos(x) + \sin(x)\right)$$
$$\implies \quad \int e^x\sin(x)\,dx = \boxed{\frac{e^x\left(\sin(x) - \cos(x)\right)}{2}}$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their n^{th} derivative (where n is the degree of the polynomial) is identically 0.

Example 6 - Challenge: Find the integral

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos(nx)\, dx,$$

where n is a positive integer.

Let $f(x) = x^2$ and $g(x) = \cos(nx)$. Then,

Differentiate $f(x)$		Integrate $g(x)$
x^3	$+$	$\cos(nx)$
$3x^2$	$-$	$\frac{1}{n}\sin(nx)$
$6x$	$+$	$-\frac{1}{n^2}\cos(nx)$
6	$-$	$-\frac{1}{n^3}\sin(nx)$
0		$\frac{1}{n^4}\cos(nx)$

Then the integral is,

$$\begin{aligned}
\frac{1}{\pi}\int x^3\cos(nx)\,dx &= \frac{1}{\pi}\left[+x^3\cdot\frac{1}{n}\sin(nx) - 3x^2\cdot\left(-\frac{1}{n^2}\right)\cos(nx) + 6x\cdot\left(-\frac{1}{n^3}\right)\sin(nx) - 6\cdot\frac{1}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{x^3}{n}\sin(nx) + \frac{3x^2}{n^2}\cos(nx) - \frac{6x}{n^3}\sin(nx) - \frac{6}{n^4}\cos(nx)\right]\Bigg|_0^{\pi} \\
&= \frac{1}{\pi}\left[\left(0 + \frac{3\pi^2}{n^2}\cos(n\pi) - 0 + \frac{6}{n^4}\right) - \left(0 + 0 - 0 - \frac{6}{n^4}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{3\pi^2(-1)^n}{n^2} - \frac{6(-1)^n}{n^4} + \frac{6}{n^4}\right] \\
&= \boxed{\frac{3}{\pi}\,\frac{\pi^2 n^2(-1)^n - 2(-1)^n + 2}{n^4}}
\end{aligned}$$

Section 8.3: Trigonometric Integrals - Worksheet

Goal: By using trig identities combined with u-substitution, we'd like to find antiderivatives of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(for integer values of m and n). The goal of this worksheet[1] is for you to work together in groups of 2-3 to discover the techniques that work for these anti-derivatives.

Example 1 - Warm-up: Find

$$\int \cos^4(x)\sin(x)\,dx.$$

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \cos^4(x)\sin(x)\,dx &= -\int u^4\,du \\ &= -\frac{u^5}{5} + C \\ &= \boxed{-\frac{\cos^5(x)}{5} + C}\end{aligned}$$

Example 2: Find

$$\int \sin^3(x)\,dx.$$

(Hint: Use the identity $\sin^2(x) + \cos^2(x) = 1$, then make a substitution.)

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \left(1 - \cos^2(x)\right)\sin(x)\,dx \\ &= -\int \left(1 - u^2\right)\,du \\ &= -u + \frac{u^3}{3} + C \\ &= \boxed{-\cos(x) + \frac{\cos^3(x)}{3} + C}\end{aligned}$$

[1] Worksheet adapted from BOALA, `math.colorado.edu/activecalc`

Example 3: Find

$$\int \sin^5(x)\cos^2(x)\,dx.$$

(Hint: Write $\sin^5(x)$ as $\left(\sin^2(x)\right)^2\sin(x)$.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= \int \left(\sin^2(x)\right)^2\cos^2(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^2\cos^2(x)\sin(x)\,dx\\ &= -\int \left(1-u^2\right)^2 u^2\,du\\ &= -\int \left(1-2u^2+u^4\right)\,du\\ &= -\int u^2-2u^4+u^6\,du\\ &= -\frac{u^3}{3}+\frac{2u^5}{5}-\frac{u^7}{7}+C\\ &= \boxed{-\frac{\cos^3(x)}{3}+\frac{2\cos^5(x)}{5}-\frac{\cos^7(x)}{7}+C}\end{aligned}$$

Example 4: Find

$$\int \sin^7(x)\cos^5(x)\,dx.$$

(The algebra here is long. Only set up the substitution - you do not need to fully evaluate.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^7(x)\cos^5(x)\,dx &= \int \left(\sin^2(x)\right)^3\cos^5(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^3\cos^5(x)\sin(x)\,dx\\ &= \boxed{-\int \left(1-u^2\right)^3 u^5\,du}\end{aligned}$$

Example 5: In general, how would you go about trying to find

$$\int \sin^m(x)\cos^n(x)\,dx,$$

where m is odd? (Hint: consider the previous three problems.)

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$\begin{aligned}\int \sin^m(x)\cos^n(x)\,dx &= \int \left(\sin^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx\\ &= \int \left(1-\cos^2(x)\right)^{(m-1)/2}\cos^n(x)\sin(x)\,dx\\ &= \boxed{-\int \left(1-u^2\right)^{(m-1)/2} u^n\,du}\end{aligned}$$

Example 6: Note that the same kind of trick works when the power on $\cos(x)$ is odd. To check that you understand, what trig identity and what u-substitution would you use to integrate

$$\int \cos^3(x)\sin^2(x)\,dx?$$

$\sin^2(x) + \cos^2(x) = 1$

$$\boxed{\cos^2(x) = 1 - \sin^2(x)}$$

$$\boxed{\begin{aligned} u &= \sin(x) \\ du &= \cos(x)\,dx \end{aligned}}$$

$$\begin{aligned} \int \cos^3(x)\sin^2(x)\,dx &= \int \cos^2(x)\sin^2(x)\cos(x)\,dx \\ &= \int \left(1 - \sin^2(x)\right)\sin^2(x)\cos(x)\,dx \\ &= \boxed{\int \left(1 - u^2\right)u^2\,du} \end{aligned}$$

Example 7: Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find

$$\int \sin^2(x)\,dx,$$

in each of the following two ways:

(a) Use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

$$\begin{aligned} \int \sin^2(x)\,dx &= \int \frac{1}{2}(1 - \cos(2x))\,dx \\ &= \frac{1}{2}\int 1 - \cos(2x)\,dx \\ &= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C} \end{aligned}$$

(b) Integrate by parts, with $u = \sin(x)$ and $dv = \sin(x)\,dx$.

$$\begin{aligned} u &= \sin(x) & dv &= \sin(x)\,dx \\ du &= \cos(x)\,dx & v &= -\cos(x) \end{aligned}$$

$$\begin{aligned} \int \sin^2(x)\,dx &= \int \sin(x)\sin(x)\,dx \\ &= -\sin(x)\cos(x) - \int -\cos(x)\cos(x)\,dx \\ &= -\sin(x)\cos(x) + \int \cos^2(x)\,dx \\ &= -\sin(x)\cos(x) + \int 1 - \sin^2(x)\,dx \\ &= -\sin(x)\cos(x) + x - \int \sin^2(x)\,dx \end{aligned}$$

$$\Longrightarrow 2\int \sin^2(x)\,dx = -\sin(x)\cos(x) + x + C$$

$$\Longrightarrow \int \sin^2(x)\,dx = \boxed{\frac{x - \sin(x)\cos(x)}{2} + C}$$

(c) Show that your answers to parts (a) and (b) above are the same by giving a suitable trig identity.

$$\sin(x)\cos(x) = \frac{1}{2}2\sin(x)\cos(x) = \frac{1}{2}\sin(2x).$$

(d) How would you evaluate the integral

$$\int \sin^2(x)\cos^2(x)\,dx?$$

$$\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}(1-\cos(2x)) \cdot \frac{1}{2}(1+\cos(2x))\,dx \\
&= \frac{1}{4}\int 1-\cos^2(x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\,dx \\
&= \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1+\cos(4x))\,dx \\
&= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos(4x)\,dx \\
&= \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}
\end{aligned}$$

Example 8: Evaluate the integral in problem (2) above, again, but this time by parts using $u = \sin^2(x)$ and $dv - \sin(x)\,dx$. (After this, you'll probably need to do a substitution.)

$u = \sin^2(x)$ $\quad dv = \sin(x)\,dx$

$du = 2\sin(x)\cos(x)\,dx$ $\quad v = -\cos(x)$

$w = \cos(x)$

$dw = -\sin(x)\,dx$

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin^2(x)\sin(x)\,dx \\
&= -\sin^2(x)\cos(x) - \int -\cos(x)\cdot 2\sin(x)\cos(x)\,dx \\
&= -\sin^2(x)\cos(x) + 2\int \cos^2(x)\sin(x)\,dx \\
&= -\sin^2(x)\cos(x) - 2\int w^2\,dw \\
&= -\sin^2(x)\cos(x) - \frac{2u^3}{3} + C \\
&= \boxed{-\sin^2(x)\cos(x) - \frac{2\cos^3(x)}{3} + C}
\end{aligned}$$

Example 9 - For fun: Can you show your answers to problem (2) and (8) above are the same? It's another great trigonometric identity.

$$-\sin^2(x)\cos(x) - \frac{2\cos^3(x)}{3} = -\left(1-\cos^2(x)\right)\cos(x) - \frac{2}{3}\cos^3(x) = -\cos(x) + \cos^3(x) - \frac{2}{3}\cos^3(x) = -\cos(x) + \frac{\cos^3(x)}{3}$$

Example 10 - Further investigations: (especially for mathematics, physics and engineering majors) We also would like to be able to solve integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$, the derivative of $\sec(x)$ is $\sec(x)|tan(x)$ and since there is a Pythagorean identity relating them. It sometimes works to use $u = \tan(x)$ and it sometimes works to use $u = \sec(x)$. Based on the values of m and n, which substitution should you use? Are there cases for which neither substitution works? (See page 472 of the text.)

Section 8.4: Trigonometric Substitution

Motivation: If we want to find the area of a circle or ellipse, we have an integral of the form

$$\int \sqrt{a^2 - x^2}\, dx$$

where $a > 0$. Regular substitution will not work here, observe:

$$u = a^2 - x^2$$
$$du = -2x\, dx \longleftarrow \text{ extra factor of } x \ldots$$

Solution: Parametrise! We change x to a function of θ by letting $x = a\sin(\theta)$ so,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin(\theta))^2} = \sqrt{1^2 - a^2\sin^2(\theta)} = \sqrt{a^2\left(1 - \sin^2(\theta)\right)} = \sqrt{a^2\cos^2(\theta)} = a\left|\cos(\theta)\right|.$$

Generally, we use an injective (one-to-one) function (so it has an inverse) to simplify calculations. Above, we ensure $a\sin(\theta)$ is invertible by restricting the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \sin^2(\theta)$$
$$\left(\div \cos^2(\theta)\right) \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \implies \tan^2(\theta) = \sec^2(\theta) - 1$$

Example 1: Evaluate

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx.$$

$x = 3\sin(\theta)$

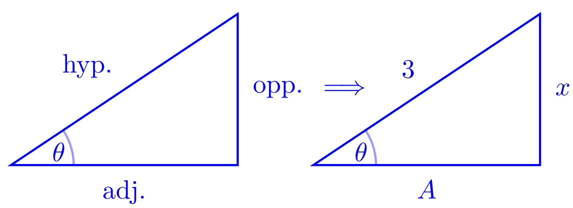
$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$dx = 3\cos(\theta)\ d\theta$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{3^2 - 3^2\sin^2(\theta)}}{3^2\sin^2(\theta)} \cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{\cancel{3}\sqrt{1-\sin^2(\theta)}}{\cancel{3^2}\sin^2(\theta)} \cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{\sqrt{\cos^2(\theta)}}{\sin^2(\theta)} \cdot \cos(\theta)\ d\theta \\
&= \int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \int \cot^2(\theta)\ d\theta \\
&= \int \csc^2(\theta) - 1\,d\theta \\
&= -\cot(\theta) - \theta + C \\
&= \boxed{-\frac{\sqrt{3^2-x^2}}{x} - \arcsin(\theta) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \implies \frac{x}{3} = \sin(\theta)$$



$A^2 + x^2 = 3^2$

$A^2 = 3^2 - x^2$

$A = \sqrt{3^2 - x^2}$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{3^2-x^2}}{x}$$

This is a common process in trig substitution. When you substitute back for your original variable, in this case x, you will always be able to find the correct substitutions by drawing out and labelling a right triangle correctly.

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

$x = 2\tan(\theta)$

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$dx = 2\sec^2(\theta)\,d\theta$

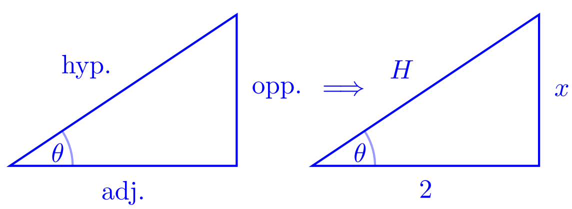
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \int \frac{2\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{2^2\tan^2(\theta)+2^2}}\,d\theta \\
&= \int \frac{\cancel{2}\sec^2(\theta)}{2^2\tan^2(\theta)\,\cancel{2}\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{2^2\tan^2(\theta)\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec(\theta)}{2^2\tan^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{4}\frac{1}{u} + C \\
&= -\frac{1}{4\sin(\theta)} + C \\
&= -\frac{1}{4}\csc(\theta) + C \\
&= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}
\end{aligned}$$

$u = \sin(\theta)$

$du = \cos(\theta)$

How did we recover x?

$x = 2\tan(\theta) \Longrightarrow \frac{x}{2} = \tan(\theta)$



$H^2 = x^2 + 2^2$

$H = \sqrt{x^2+4}$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hyp.}}{\text{opp.}} = \frac{\sqrt{x^2+4}}{x}$$

Example 3: Evaluate

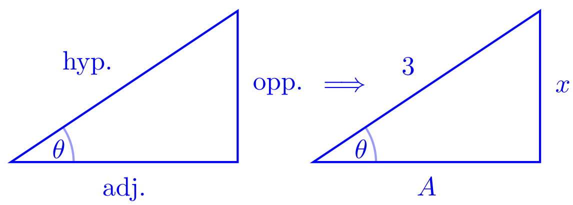
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$$x = 3\sin(\theta)$$
$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$dx = 3\cos(\theta)\ d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{3^2\sin^2(\theta)}{\sqrt{3^2-3^2\sin^2(\theta)}}\cdot 3\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\cancel{3}\sqrt{1-\sin^2(\theta)}}\cdot \cancel{3}\cos(\theta)\ d\theta \\
&= \int \frac{3^2\sin^2(\theta)}{\sqrt{\cos^2(\theta)}}\cdot \cos(\theta)\ d\theta \\
&= 9\int \sin^2(\theta) \\
&= \frac{9}{2}\int 1-\cos(2\theta)\ d\theta \\
&= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{9}{2}\left(\theta - \sin(\theta)\cos(\theta)\right) + C \\
&= \boxed{\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C}
\end{aligned}$$

How did we recover x?

$$x = 3\sin(\theta) \Longrightarrow \frac{x}{3} = \sin(\theta)$$



$$3^2 = x^2 + A^2$$
$$A = \sqrt{9-x^2}$$

$$\cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{9-x^2}}{x}$$

Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\,dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\,dx = \ln|x+1| + C$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} = \frac{2(x-3)-(x+1)}{(x+1)(x-3)} = \frac{2x-6-x-1}{(x+1)(x-3)} = \frac{x-7}{(x+1)(x-3)}$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1} - \frac{1}{x-3}\,dx = 2\ln|x+1| - \ln|x-3| + C$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\,dx$.

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} = \frac{7(x-4)+3(x-1)}{(x-1)(x-4)} = \frac{10x-31}{(x-1)(x-4)}$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\,dx = \int \frac{7}{x-1} + \frac{3}{x-4}\,dx = 7\ln|x-1| + 3\ln|x-4| + C$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Our first step is to decompose $\frac{x+14}{(x+5)(x+2)}$ as

$$\frac{x+14}{(x+5)(x+2)} = \frac{?}{x+5} + \frac{?}{x+2}.$$

There is no indicator of what the numerators should be, so there is work to be done to find them. If we let the numerators be variables, we can use algebra to solve. That is, we want to find constants A and B that make the equation below true for all $x \neq -5, -2$.

$$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$

We solve for A and B by cross multiplying and equating the numerators.

$$\begin{aligned}\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} = \frac{A(x+2)+B(x+5)}{(x+5)(x+2)} \Longrightarrow x+14 &= A(x+2)+B(x+5)\\ &= Ax+2A+Bx+5B\\ &= (A+B)x+2A+5B\end{aligned}$$

$$\begin{aligned}1 &= A+B \Longrightarrow B = 1-A\\ 14 &= 2A+5B\\ &= 2A+5(1-A)\\ &= 2A+5-5A\\ &= 5-3A\\ \Longrightarrow 9 &= -3A\\ \Longrightarrow -3 &= A\\ \Longrightarrow B &= 1-(-3) = 4\end{aligned}$$

$$\begin{aligned}\int \frac{x+14}{(x+5)(x+2)}\,dx &= \int \frac{-3}{x+5} + \frac{4}{x+2}\,dx\\ &= \boxed{-3\ln|x+5| + 4\ln|x+2| + C}\end{aligned}$$

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

$$\begin{aligned}\frac{x+15}{(3x-4)(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(3x-4)}{(3x-4)(x+1)} \Longrightarrow x+15 &= A(x+1)+B(3x-4)\\ &= Ax+A+3Bx-4B\\ &= (A+3B)x+A-4B\end{aligned}$$

$$\begin{aligned}1 &= A+3B \Longrightarrow A = 1-3B\\ 15 &= A-4B\\ &= (1-3B)-4B\\ &= 1-7B\\ \Longrightarrow 14 &= -7B\\ \Longrightarrow -2 &= B\\ \Longrightarrow A &= 1-3(-2) = 7\end{aligned}$$

$$\begin{aligned}\int \frac{x+15}{(3x-4)(x+1)}\,dx &= \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx\\ &= \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}\end{aligned}$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

$$\frac{x+15}{(3x-4)(x+1)} = \frac{A}{3x-4} + \frac{B}{x+1} = \frac{A(x+1)+B(3x-4)}{(3x-4)(x+1)} \Longrightarrow x+15 = A(x+1)+B(3x-4)$$

Instead of expanding everything, comparing coefficients and solving a system of linear equations, sometimes it may be helpful to plug in strategic values of x to solve. Good values to choose are those that are roots of the polynomials that appear on the denominators of the fraction. Observe,

$$\begin{aligned} x=-1: &\quad (-1)+15 = A((-1)+1)+B(3(-1)-4) \\ \Longrightarrow &\quad 14 = 0-7B \\ \Longrightarrow &\quad -2 = B \end{aligned}$$

$$\begin{aligned} x=\tfrac{4}{3}: &\quad \left(\tfrac{4}{3}\right)+15 = A\left(\left(\tfrac{4}{3}\right)+1\right)+B\left(3\left(\tfrac{4}{3}\right)-4\right) \\ \Longrightarrow &\quad \tfrac{49}{3} = \tfrac{7}{3}A+0 \\ \Longrightarrow &\quad 7 = A \end{aligned}$$

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx = \int \frac{7}{3x-4} - \frac{2}{x+1}\,dx = \boxed{\frac{7}{3}\ln|3x+5| - 2\ln|x+1| + C}$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2} \Longrightarrow 5x-2 = A(x+3)+B$$

$$\begin{aligned} x=-3: &\quad 5(-3)-2 = A((-3)+3)+B \\ \Longrightarrow &\quad -17 = 0+B \\ \Longrightarrow &\quad -17 = B \end{aligned}$$

$$\begin{aligned} 5x-2 &= A(x+3)-17 \\ &= Ax+3A-17 \\ \Longrightarrow \quad 5x &= Ax \\ \Longrightarrow \quad 5 &= A \end{aligned}$$

$$\int \frac{5x-2}{(x+3)^2}\,dx = \int \frac{5}{x+3} - \frac{17}{(x+3)^2}\,dx = \boxed{5\ln|x+3| + \frac{17}{x+3} + C}$$

Example 6: What if the denominator is an irreducible quadratic of the form $x^2 + px + q$? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2 + px + q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_1 x + C_1}{(x^2 + px + q)^2} + \frac{B_1 x + C_1}{(x^2 + px + q)^3} + \cdots + \frac{B_1 x + C_1}{(x^2 + px + q)^n}.$$

Compute $\displaystyle\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$.

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2} = \frac{(Ax + B)\,(x - 1)^2 + C(x^2 + 1)(x - 1) + D(x^2 + 1)}{(x^2 + 1)(x - 1)^2}$$

$$\Longrightarrow -2x + 4 = (Ax + B)\,(x - 1)^2 + C(x^2 + 1)(x - 1) + D(x^2 + 1)$$

There are four unknowns here, A, B, C and D. In this case we're going to want to minimise the amount of work we do here. In general it is going to be beneficial to solve for as many coefficients as we can by plugging in numbers, and then expand everything to compare coefficient after reducing the workload.

$$\begin{aligned}
x = 1: \quad & -2(1) + 4 = (Ax + B)((1) - 1)^2 + C((1)^2 + 1)((1) - 1) + D((1)^2 + 1)\\
\Longrightarrow \quad & 2 = 0 + 0 + 2D\\
\Longrightarrow \quad & 1 = D
\end{aligned}$$

So we got one coefficient this way. That's better than nothing! Now if we use this new information and then rearrange a little we end up with less solving to do. This *does* require you however to be comfortable with algebra.

$$\begin{aligned}
\Longrightarrow \quad & -2x + 4 = (Ax + B)\,(x - 1)^2 + C(x^2 + 1)(x - 1) + (x^2 + 1)\\
\Longrightarrow \quad & -x^2 - 2x + 3 = (Ax + B)\,(x - 1)^2 + C(x^2 + 1)(x - 1)\\
& -(x^2 + 2x - 3) =\\
& -(x - 1)(x + 3) =
\end{aligned}$$

Now we have already seen what happens when $x = 1$, so we can go right ahead and divide by the $(x-1)$ term that appears on both sides.

$$\begin{aligned}
\Longrightarrow \quad -x - 3 &= (Ax + B)\,(x - 1) + C(x^2 + 1)\\
&= Ax^2 + Bx - Ax - B + Cx^2 + C\\
&= (A + C)x^2 + (B - A)x + C - B
\end{aligned}$$

Now we can go through and set up equations and solve by coefficients. When there are lots of coefficients it is a good idea of coming up with a way to book-keep your algebra - it can get *very* messy if you don't. Below is just one way you can do it.

$$\begin{pmatrix} (1) & 0 & = & A & & +C \\ (2) & -1 & = & -A & +B & \\ (3) & -3 & = & & -B & +C \end{pmatrix} \overset{(2)+(3)}{\Longrightarrow} \begin{pmatrix} (1) & 0 & = & A & & +C \\ (2) & -4 & = & -A & & +C \\ (3) & -3 & = & & -B & +C \end{pmatrix} \overset{(1)+(2)}{\Longrightarrow} \begin{pmatrix} (1) & -4 & = & & & 2C \\ (2) & -4 & = & -A & & +C \\ (3) & -3 & = & & -B & +C \end{pmatrix}$$

$$\begin{aligned}
\Longrightarrow \quad -4 = 2C \quad & \Longrightarrow \quad -2 = C\\
\Longrightarrow \quad -4 = -A - 2 \quad & \Longrightarrow \quad 2 = A\\
\Longrightarrow \quad -3 = -B - 2 \quad & \Longrightarrow \quad 1 = B, \text{ So}\ldots
\end{aligned}$$

$$\begin{aligned}
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx &= \int \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\,dx\\
&= \int \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\,dx\\
&= \boxed{\ln(x^2 + 1) + \tan^{-1}(x) - 2\ln|x - 1| - \frac{1}{x - 1} + C}
\end{aligned}$$

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper ($\deg f(x) < \deg g(x)$)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_1x + C_1}{(x^2+px+q)^2} + \frac{B_1x + C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1x + C_1}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are

   (i) One fraction for each power of the irreducible factor that appears

   (ii) The degree of the numerator should be one less than the degree of the denominator

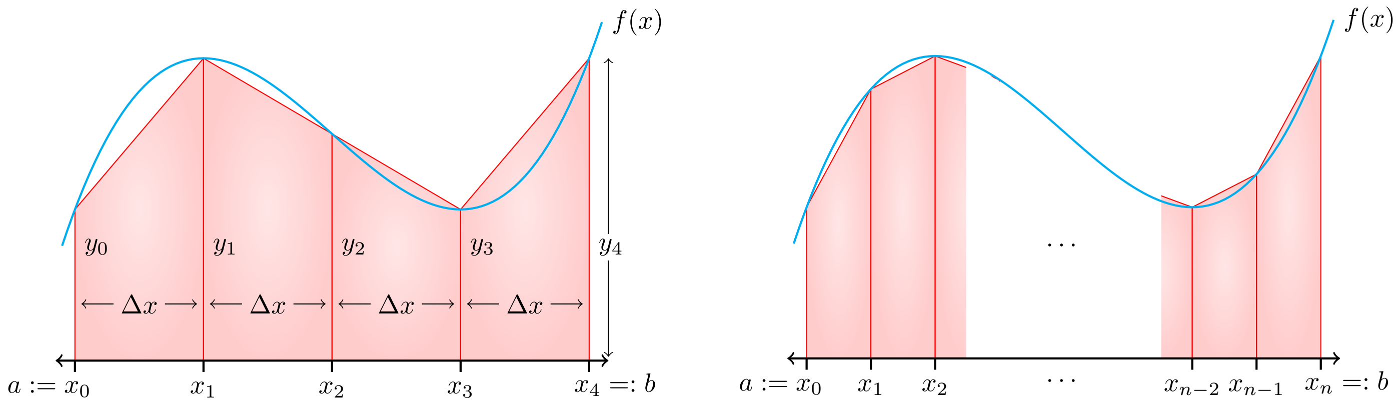
4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.

Section 8.7: Numerical Integration

What to do when there's no nice antiderivative? The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln(x)$ and $\sqrt{1+x^4}$ have no elementary formulas/ When we cannot find a workable antiderivative for a function $f(x)$ that we have to integrate, we can partition the interval of integration, replace $f(x)$ by a closely fitting polynomial on each subinterval, integrate the poynomials and add the results to *approximate* the definite integral of $f(x)$. This is an example of numerical integration. There are many methods of numerical integration but we will study only two: the *Trapezium Rule* and *Simpson's Rule*.

Trapezoidal Approximations: As the name implies, the Trapezium Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapeziums instead of rectangles - which, if you recall, we studied when we looked at Riemann integration in Calculus I.



Assume the length of each subinterval is $\Delta x = \dfrac{b-a}{n}$. Then the area of the trapezium that lies above the x-axis in the i^{th} subinterval is $T_i = \dfrac{\delta x}{2}\left(y_{i-1}+y_i\right)$ where $y_{i-1} = f\left(x_{i-1}\right)$ and $y_i = f\left(x_i\right)$. Then the area of the under the curve and above the x-axis is approximated by the sum of the trapeziums:

$$\begin{aligned}
T &= \frac{\Delta x}{2}\left(y_0+y_1\right) + \frac{\Delta x}{2}\left(y_1+y_2\right) + \cdots + \frac{\Delta x}{2}\left(y_{n-1}+y_n\right)\\
&= \frac{\Delta x}{2}\left(y_0+y_1+y_1+y_2+\cdots+y_{n-2}+y_{n-1}+y_{n-1}+y_n\right)\\
&= \frac{\Delta x}{2}\left(y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n\right)\\
&= \frac{\Delta x}{2}\left(y_0+y_n+2\sum_{i=1}^{n-1} y_i\right)\\
&= \frac{\Delta x}{2}\left(f(x_0)+f(x_n)+2\sum_{i=1}^{n-1} f\left(x_i\right)\right)
\end{aligned}$$

The Trapezium Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f\left(x_i\right)\right), \end{aligned}$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$.

Example 1: Use the Trapezium Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Partition the interval $[1, 2]$ into 4 subintervals:

$$\begin{aligned} \Delta x &= \frac{2-1}{4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} x_0 &= a & x_1 &= a + \Delta x & x_2 &= a + 2\Delta x & x_3 &= a + 3\Delta x & x_4 &= a + 4\Delta x \\ &= 1 & &= 1 + 1\cdot\frac{1}{4} & &= 1 + 2\cdot\frac{1}{4} & &= 1 + 3\cdot\frac{1}{4} & &= 1 + 4\cdot\frac{1}{4} \\ &= \frac{4}{4} & &= \frac{5}{4} & &= \frac{6}{4} & &= \frac{7}{4} & &= \frac{8}{4} \end{aligned}$$

Now use these points together with the formula for the Trapezium Rule:

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1/4}{2}\left(f\left(\frac{4}{4}\right) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 2f\left(\frac{7}{4}\right) + f\left(\frac{8}{4}\right)\right) \\ &= \frac{1}{8}\left(\frac{16}{16} + 2\frac{25}{16} + 2\frac{36}{16} + 2\frac{49}{16} + \frac{64}{16}\right) \\ &= \frac{1}{128}\left(16 + 50 + 72 + 98 + 64\right) \\ &= \frac{1}{128}(300) \\ &= \boxed{\frac{75}{32}} \end{aligned}$$

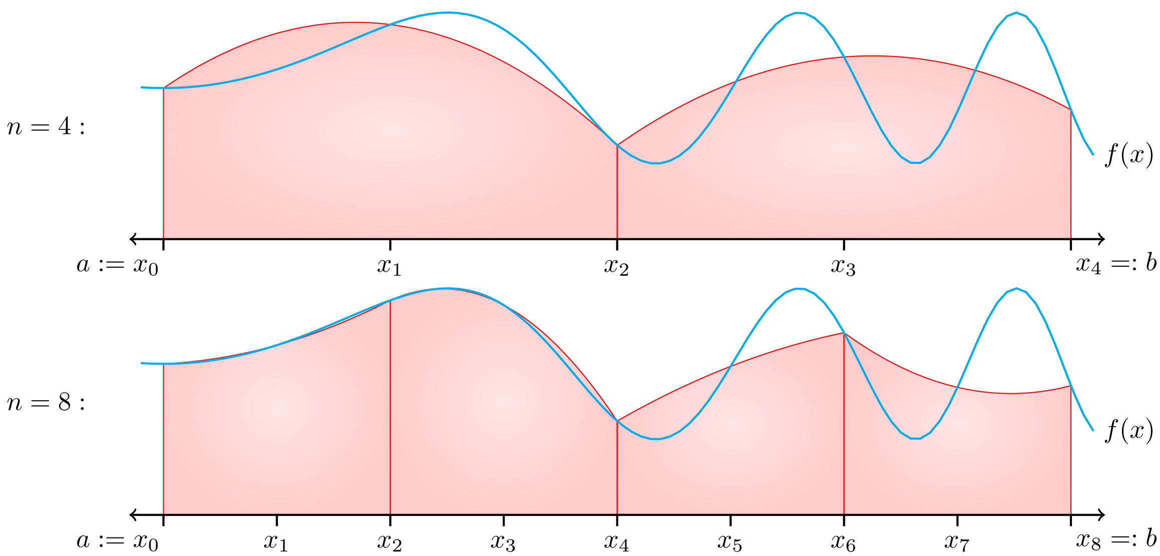
$$\begin{aligned} \int_1^2 x^2\,dx &= \frac{1}{3}x^3\bigg|_1^2 \\ &= \frac{1}{3}\left(2^3 - 1^3\right) \\ &= \frac{1}{3}(8-1) \\ &= \boxed{\frac{7}{3}} \end{aligned}$$

$$\frac{75}{32} - \frac{7}{3} = \frac{225}{96} - \frac{224}{96} = \frac{1}{96}.$$

So the approximation overestimated the actual area by $\frac{1}{96}$, which is pretty good considering we only used 4 trapeziums.

Just like when we looked at Riemann sums, using more trapeziums results in a better approximation.

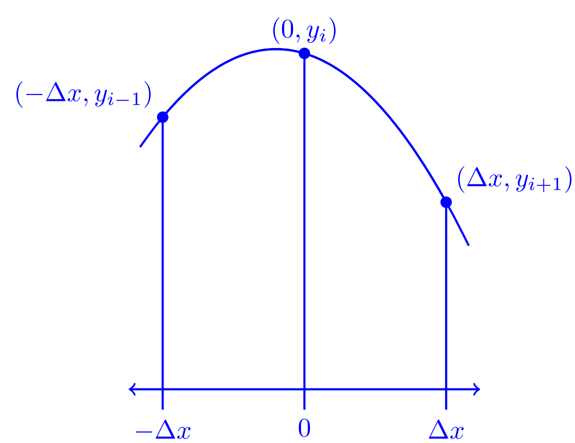
Parabolic Approximations: Instead of using the straight-line segments that produced the trapeziums, we can use parabolas to approximate the definite integral of a continuous function. We partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ but this time we require n to be an even number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passed through three consecutive points: (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) on the curve.



So how do we compute the area under each parabola $y = Ax^2 + Bx + C$? By translating we can assume that the centre point of our parabola is at $x_i = 0$

The area under the parabola and above the x-axis is given by

$$\begin{aligned} S_i &= \int_{-\Delta x}^{\Delta x} Ax^2 + Bx + C\, dx \\ &= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big|_{-\Delta x}^{\Delta x} \\ &= \frac{A(\Delta x)^3}{3} + \frac{B(\Delta x)^2}{2} + C(\Delta x) - \left[\frac{A(-\Delta x)^3}{3} + \frac{B(-\Delta x)^2}{2} + C(-\Delta x)\right] \\ &= \frac{2A\Delta x^3}{3} + 2C\Delta x \\ &= \frac{\Delta x}{3}\left(2A\Delta x + 6C\right) \end{aligned}$$



$$\left.\begin{aligned} y_{i-1} &= A\Delta x^2 - B\Delta x + C \\ y_i &= C \\ y_{i+1} &= A\Delta x^2 + B\Delta x + C \end{aligned}\right\} \implies y_{i-1} + 4y_i + y_{i+1} = \left(A\Delta x^2 - B\Delta x + C\right) + 4C + \left(A\Delta x^2 + B\Delta x + C\right) = 2A\Delta x + 6C$$

$$\implies \boxed{S_i = \frac{\Delta x}{3}\left(y_{i-1} + 4y_i + y_{i+1}\right)}$$

So if we sum up the areas under all of the parabolas, we obtain our approximation.

Simpson's Rule: To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$
$$= \frac{\Delta x}{3}\left(f\left(x_0\right) + f\left(x_n\right) + 2\left(\sum_{i=1}^{\frac{n-1}{2}} f\left(x_{2i-1}\right) + 2f\left(x_{2i}\right)\right)\right),$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$ with n an *even* number.

Example 2: Use the Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$. Compare the estimate with the exact value.

Partition the interval $[1, 2]$ into 4 subintervals:

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$x_0 = a = 0 = \frac{0}{2}$$
$$x_1 = a + \Delta x = 0 + 1\cdot\frac{1}{2} = \frac{1}{2}$$
$$x_2 = a + 2\Delta x = 0 + 2\cdot\frac{1}{2} = \frac{2}{2}$$
$$x_3 = a + 3\Delta x = 0 + 3\cdot\frac{1}{2} = \frac{3}{2}$$
$$x_4 = a + 4\Delta x = 0 + 4\cdot\frac{1}{2} = \frac{4}{2}$$

Now use these points together with the formula for Simpson's Rule:

$$\begin{aligned}
S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)\\
&= \frac{1/2}{3}\left(f\left(\frac{0}{2}\right) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right)\right)\\
&= \frac{1}{6}\left(5\frac{0}{16} + 4\cdot 5\frac{1}{16} + 2\cdot 5\frac{16}{16} + 4\cdot 5\frac{81}{16} + 5\frac{128}{16}\right)\\
&= \frac{5}{96}\left(0 + 4 + 32 + 324 + 256\right)\\
&= \frac{5}{96}(616)\\
&= \boxed{\frac{385}{12}}
\end{aligned}$$

$$\int_0^2 5x^4\,dx = x^5\Big|_0^2 = 2^5 - 0^5 = 32 - 0 = \boxed{32}$$

$$\frac{385}{12} - 32 = \frac{385}{12} - \frac{384}{12} = \frac{1}{12}.$$

So the approximation overestimated the actual area by $\frac{1}{12}$, which is pretty good considering we only used 2 parabolas.

Just like Riemann sums and the Trapezium rule, using more parabolas results in a better approximation. In fact, of the three rules Simpson's Rule gives the best approximation. This can be seen by looking at the *error estimates*.

Error Estimates in the Trapezium and Simpson's Rules If $f''(x)$ is continuous and M is any upper bound for the values of $|f''(x)|$ on $[a,b]$, then the error E_T in the Trapezium Rule for approximating the definite integral of $f(x)$ over the interval $[a,b]$ using n trapeziums satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

If $f^{(4)}(x)$ is continuous and M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a,b]$, then the error E_S in Simpson's Rule for approximating the definite integral of $f(x)$ over the interval $[a,b]$ using $\frac{n}{2}$ parabolas satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3: Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$. What value of n should we pick so that the error is within 0.001 of the true value?

First we differentiate $f(x)$ 4 times and check that it is continuous on the interval $[0,2]$.

$$\begin{aligned} f(x) &= 5x^4 \\ f'(x) &= 20x^3 \\ f''(x) &= 60x^2 \\ f'''(x) &= 120x \\ f^{(4)}(x) &= 120 \end{aligned}$$

This is a constant function, so it *is* continuous on our interval. Further

$$|f^{(4)}(x)| = 120 \leq 120 \text{ for all } x \in [0,2].$$

Thus $M = 120$ works as a bound. So, with $n = 4$, the error is bounded by:

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2-0)^5}{180(4)^4} = \frac{120 \cdot 2^5}{180 \cdot 2^8} = \frac{1}{3 \cdot 2^2} = \frac{1}{12}.$$

To achieve an approximation with $|E_S| \leq 0.001$, we again find a bound for M but this time we solve the inequality for n.

$$\frac{M(b-a)^5}{180n^4} = \frac{120(2-0)^5}{180n^4} = \frac{2^6}{3n^4} \leq 0.001$$

$$\begin{aligned} \implies \quad & \frac{2^6}{3} \leq \frac{1}{1000}n^4 \\ \implies \quad & \frac{2^6 \cdot 1000}{3} \leq n^4 \\ \implies \quad & \frac{2^8 \cdot 2 \cdot 5^3}{3} \leq n^4 \\ \implies \quad & 4\sqrt[4]{\frac{2 \cdot 5^3}{3}} \leq n \end{aligned}$$

So setting $n \geq 4\sqrt[4]{\frac{2\cdot 5^3}{3}} \approx 12.086$ would ensure an approximation of the desired accuracy.

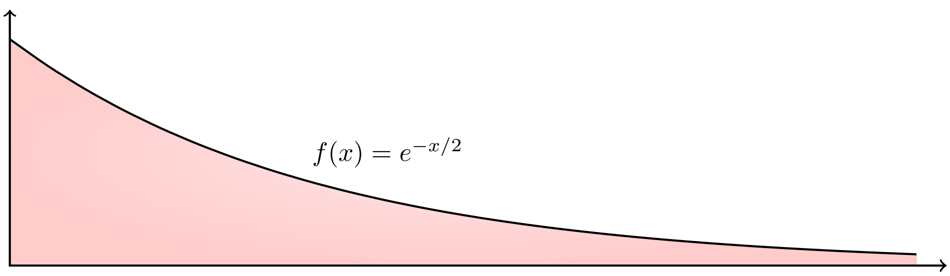
Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



First, we examine what the area looks like over finite intervals. That is, we integrate over $[0, b]$.

$$A(b) := \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} - \left[-2e^{-0/2}\right] = 2\left(1 - e^{-b/2}\right).$$

Now we have an expression for the area over a finite integral, we can let $b \longrightarrow \infty$ by calculating the limit of this expression.

$$A = \lim_{b\to\infty} A(b) = \lim_{b\to\infty} 2\left(1 - e^{-b/2}\right) = 2\,(1 - 0) = 2.$$

So,

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\,dx = 2.$$

So this is how we deal with infinite limits of integration - with a limit! Remember those?

Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty}\int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we sat that the improper integral ___converges___ and that the limit is the ___value___ of the improper integral. If the limit fails to exist, the improper integral ___diverges___

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

$$u = \ln(x) \qquad dv = \frac{1}{x^2}\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$$

$$\begin{aligned}\int_1^b \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln(x)}{x}\bigg|_1^b - \int_1^b -\frac{1}{x^2}\,dx \\ &= -\frac{\ln(x)}{x} - \frac{1}{x}\bigg|_1^b \\ &= -\frac{\ln(b)}{b} - \frac{1}{b} - \left[-\frac{\ln(1)}{1} - \frac{1}{1}\right] \\ &= -\frac{\ln(b)}{b} - \frac{1}{b} + 1\end{aligned}$$

Now we take a limit,

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} - \frac{1}{b} + 1\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b}\right] - 0 + 1 \overset{\text{L'H}}{=} \lim_{b\to\infty}\left[-\frac{1/b}{1}\right] + 1 = 0 + 1 = \boxed{1}$$

L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that $f(x)$ and $g(x)$ are differentiable on an open interval I containing a and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the left and right both exist.

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

According to part 3 of our definition, we can choose any real number c and split this integral into two integrals and then apply parts 1 and 2 to each piece. Let's choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx.$$

Now we will evaluate each piece separately.

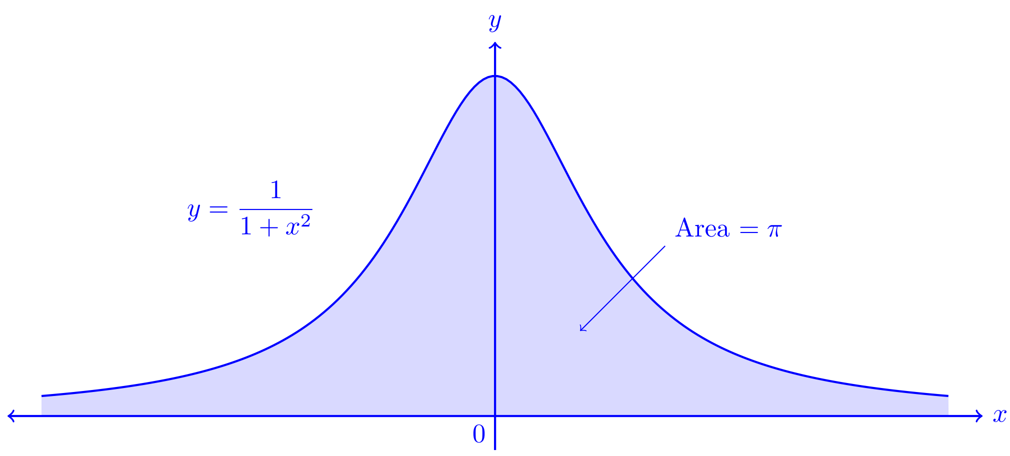
$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_{a}^{0} \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1}(x)\Big|_{1}^{0}\\ &= \lim_{a\to-\infty} \tan^{-1}(0)-\tan^{-1}(a)\\ &= \lim_{a\to-\infty} -\tan^{-1}(a)\\ &= \frac{\pi}{2},\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_{0}^{b} \frac{1}{1+x^2}\,dx\\ &= \lim_{b\to\infty} \tan^{-1}(x)\Big|_{0}^{b}\\ &= \lim_{b\to\infty} \tan^{-1}(b)-\tan^{-1}(0)\\ &= \lim_{b\to\infty} \tan^{-1}(b)\\ &= \frac{\pi}{2}.\end{aligned}$$

So,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \boxed{\pi}$$

Since $1/(1+x^2) > 0$ on $\mathbb{R}$, the improper integral can be interpreted as the (finite) area between the curve and the x-axis.



A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

We split this investigation into two cases; when $p \neq 1$ and when $p = 1$.

If $p \neq 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^b \\ &= \lim_{b\to\infty}\left.\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\right|_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p<1.\end{cases}\end{aligned}$$

If $p = 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty}\ln(x)\Big|_1^b \\ &= \lim_{b\to\infty}[\ln(b)-\ln(1)] \\ &= \lim_{b\to\infty}\ln(b) = \infty\end{aligned}$$

Combining these two results we have

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases}\frac{1}{p-1}, & p>1\\ \infty, & p\leq 1\end{cases}}$$

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

First we find the integral over the region $[a, 1]$ where $0 < a \leq 1$.

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_a^1 = 2\sqrt{x}\Big|_a^1 = 2 - 2\sqrt{a} = 2(1-\sqrt{a}).$$

Then we find the limit as $a \to 0^+$:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\left(1-\sqrt{a}\right) = 2.$$

Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ___converges___ and that the limit is the ___value___ of the improper integral. If the limit fails to exist, the improper integral ___diverges___

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$\begin{aligned}
\int_0^1 \frac{1}{1-x}\,dx &= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx \\
&= \lim_{b\to 1^-} -\int_0^b \frac{1}{x-1}\,dx \\
&= \lim_{b\to 1^-} -\ln|x-1|\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(x-1)\Big|_0^b \\
&= \lim_{b\to 1^-} -\ln(1-b) \\
&= -(-\infty) \\
&= \boxed{\infty}
\end{aligned}$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Why does this make sense?

1. If the area under the curve of $g(x)$ is *finite* and $f(x)$ is bounded above by $g(x)$ (and below by 0), then the area under the curve of $f(x)$ must be *less than or equal to* the area under the curve of $g(x)$. A positive number less that a *finite* number is also *finite*.
2. If the area under the curve of $f(x)$ is *infinite* and $g(x)$ is bounded below by $f(x)$, then the area under the curve of $g(x)$ must be *"less than or equal to"* the area under the curve of $g(x)$. Since there is no finite number "greater than" infinity, the area under $g(x)$ must also be *infinite*.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

We want to find a function $g(x)$ such that for some $a \in \mathbb{R}$, $f(x) = \dfrac{\cos^2(x)}{x^2} \leq g(x)$ or $f(x) = \dfrac{\cos^2(x)}{x^2} \geq g(x)$ for all $x \geq a$. One way we can do this is by finding *bounds* for $f(x)$. Since $0 \leq \cos^2(x) \leq 1$ for all x,

$$\frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}.$$

So then we can use $g(x) := \dfrac{1}{x^2}$. So,

$$0 \leq \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx \leq \int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}.$$

So $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ converges.

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Since $x \geq x - e^{-x}$, $f(x) := \dfrac{1}{x} \leq \dfrac{1}{x - e^{-1}} =: g(x)$ for all $x \geq 3$. So,

$$0 \leq \int_3^\infty f(x)\,dx \leq \int_3^\infty g(x)\,dx.$$

By the Direct Comparison Test then, $\displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,dx$ *diverges* since $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges.

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Why does this make sense? The convergence is really only dependent on the "tail" of the integral. That is, the convergence is dictated by what happens "at infinity." If for sufficiently large values of x, $f(x) \approx Lg(x)$ and one of the two integrals converges, then the other one should also converge, since it is only off by "about a scalar multiple." The same goes for diverging, if one diverges, then multiplying it by a positive number won't suddenly make it converge, so the other one should also diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Let $f(x) := \dfrac{1}{1+x^2}$ and $g(x) := \dfrac{1}{x^2}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1+x^2-1}{1+x^2} = \lim_{x\to\infty} \left(1 - \frac{1}{1+x^2}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ *converges.*

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Let $f(x) := \dfrac{1-e^{-x}}{x}$ and $g(x) := \dfrac{1}{x}$. Then,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1 - e^{-x}\right) = 1.$$

So, by the Limit Comparison Test, the integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges.*

Section 10.1: Sequences

Definition: A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The order is important here, for example 2, 4, 6, 8, ... is *not* the same as 4, 2, 6, 8, ...

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_1 = \dfrac{1}{1+1} = \dfrac{1}{2},\ a_2 = \dfrac{2}{2+1} = \dfrac{2}{3},\ a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}, \ldots$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ $\quad a_1 = \dfrac{(-1)^1(1+1)}{3^1} = \dfrac{-2}{3},\ a_2 = \dfrac{(-1)^2(2+1)}{3^2} = \dfrac{1}{3},\ a_3 = \dfrac{(-1)^3(3+1)}{3^3} = \dfrac{-4}{27}, \ldots$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8, \quad \ldots$$

Definition: (**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$\text{for all } n \geq N \qquad |a_n - L| < \varepsilon.$$

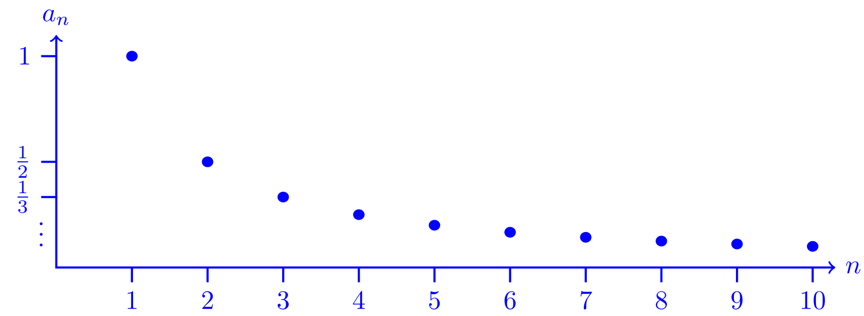
If no such number L exists, we say that $\{a_n\}$ **diverges**.

Definition: (**Friendly Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if

$$\lim_{n\to\infty} a_n = L.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?



From the plot above it looks as if the sequence is tending towards 0. It seems that plotting sequences looks a lot like plotting a function. In fact, we can use our knowledge of functions to infer things about sequences.

Theorem: (Continuous Function Theorem) If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ whenever n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

We know that $f(x) = \frac{1}{x}$ satisfies $f(n) = a_n$ for every positive integer n, so then

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

In truth, the limit of this sequence is clear without invoking the power of this theorem. But, the theorem is still a great tool that we can use for more complicated sequences.

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences: (i.e. the limit rules you already know for functions)

If $a_n \longrightarrow L$, $b_n \longrightarrow M$, then:

1. Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = L + M$,
2. Constant Rule: $\lim_{n\to\infty} c = c$ for any $c \in \mathbb{R}$,
3. Product Rule: $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$,
4. Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, if $M \neq 0$
5. Power Rule: $\lim_{n\to\infty} a_n^p = L^p$, if $p > 0$, $a_n > 0$

Squeeze Theorem: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \boxed{1}$$

2. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

Note that $f(x) := \dfrac{\ln(x)}{x}$ satisfies $f(n) = a_n$ for each positive integer n. So,

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = \boxed{0}$$

3. $\left\{\dfrac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

Since $-1 \leq \cos(n) \leq 1$ for all $n \in \mathbb{N}$, we have $-\dfrac{1}{n} \leq \dfrac{\cos(n)}{n} \leq \dfrac{1}{n}$ and since

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

we have $\lim_{n\to\infty} \dfrac{\cos(n)}{n} = 0$, by the Squeeze Theorem.

4. $\left\{\dfrac{(-1)^n}{n}\right\}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \boxed{0}$$

Examples of Divergent Sequences:

1. $\{(-1)^n\}_{n=1}^{\infty}$

2. $\{(-1)^n n\}_{n=1}^{\infty}$

3. $\{\sin(n)\}_{n=1}^{\infty}$

Definition: The product of the first n positive integers,

$$n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is denoted by $n!$ (read n **factorial**.)

Convention: $0! = 1$

Example 1: Find the limit of the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Observe,

$$a_1 = \frac{1!}{1^1} = \frac{1}{1} \leq \frac{1}{1}$$

$$a_2 = \frac{2!}{2^2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{2} \cdot \frac{1}{2} \leq \frac{1}{2}$$

$$a_3 = \frac{3!}{3^3} = \frac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3} = \underbrace{\frac{3}{3} \cdot \frac{2}{3}}_{\leq 1} \cdot \frac{1}{3} \leq \frac{1}{3}$$

$$\vdots$$

$$a_n = \frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n} = \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n}}_{\leq 1} \cdot \frac{1}{n} \leq \frac{1}{n}$$

So we have $0 \leq a_n \leq \frac{1}{n}$, so by the Squeeze Theorem $\lim_{n \to \infty} a_n = 0$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

- If $r > 1$, $\lim_{n \to \infty} r^n = \infty$
- If $r = 1$, $\lim_{n \to \infty} r^n = 1$
- If $0 < r < 1$, $\lim_{n \to \infty} r^n = 0$
- If $r = 0$, $\lim_{n \to \infty} r^n = 0$
- If $-1 < r < 0$, $\lim_{n \to \infty} r^n = 0$
- If $r \leq -1$, $\{r^n\}_{n=1}^{\infty}$ diverges

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______bounded from above______ if there exists a number M such that $a_n \leq M$ for all n.

The number M is an ______upper bound______ for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the

______least upper bound (supremum)______ of $\{a_n\}_{n=1}\infty$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______bounded from below______ if there exists a number m such that $a_n \geq m$ for all n.

The number m is a ______lower bound______ for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the

______greatest lower bound (infimum)______ of $\{a_n\}_{n=1}\infty$.

(c) **Completeness Axiom**: If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound b. Similarly for least upper bound.

(d) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is ______bounded______.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an ______unbounded______ sequence.

(e) Every ______convergent______ sequence is ______bounded______ but **not** every ______bounded sequence______ ______converges______. (consider ______$a_n = (-1)^n$______).

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ is ______non-decreasing______ if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ______non-increasing______ if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ______monotonic______ if it is either non-decreasing or non-increasing.

The Monotone Convergence Theorem: Every bounded, monotonic sequence converges.

Note: The Monotone Convergence Theorem ONLY tells us that the limit exists, NOT the value of the limit. It also tells us that a non-decreasing sequence converges when it is bounded from above, but diverges to infinity otherwise.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

$$a_1 = 2, \qquad a_2 = \frac{1}{2}(2+6) = 4, \qquad a_3 = \frac{1}{2}(4+6) = 5, \qquad \frac{11}{2}, \qquad \frac{23}{4}, \ldots.$$

It seems that the sequence is increasing. Lets prove this by *induction*. Suppose that $a_{k-1} > a_k$ for some $k > 2$. If we can show $a_{k+1} > a_k$ then we are done. Indeed,

$$a_{k-1} < a_k \Longrightarrow a_{k-1} + 6 < a_k + 6 \Longrightarrow a_k = \frac{1}{2}(a_{k-1} + 6) < \frac{1}{2}(a_k + 6) = a_{k+1}.$$

Thus $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence. If we show that the sequence is bounded we can use the Monotone Convergence Theorem. We know that it is bounded below by 2, since we just showed it was an increasing sequence. Note too that, at least for the ones we checked, $a_k < 6$. So,

$$a_{k+1} = \frac{1}{2}\left(a_k + 6\right) < \frac{1}{2}(6+6) = 6.$$

So we have $\{a_n\}_{n=1}^{\infty}$ is bounded above by 6. So, by the Monotone Convergence Theorem $\{a_n\}_{n=1}^{\infty}$ converges.

To find the limit, let $L := \lim_{n\to\infty} a_n$. Then,

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L+6) \Longrightarrow 2L = L + 6 \Longrightarrow L = 6.$$

So $\displaystyle\lim_{n\to\infty} a_n = 6$.

Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	$1 = \frac{2^1 - 1}{2^{1-1}}$
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2} = \frac{2^2 - 1}{2^{2-1}}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4} = \frac{2^3 - 1}{2^{3-1}}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^{n-1}} = \lim_{n\to\infty} \left(\frac{2^n}{2^{n-1}} - \frac{1}{2^{n-1}}\right) = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right) 2.$$

Since the sequence of partial sums converges, the *infinite series* converges. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ___infinite series___. The number a_n is the ___n^{th} term___ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the ___sequence of partial sums___ of the series, the number S_n being the ___n^{th} partial sum___.

If the sequence of partial sums converges to a limit L, we say that the series ___converges___ and that the

___sum___ is L. In this case we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sums of the series does not converge, we say that the series ___diverges___.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

in which a and r are fixed real numbers and $a \neq 0$. The ___ratio___ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

$$S_n = a_a(1) + a(1)^2 + a(1)^3 + \cdots + a(1)^{n-1} = na$$

and the series diverges since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \pm\infty$ (depending on the sign of a).

If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

$$S_1 = a, \quad S_2 = a + a(-1) = 0, \quad a + a(-1) = a(-1)^2 = a, \quad \ldots$$

If $|r| \neq 1$, then we use the following "trick":

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + sr^{n-1} \\ \Longrightarrow \quad rS_n &= ar + ar^2 + ar^3 + \cdots + ar^n \\ \Longrightarrow S_n - rS_n &= a - ar^n \\ \Longrightarrow \quad S_n &= \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}. \end{aligned}$$

If $|r| < 1$ then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$, so $S_n \longrightarrow \dfrac{a}{1-r}$. If $|r| > 1$ then $|r^n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and the series diverges.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 5}{4^{n-1}} = \sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}.$$

So this series is a geometric series with $a = 5$ and $r = -\dfrac{1}{4}$. Since $|r| < 1$ the series converges and so,

$$\sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1} = \frac{5}{1-(-\frac{1}{4})} = \boxed{4}$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

$$\begin{aligned} 5.232323\ldots &= 5 + \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \cdots \\ &= 5 + \frac{23}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\ &= 5 + \frac{23}{100}\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^{n-1} \qquad a = 1, \quad r = \frac{1}{100} \\ &= 5 + \frac{23}{100}\left(\frac{1}{1-\frac{1}{100}}\right) \\ &= 5 + \frac{23}{100} \cdot \frac{100}{99} \\ &= \boxed{\frac{518}{99}} \end{aligned}$$

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

If we take the partial sum decomposition,

$$\sum_{n=1} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right),$$

then its easy to see that the partial sums are,

$$S_n = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \xrightarrow{n\to\infty} 1.$$

Since the sequence of partial sums converges, the series converges and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \boxed{1}$

Theorem: If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Suppose $\{S_n\}_{n=1}^{\infty}$ converges to L. Then note that $\{S_{n+1}\}_{n=1}^{\infty}$ also converges to L. So then,

$$0 = L - L = \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n+1} - S_n) = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

The n^{th} Term Test for Divergence: The series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* if $\displaystyle\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

SUPER IMPORTANT NOTE: This theorem does **NOT** say that if $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

1. $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges since $\displaystyle\lim_{n\to\infty} n^2 = \infty$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since $\displaystyle\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B,$

2) Constant Multiple Rule : $\displaystyle\sum_{n=1}^{\infty} ca_n = cA,$ for any $c \in \mathbb{R}$.

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Caution! $\sum (a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ diverge!.

$$\begin{cases} \sum_{n=1}^{\infty} 1 \text{ diverges} \\ \sum_{n=1}^{\infty} (-1) \text{ diverges} \\ \sum_{n=1}^{\infty} (1 + (-1)) = 0 \end{cases}$$

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.

Section 10.3: The Integral Test

Tests for Convergence: The most basic question we can ask about a series is whether or not it converges. In the next few sections we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series we need to investigate the error involved when using a partial sum to approximate its total sum.

Non-decreasing Partial Sums: Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $S_{n+1} = S_n + a_{n+1}$, so

$$S_1 \leq S_2 \leq S_3 \leq \cdots \leq S_n \leq S_{n+1} \leq \ldots$$

Since the partial sums form a non-decreasing sequence, the Monotone Convergence Theorem give us the following result:

Corollary Of MCT: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Example 1: Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

n^{th} term test:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow \ n^{\text{th}} \text{ term test is inconclusive.}$$

Note however,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1+\frac{1}{2}}_{\frac{3}{2}} + \underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{2}{4}=\frac{1}{2}} + \underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{4}{8}=\frac{1}{2}} + \underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{>\frac{8}{16}=\frac{1}{2}} + \cdots$$

In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{1}{2}$. If $n = 2^k$, the sum S_n is greater than $\frac{k}{2}$, so S_n is not bounded from above. So **the Harmonic Series diverges**. Another way of seeing this is

$$S_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} > \frac{k}{2} \overset{k\to\infty}{\longrightarrow} \infty,$$

so then $S_n \longrightarrow \infty$ and the series diverges.

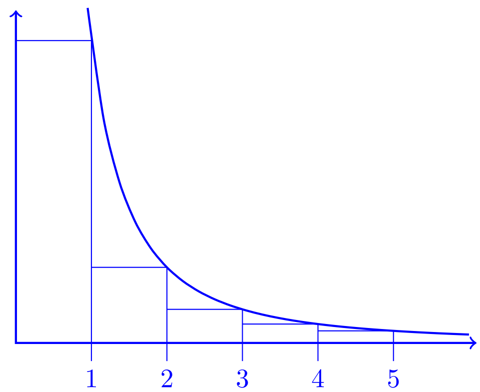
We now introduce the Integral Test with a series that is related to the harmonic series, but whose n^{th} term is $1/n^2$ instead of $1/n$.

Example 2: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We will compare the series to $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\begin{aligned} S_n &= \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2}\,dx \\ &< f(1) + \int_1^\infty \frac{1}{x^2}\,dx \\ &= 1 + 1 \\ &= 2 \end{aligned}$$



Since the partial sums are bounded above by 2, the sum *converges*.

The Integral Test: Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a ___positive___, ___continuous___, ___decreasing___ function of x. Then the series $\sum_{n=\underline{N}}^{\infty} a_n$ and the integral $\int_{\underline{N}}^\infty f(x)\,dx$ both converge or diverge.

Example 3: Show that the p**-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

If $p > 1$ then $f(x) = \frac{1}{x^p}$ is a positive, continuous, decreasing function of x. Since $\int_1^\infty f(x)\,dx = \frac{1}{p-1}$, the series converges by the Integral Test. Note that the sum of this series is <u>not</u> generally $\frac{1}{p-1}$. If $p \leq 0$, the sum diverges by the n^{th} term test. If $0 < p < 1$ then $1 - p > 0$ and

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \frac{1}{p-1}\left(\lim_{b\to\infty} b^{1-p} - 1\right) = \infty.$$

Example 4: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} n e^{-n^2}.$$

$f(x) = xe^{-x^2}$ is positive, continuous, decreasing and $f(n) = a_n$ for all n. Further,

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx = \frac{1}{2}\lim_{b\to\infty}\left[-e^{-b^2} - (-e^{-1})\right] = \frac{1}{2e}.$$

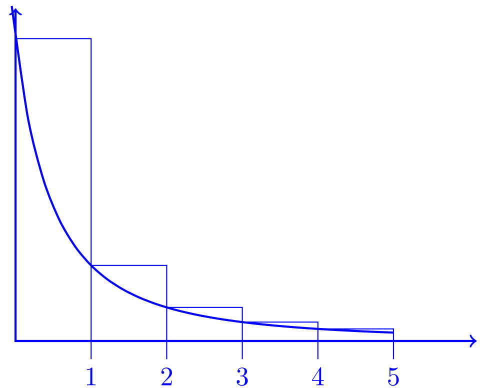
Since the integral converges, the series also converges.

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Suppose a series $\sum a_n$ is shown to be convergent by the integral test and we want to estimate the size of the remainder R_n measuring the difference between the total sum S and its n^{th} partial sum S_n.

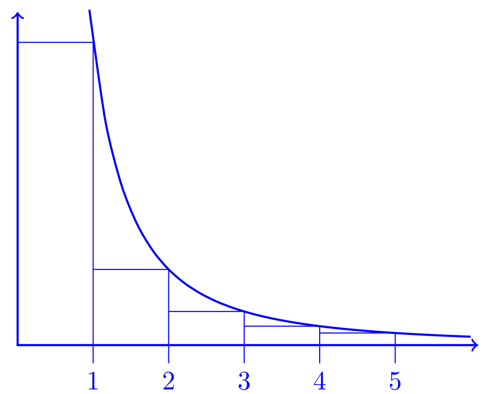
$$R_n = S - S_n = a_{n+1} + a_{n+1} + a_{n+1} + \cdots$$

Lower Bound: Shift the integral test function left 1 unit.



$$R_n \geq \int_{n+1}^{\infty} f(x)\, dx$$

Upper Bound: The integral test function.



$$R_n \leq \int_{n}^{\infty} f(x)\, dx$$

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx.$$

Example 5: Estimate the sum, S, of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.

$$\int_{n}^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty} \left[-\frac{1}{b} + \frac{1}{n}\right] = \frac{1}{n} \quad \Longrightarrow \quad S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}$$

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdot + \frac{1}{100} \approx 1.54977 \quad \Longrightarrow \quad 1.64068 \leq S \leq 1.64977$$

It seems reasonable that taking the midpoint of this interval would give a good estimate, so

$$S \approx 1.6452.$$

It turns out that using fancy advanced calculus (Fourier Analysis) we actually know that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493.$$

Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \le a_n \le \dfrac{1}{n}$ for all n, we can conclude ____nothing____.

(b) If $\dfrac{1}{n} \le a_n$ for all n, we can conclude ____$\sum_{n=1}^{\infty} a_n$ diverges____.

(c) If $0 \le a_n \le \dfrac{1}{n^2}$ for all n, we can conclude ____$\sum_{n=1}^{\infty} a_n$ converges____.

(d) If $\dfrac{1}{n^2} \le a_n$ for all n, we can conclude ____nothing____.

(e) If $\dfrac{1}{n^2} \le a_n \le \dfrac{1}{n}$ for all n, we can conclude ____nothing____.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

(a) (i) $\dfrac{1}{n^2} \le \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges. (ii) $\dfrac{1}{n+1} \le \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n+1}$ diverges.

(d) (i) $\dfrac{1}{n^2} \le \dfrac{1}{n^2-1}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2-1}$ converges. (ii) $\dfrac{1}{n^2} \le \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

(e) (i) $\dfrac{1}{n^2} \le \dfrac{1}{n^2-1} \le \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2-1}$ converges. (ii) $\dfrac{1}{n^2} \le \dfrac{1}{n+1} \le \dfrac{1}{n}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n+1}$ diverges.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ ____converges____, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ ____diverges____, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

Converges - its a Geometric Series with $r = \dfrac{1}{2}$.

(b) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{2^n + n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = b_n.$$

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

It converges!

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

$$b_n = \frac{1}{n^2}$$

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

Converges - its a $p-$series with $p = 2$

(c) How do the sizes of the terms a_n and b_n compare?

$$a_n = \frac{1}{n^2 + n + 1} \leq \frac{1}{n^2} = b_n$$

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

It converges!

Problem 5: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^5+3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

The dominant terms of a_n are $\frac{\sqrt{n^4}}{n^5} = \frac{n^2}{n^5} = \frac{1}{n^3}$.

- Choose $b_n = \frac{1}{n^3}$.
- $a_n = \frac{\sqrt{n^4-1}}{n^5+3} < \frac{\sqrt{n^4}}{n^5+3} = \frac{n^2}{n^5+3} < \frac{n^2}{n^5} = \frac{1}{n^3} = b_n$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = 3 > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 6: Use the Direct Comparison Test to determine if $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3+n}}$ converges or diverges.

- $\cos^2(n) \leq 1 \implies \frac{\cos^2(n)}{\sqrt{n^3+n}} \leq \frac{1}{\sqrt{n^3+n}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}} \implies$ choose $b_n = \frac{1}{n^{3/2}}$.
- $\sum_{n=1}^{\infty} b_n$ is a p-series with $p = \frac{3}{2} > 1$, so it converges.
- Since $a_n < b_n$, $\sum_{n=1}^{\infty} a_n$ also *converges.*

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2-1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\sum_{n=2}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2} \leq \frac{1}{n^2-1} \implies \text{So Direct Comparison is inconclusive.}$$

(b) Show that $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{n^2-1+1}{n^2-1} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2-1}\right) = 1$$

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \le 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

$$\frac{1}{n^2-1} \le \frac{2}{n^2} \iff n^2 \le 2(n^2-1) \iff n^2 \le 2n^2 - 2 \iff 2 \le n^2 \iff 1 < n.$$

Yes!

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \le \sum_{n=2}^{\infty} \frac{2}{n^2} = 2\sum_{n=2}^{\infty} \frac{1}{n^2} \text{ converges since it is a } p\text{-series} \implies \sum_{n=1}^{\infty} b_n \text{ converges!}$$

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ____converge____ or both ____diverge____.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ converges or diverges.

$$\frac{n^3-2n}{n^4+3} > \frac{n^3}{n^4+3} \text{ which behaves like } \frac{1}{n}.$$

Let $b_n = \frac{1}{n}$ and use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n^3-2n}{n^4+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-2n}{n^4+3} \cdot n = \lim_{n\to\infty} \frac{n^4-2n^2}{n^4+3} = 1 > 0$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{n^3-2n}{n^4+3}$ also *diverges.*

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ converges or diverges.

$$0 < \frac{10n+1}{n(n+1)(n+2)} \approx \frac{10n}{n^3} = 10\frac{1}{n^2} \text{ so let } b_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{10n+1}{n^3+2n^2+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{10n^3+n^2}{n^3+2n^2+2n} = \lim_{n\to\infty} \frac{10+1\frac{1}{n}}{1+\frac{2}{n}+\frac{2}{n^2}} = 10 > 0.$$

So $\sum_{n=1}^{\infty} a_n$ behaves the same way $\sum_{n=1}^{\infty} b_n$ does. Thus by the limit comparison test, $\sum_{n=1}^{\infty} a_n$ *converges.*

Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

This is a geometric series with $|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$, so it converges.

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

This is a geometric series with $|r = \left|-\frac{5}{4}\right| = \frac{5}{4} > 1$, so it diverges.

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

$a_n = (-1)^{n+1} \frac{1}{n^2} \Longrightarrow |a_n| = \frac{1}{n^2}$:

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$,

so $\sum_{n=1}^{\infty} a_n$ converges absolutely

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3^{n+1}}\cdot\frac{3^n}{2^n+5}\right| \\ &= \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{3\left(2^n+5\right)}\right| \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2^{n+1}+5}{2^n+5} \\ &= \frac{1}{3}\lim_{n\to\infty}\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}} \\ &= \frac{2}{3} < 1\end{aligned}$$

So, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ converges absolutely by the Ratio Test.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2(n+1))!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}\right| &= \lim_{n\to\infty}\left|\frac{(2n+2)!}{(n+1)!\cdot(n+1)!}\cdot\frac{n!\cdot n!}{(2n)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}}{(n+1)\cdot\cancel{n!}\cdot(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{n!}\cdot\cancel{n!}}{\cancel{(2n)!}}\right| \\ &= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\ &= \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} \\ &= 4 > 1\end{aligned}$$

So, $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

The ratio test is super useful for factorials

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2}{2^n}\right|} &= \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{2} \\ &= \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2} \\ &= \frac{1^2}{2} \\ &= \frac{1}{2} < 1\end{aligned}$$

So, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges absolutely by the Root Test.

The ratio test is super useful for a^n

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln(\sqrt[n]{n})} = e^{\lim_{n\to\infty} \frac{\ln(n)}{n}} \overset{\text{L'H}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1$$

Section 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim_{n\to\infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the requirements with $N = 1$ and therefore converges.

Instead of verifying $b_n \geq b_{n+1}$, we can follow the steps we did in the integral test to verify the sequence is decreasing. Define a differentiable function $f(x)$ satisfying $f(n) = b_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is non-increasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $b_n \geq b_{n+1}$ for all N.

Example 2: Consider the sequence where $b_n == \frac{10n}{n^2+16}$. Define $f(x) = \frac{10x}{x^2+16}$. Then $f'(x) = \frac{10(16-x^2)}{(x^2+16)} \geq 0$ when $x \geq 4$. It follows that $b_n \geq b_{n+1}$ for $n \geq 4$.

The Alternating Series Test Estimation Theorem: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies the conditions of the AST, then for $n \geq N$,

$$S_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n$$

approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term.
Furthermore, the sum L lies between any two successive partial sums S_n and S_{n+1}, and the remainder, $L - S_n$, has the same sign as the first unused term.

Example 3: Let's apply the Estimation Theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

n	Sum	S_n	$L-S_n$
0	1	1	$-\frac{1}{3}$
1	$1-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$
2	$1-\frac{1}{2}+\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{12}$
3	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{24}$
4	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$	$\frac{11}{16}$	$-\frac{1}{48}$
5	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$	$\frac{21}{32}$	$\frac{1}{96}$
6	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}$	$\frac{43}{64}$	$-\frac{1}{192}$
7	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$	$\frac{85}{128}$	$\frac{1}{384}$
8	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}$	$\frac{171}{256}$	$-\frac{1}{768}$

Example 4 - Conditional Convergence: We have seen that in absolute value, the Alternating Harmonic Series diverges. The presence of infinitely many negative terms is essential to its convergence. We say the Alternating Harmonic Series if **conditionally convergent**. We can extend this idea to the alternating p-series.
If p is a positive constant, the sequence $\dfrac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p} = 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots, \qquad p>0$$

converges.

- If $p>1$, the series converges absolutely.
- If $0<p\leq 1$, the series converges conditionally.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \ldots, b_n \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Example 5: We know $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to some number L.
By the Estimation Theorem, we know $L \neq 0$ (our partial sums never "hop" over 0). So,

$$\begin{aligned}
2L &= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\right)\\
&= 2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\cdots\\
&= (2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\cdots\\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\
&= L
\end{aligned}$$

(group all the terms with odd denominators together, leaving the even denominator terms alone)

So $2L = L \ldots$ so $L=0$? But $L\neq 0\ldots$ oops. Thus we cannot rearrange the sum in a conditionally convergent sequence.

Section 10.7: Power Series

Definition: A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

$$\begin{aligned} S_n &= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n \\ \implies (1-x)S_n &= (1-x)\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) \\ &= \left(1 + x + x^2 + x^3 + x^4 + \cdots + x^n\right) - \left(x + x^2 + x^3 + x^4 + x^5 \cdots + x^{n+1}\right) \\ &= 1 - x^{n+1} \\ \implies S_n &= \frac{1-x^n}{1-x} \end{aligned}$$

So,

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} \text{ which converges if and only if } |x| < 1$$

Instead of focussing on finding a formula for the sum of a power series, we are now going to think of the partial sums of the series as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need only take a few terms of the series to get a good approximation. As we move toward $x = 1$ or $x = -1$, we need more terms.

One of the most important questions we can ask about a power series is "for what values of x will the series converge?" Since power series are functions, what we are really asking here is "what is the **domain** of the power series?"

Example 2: Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

$$\text{Centre: } a = 2, \qquad c_0 = 1,\ c_1 = -\frac{1}{2},\ c_2 = \frac{1}{4},\ \ldots,\ c_n = \left(-\frac{1}{2}\right)^n,$$

$$\text{Ratio: } r = \frac{c_{n+1}(x-2)^{n+1}}{c_n(x-2)^n} = \frac{c_1(x-2)}{c_0} = \frac{-\frac{1}{2}(x-2)}{1} = -\frac{x-2}{2}$$

The series converges when $|r| < 1$, that is,

$$\left|-\frac{x-2}{2}\right| < 1 \Longrightarrow \left|\frac{x-2}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow -2 < x-2 < 2 \Longrightarrow 0 < x < 4.$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^n \frac{x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1}x^n}\right| = \left|\frac{nx}{x+1}\right| = |x|\frac{n}{n+1} \overset{n\to\infty}{\longrightarrow} |x|$$

The series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. It remains to see what happens at the endpoints; $x = -1$ and $x = 1$.

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} \Longrightarrow$ the series diverges at $x = -1$.

$x = 1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ = the Alternating Harmonic Series $\Longrightarrow$ the series converges at $x = 1$.

So, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges for $-1 < x \leq 1$ and diverges elsewhere.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We will use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| = \frac{|x|}{n+1} \overset{n\to\infty}{=\longrightarrow} 0$$

Since the value of the limit is 0, no matter what real number we choose for x and $0 < 1$, the series converges absolutely for all values of x. $\quad (x \in \mathbb{R},\ \ -\infty < x < \infty,\ \ (-\infty, \infty))$.

Fact: There is always at least one point for which a power series converges: the point $x = a$ at which the series is centred.

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Convergence Theorem and the previous examples lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways;

- If might converge on some interval of *radius* R. an interval has radius R if its length is $2R$
- It might converge everywhere.
- It might converge only at $x = a$.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$)
3. The series converges only at $x = a$ and diverges elsewhere ($R = 0$)

R is called the **radius of convergence** of the power series, and the interval of radius R centred at $x = 1$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series.

How to test a Power Series for Convergence:

1. Use the Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test fo convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the n^{th} term does not approach zero for those values of x.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^{3/2}3^{n+1}} \cdot \frac{n^{3/2}3^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{xn^{3/2}}{(n+1)^{3/2}3} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{3/2} = \frac{|x|}{3}.$$

So the series converges absolutely when $\frac{|x|}{3} < 1 \Longrightarrow |x| < 3 \Longrightarrow -3 < x < 3$.

Check the endpoints:

$x = -3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ which is an alternating p-series with $p = \frac{3}{2}$, so it *converges.*

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^{3/2}3^n} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2}$, so it *converges.*

Thus the interval of convergence is $[-3, 3]$ and the radius of convergence is $R = 3$.

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants. They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product.

The Series Multiplication Theorem for Power Series: If $A(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $B_n(x) = \displaystyle\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\displaystyle\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$. For example:

Since $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

Term-by-Term Differentiation Theorem: If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n\,(x-a)^n$$

on the interval $a - R < x < a + R$. This function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n\,(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.
Note: When we differentiate we may have to start our index at one more than it was before. This is because we lose the constant term (if it exists) when we differentiate.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=0}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

which *diverges* for all x. This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find a series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

$$f'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots = \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1.$$

$$f''(x) = \frac{2}{(1-x)^3} = 0 + 0 + 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

converges for $a - R < x < a + R$ for $R > 0$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$.

Example 6: Given $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

$$\begin{aligned} \ln(1+x) = \int_0^x \frac{1}{1+t}\,dt &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x \\ ` &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad -1 < x < 1. \end{aligned}$$

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$

Differentiate

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Now we can integrate to find $f(x)$:

$$f(x) = \int f'(t)\,dt = \arctan(x) + C.$$

Since $f(0) = 0$, we have $0 = \arctan(0) + C = C$, so then

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\arctan(x)} \qquad -1 < x < 1$$

Section 10.8: Taylor and Maclaurin Series

Series Representations: We've seen that geometric series can be used to generate a power series for functions having a special form, such as $f(x) = \dfrac{1}{1-x}$ or $g(x) = \dfrac{3}{x-2}$. Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

$$f'(x) = 1 \cdot c_1 + 2 \cdot c_2(x-a) + 3 \cdot c_3(x-a)^2 + 4 \cdot c_4(x-a)^3 + \cdots + n \cdot c_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2 \cdot 1 \cdot c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \cdots + n \cdot (n-1) \cdot c_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 3 \cdot 2 \cdot 1 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + \cdots + n \cdot (n-1) \cdot (n-2) \cdot c_n(x-a)^{n-2} + \cdots$$

$$\vdots$$

Since $x = a$ is in the assumed interval of convergence, all of the above equations are valid when $x = a$:

$$f(a) = c_0, \qquad f'(a) = 1 \cdot c_1, \qquad f''(a) = 2 \cdot 1 \cdot c_2, \qquad f'''(a) = 3 \cdot 2 \cdot 1 \cdot c_3, \qquad f^{(n)}(a) = n! \cdot c_n$$

Solving for each c_k gives:

$$c_0 = f(a), \qquad c_1 = \frac{f'(a)}{1}, \qquad c_2 = \frac{f''(a)}{2 \cdot 1}, \qquad c_3 = \frac{f'''(a)}{3 \cdot 2 \cdot 1}, \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

Thus, if $f(x)$ has such a series representation, it must have the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

On the other hand, if we start with an arbitrary function $f(x)$ that is infinitely differentiable on an interval containing $x = a$ and use it to generate the series above, will the series then converge to $f(x)$ at each x in the interval of convergence? The answer is maybe.

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \dfrac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\frac{1}{x}$	$\frac{1}{2}$
1	$(-1)\cdot\frac{1}{x^2}$	$(-1)\frac{1}{2^2}$
2	$(-1)^2\cdot\frac{2\cdot 1}{x^3}$	$(-1)^2\frac{2\cdot 1}{2^3}$
3	$(-1)^3\cdot\frac{3\cdot 2\cdot 1}{x^4}$	$(-1)^3\frac{3\cdot 2\cdot 1}{2^4}$
4	$(-1)^4\cdot\frac{4\cdot 3\cdot 2\cdot 1}{x^5}$	$(-1)^4\frac{4\cdot 3\cdot 2\cdot 1}{2^5}$
n	$(-1)^n\cdot\frac{n!}{x^{n+1}}$	$(-1)^n\frac{n!}{2^{n+1}}$

The key thing to do when looking for the general term is to not simplify everything. You should try and only group those terms that come from the "same place." For example, when $n = 2$ we could have cancelled a 2 from the numerator and denominator of $f''(2)$. But since the 2 in the numerator came from differentiating and the 2 on the denominator came from plugging in $x = a$, we leave them alone. Leaving factors alone this way will help you more easily see where each number in the factor is coming from and its relation to the value of n.

So, the Taylor Series generated by $f(x) = \dfrac{1}{x}$ centred at $a = 2$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n}$$

Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + \frac{(-1)^n(x-2)^n}{2^{n+1}}$$

is geometric with first term $\dfrac{1}{2}$ and ratio $r = -\dfrac{(x-2)}{2}$. So it converges (absolutely) for

$$\left|-\frac{(x-2)}{2}\right| < 1 \Longrightarrow |x-2| < 2 \Longrightarrow 0 < x < 4.$$

Now we check the endpoints:

$x = 0$: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(0-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2} \Longrightarrow \text{diverges.}$$ (Also clear since $f(x) = \frac{1}{x}$ is not defined at $x = 0$)

$x = 4$: $$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(4-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \Longrightarrow \text{diverges.}$$

Thus the only values of x for which this Taylor Series converges are $\boxed{0 < x < 4}$.

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial** of order n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as the linearisation of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$2n$	$(-1)^n\cos(x)$	$(-1)^n$
$2n+1$	$(-1)^{n+1}\sin(x)$	$(-1)^n 0$

When terms are alternating between 0s and non-zero terms, take a look at the parity of the values of n for which they appear. That is, check if all the 0s occur when n is odd (or when n is even). Once you figure out which is which you can ignore all the zero terms by considering $2n$ or $2n+1$.

If you are dealing with trigonometric functions, it is likely that at some point there will be some repetition happening. For example here $f^{(4)}(x) = f(x)$. So then you might be able to see what is happening by only using the terms up until the repeat.

So the Taylor Series generated by $f(x) = \cos(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}}$$

To find the interval of convergence, we can use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}}\right| = \frac{x^2}{(2n+2)(2n+1)} \overset{n\to\infty}{\longrightarrow} 0$$

So this Taylor Series converges for all $x \in \mathbb{R}$.

Finally, the Taylor polynomials are given by:

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 3: Find the Maclaurin Series generated by $f(x) = \sin(x)$.

Recall that $\cos(x)$ is an <u>even function</u> and we have just discovered in Example 2 that only <u>even</u> powers of x occur in its Maclaurin Series. One would expect then that since $f(x) = \sin(x)$ is an <u>odd function</u> that only <u>odd</u> powers of x will appear in its Maclaurin Series. Indeed this is actually the case. Doing the same calculations as in Example 2 will yield the desired result.

Here however we will just invoke the power of integration: Since $\int_0^x \cos(t)\,dt = \sin(x)$ and

$$\int_0^x \frac{(-1)^n}{(2n)!} t^{2n}\,dt = \frac{(-1)^n}{(2n)!} \cdot \frac{t^{2n+1}}{(2n+1)}\Bigg|_0^x = \frac{(-1)^n}{(2n+1)!} t^{2n+1}\Bigg|_0^x = \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

we have the Taylor Series generated by $f(x) = \sin(x)$ is

$$\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}\,dt = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

Example 4: Find the Taylor Series generated by $f(x) = e^x$.

Note that $f^{(n)}(x) = f(x) = e^x$ for every positive integer n. So $f^{(n)}(0) = e^0 = 1$ for each n, so then the Taylor Series generated by $f(x) = e^x$ at $a = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Section 10.9: Convergence of Taylor Series

Taylors Theorem: In the last section, we asked when a Taylor Series for a function can be expected to that (generating) function. That question is answered by the following theorem:

If $f(x)$ and its first n derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ are continuous on the closed interval between a and b, and $f^{(n)}(x)$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Interesting Fact: Taylor's Theorem is a generalisation of the Mean Value Theorem!

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Stating Taylor's Theorem in this way says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where the function $R_n(x)$ is determined by the value of the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ at a point c that depends on both a and x, and that it lies somewhere between them.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the ____remainder____ ____of order n____ or the ____error term____ for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

$f(x)$ has derivatives of all orders on $(-\infty, \infty)$. Using the Taylor Polynomial generated by $f(x) = e^x$ at $a = 0$ and Taylor's formula, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

where $R_n(x) = \dfrac{e^c}{(n+1)!}x^{n+1}$ for some 0 between 0 and x. Recall that e^x is an increasing function, so;

$x > 0$: $0 < c < x \Longrightarrow e^0 < e^c < e^x \Longrightarrow 1 < e^c < e^x$

$x < 0$: $x < c < 0 \Longrightarrow e^x < e^c < e^0 \Longrightarrow e^x < e^c < 1$

$x = 0$: $e^x = 1$, $x^{n+1} = 0 \Longrightarrow R_n(x) = 0$

So,

$x > 0$: $|R_n(x)| = \left|\dfrac{e^c x^{n+1}}{(n+1)!}\right| \leq \dfrac{e^x x^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

$x \leq 0$: $|R_n(x)| = \left|\dfrac{e^c x^{n+1}}{(n+1)!}\right| \leq \dfrac{|x|^{n+1}}{(n+1)!} \overset{n\to\infty}{\longrightarrow} 0$

Thus $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, so the series converges to e^x on $(-\infty, \infty)$. Thus,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

This gives us a new* definition for the number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

* Recall in Calc I we showed $e = \lim\limits_{x\to 0^+} (1+x)^{1/x}$ using L'Hôpitals Rule.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 2: Show that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ converges to $\sin(x)$ for all x.

Recall that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$. Since for each n, $\left|f^{(2n+1)}(x)\right| = |\cos(x)| \leq 1$ and $\left|f^{(2n)}\right| = |\sin(x)| \leq 1$, let $M = 1$. Then,

$$|R_{2n+1}(x)| \leq 1 \cdot \frac{|x-0|^{2n+2}}{(2n+2)!} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus the Taylor Series converges to $f(x) = \sin(x)$. That is,

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}$$

Using Taylor Series: Since every Taylor series is a power series, the operations of adding, subtracting and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 3: Using known series, find the first few terms of the Taylor series for

$$\frac{1}{3}(2x + x\cos(x))$$

using power series operations.
We have,

$$\begin{aligned}
\frac{1}{3}(2x + x\cos(x)) &= \frac{2}{3}x + \frac{x}{3}\cos(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sin(x) \\
&= \frac{2}{3}x + \frac{x}{3}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \\
&= \frac{2}{3}x + \sum_{n=0}^{\infty}\frac{(-1)^n}{3\cdot(2n)!}x^{2n+1} \\
&= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{3\cdot 2!} + \frac{x^5}{3\cdot 4!} - \cdots \\
&= x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots
\end{aligned}$$

Example 4: For what values of x can we replace $\sin(x)$ by the polynomial $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?
We use the fact that the Taylor series for $\sin(x)$ is an alternating series for every non-zero value of x. By the Alternating Series Estimation Theorem (Section 10.6), the error in truncating

$$\sin(x) = x - \frac{x^3}{3!}\Bigg| + \frac{x^5}{5!} - \cdots$$

is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

So the error will be less than 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \Longleftrightarrow \quad |x|^5 < 360 \times 10^{-4} = 0.036 \quad \Longleftrightarrow \quad |x| < \sqrt[5]{0.036} \approx 0.514.$$

So, if $= -0.514 < x < 0.514$, the error obtained from using $x - \frac{x^3}{3!}$ to approximate $\sin(x)$ will be less than 10×10^{-4}.

Moreover, by the Alternating Series Estimation Theorem, we know the estimate $x - \frac{x^3}{3!}$ is an underestimate of $\sin(x)$ when x is positive, since $\frac{x^5}{120}$ would be positive, and an overestimate if x is negative.

Section 10.10: Applications of Taylor Series

Evaluating Non-elementary Integrals: Taylor series can be used to express non-elementary integrals in terms of series. Integrals like the one in the next example arise in the study of the diffraction of light.

Example 1: Express

$$\int \sin(x^2)\,dx$$

as a power series.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \Longrightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

So,

$$\int \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!} x^{4n+3}$$

Example 2: Estimate

$$\int_0^1 \sin(x^2)\,dx$$

with an error of less than 0.001.

Using the previous example we see

$$\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!} x^{4n+3}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!} - [0] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)\cdot(2n+1)!}$$

We want to use the Alternating Series Estimation Theorem (section 10.6). So we want

$$\left|\frac{(-1)^{n+1}}{(4(n+1)+3)\cdot(2(n+1)+1)!}\right| < 0.001 \Longrightarrow \frac{1}{(4n+7)\cdot(2n+3)!} < 0.001$$
$$\Longrightarrow (4n+7)\cdot(2n+3)! > 1000$$

By trial and error we obtain $n = 1$ works. So then $\displaystyle\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} \approx \boxed{0.310}$.

If we extend this to 5 terms, we obtain

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} \approx 0.310268303.$$

This gives an error of about 1.08×10^{-9}. To guarantee this accuracy (using the error formula) for the Trapezium Rule, we would need to use about 8000 subintervals!

Euler's Identity: A complex number is a number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$. So then

$$i=\sqrt{-1} \qquad i^2=-1 \qquad i^3=-\sqrt{-1} \qquad i^4=1 \qquad i^{4n+k}=i^k \qquad i^{2n+k}=(-1)^n i^k.$$

If we substitute $x=i\theta$ into the Taylor series for e^x ans use the relations above, we obtain

$$\begin{aligned}
e^{i\theta} = \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{(i\theta)^{2n}}{(2n)!}+\frac{(i\theta)^{2n+1}}{(2n+1)!}\right) && \text{(split into even and odd terms)}\\
&= \sum_{n=0}^{\infty}\left(\frac{(-1)^n\theta^{2n}}{(2n)!}+\frac{(-1)^n i\theta^{2n+1}}{(2n+1)!}\right) && \text{(apply the indentities of } i\text{)}\\
&= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}\theta^{2n}+i\frac{(-1)^n}{(2n+1)!}\theta^{2n+1}\right) && \text{(rewrite for foreshadowing)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\theta^{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1} && \text{(break up sum)}\\
&= \cos(\theta)+i\sin(\theta). && \text{(know things)}
\end{aligned}$$

$$\boxed{\text{Euler's Identity: } e^{i\theta}=\cos(\theta)+i\sin(\theta)}$$

This identity is actually amazing. You can use this identity to derive all of the angle sum formulas, so you never need to remember them all! Also we see that $e^{i\pi}=-1$, which we can rewrite to obtain

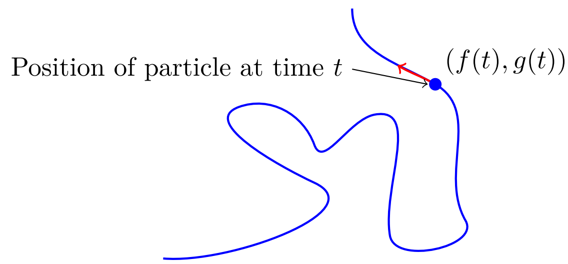
$$\boxed{e^{i\pi}+1=0}$$

which combines 5 of the most important constants in mathematics; e, π, i, 1 and 0.

Common Taylor Series			
1. $\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$-1 < x \leq 1$
7. $\tan^{-1}(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$\lvert x\rvert \leq 1$

Section 11.1: Parametrisations of Plane Curves

Parametric Equations: Below we have the path of a moving particle on the xy-plane. We can sometimes describe such a path by a pair of equations, $x = f(t)$ and $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.



Definitions: If x and y are given as functions

$$x = f(t) \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ___parametric curve___.

The equations are ___parametric equations___ for the curve.

The variable t is the ___parameter___ for the curve and its domain I is the ___parameter interval___.

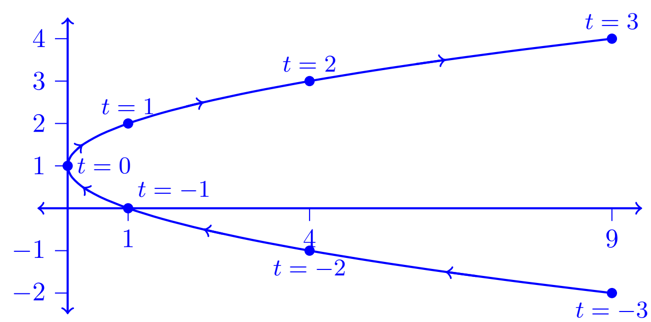
If I is a closed interval, $a \le t \le b$, the ___initial point___ of the curve is the point $(f(a), g(a))$ and the ___terminal point___ of the curve is $(f(b), g(b))$.

Example 1: Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

The (x, y) coordinates are determined by values for t, $(t^2, t + 1)$.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



Here the arrows indicate the direction of travel.

Example 2: Identify geometrically the curve in Example 1 by eliminating the parameter t and obtaining an algebraic equation in x and y.

Since both x and y are defined in terms of t, we can use substitution to eliminate the parameter:

Option 1:

$$y = t + 1 \qquad x = t^2$$
$$\implies y - 1 = t \qquad x = (y-1)^2$$
$$\implies \boxed{x = y^2 - 2y + 1}$$

Option 2:

$$x = t^2 \qquad y = t + 1$$
$$\implies \pm\sqrt{x} = t \qquad y = \pm\sqrt{x} + 1$$
$$\implies \boxed{y = \sqrt{x} + 1, \quad y = -\sqrt{x} + 1}$$

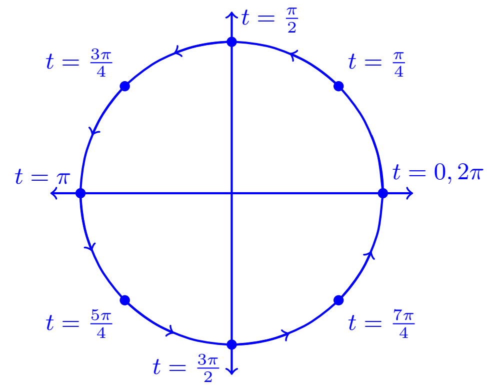
Example 3: Graph the parametric curves

(a) $x = \cos(t), \quad y = \sin(t), \quad 0 \le t \le 2\pi,$

(b) $x = a\cos(t), \quad y = a\sin(t), \quad 0 \le t \le 2\pi, \quad a \in \mathbb{R}.$

(a)

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
π	-1	0
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
2π	1	0



Here the arrows indicate the direction of travel.

We see then that these parametric equations correspond to travelling around the unit circle anticlockwise. Algebraically we can verify this to see that

$$\cos^2(t) + \sin^2(t) = x^2 + y^2 = 1$$

which is precisely the equation for a circle of radius 1, centred at the origin.

(b) It should come at no surprise that these parametric equations correspond to travelling around the circle of radius a, centred at the origin, anticlockwise.

Example 4: The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

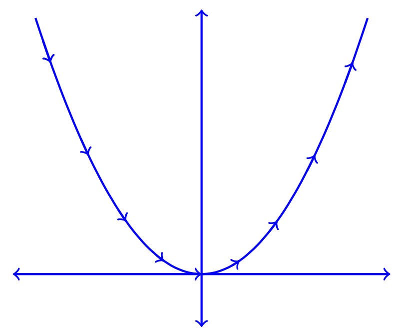
$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

We can either find a table of values and plot or we can find a Cartesian equation. The latter is more straight forward and we see that $x = \sqrt{y}$ for $y \geq 0$ (or $y = x^2$ for $x \geq 0$). So the curve is the part of $y = x^2$ lying in the first quadrant of the xy-plane.

Example 5 - Natural Parametrisation: A parametrisation of the function $f(x) = x^2$ is given by

Let $x = t$. Then $y = x^2 = t^2$ and so the *natural parametrisation* of the curve $y = x^2$ is (t, t^2) where $-\infty < t < \infty$.



Example 6: Find a parametrisation for the line through the point (a, b) having slope m.

A Cartesian equation of the line through (a, b) with slope m is

$$y - b = m(x - a).$$

Let $t = x - a$. Then $y - b = mt$ so $y = mt + b$. Therefore a parametrisation is

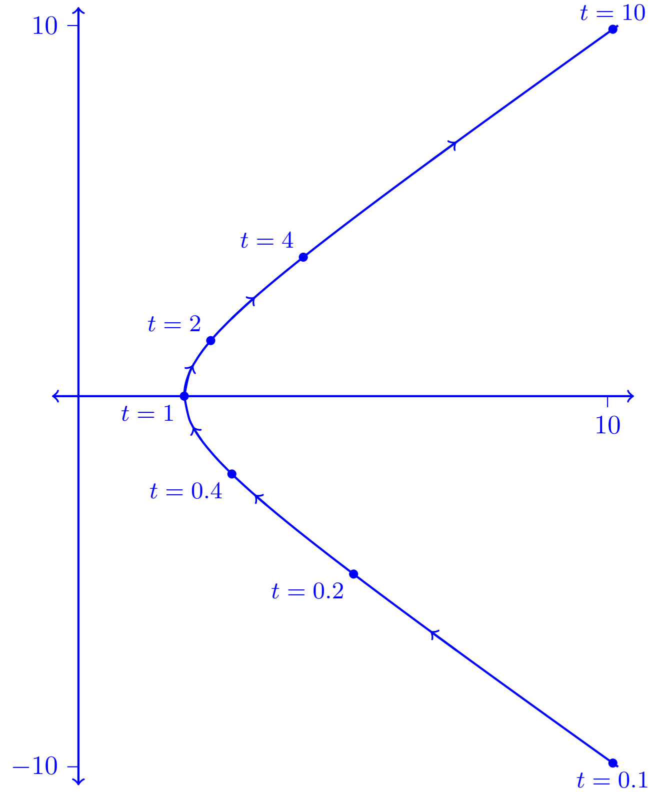
$$(x, y) = (t + a, mt + b), \quad -\infty < t < \infty.$$

It is important that the usage of the phrase "a parametrisation" is precise here since parametrisations are not unique. Here we could also use the *natural parametrisation* to obtain $(x, y) = (t, mt - (ma - b))$, $-\infty < t < \infty$.

Example 7: Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

t	x	y
0.1	10.1	-9.9
0.2	5.2	-4.8
0.4	2.9	2.1
1	2	0
2	2.5	1.5
4	4.25	3.75
10	10.1	9.9



(1) $x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t$

(2) $x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}$

(3) $x^2 - y^2 = (x+y)(x-y) = (2t)\left(\frac{2}{t}\right) = 4$

The Cartesian equation $x^2 - y^2 = 4$ is the standard form for the equation of a hyperbola.

Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

First we need to calculate the value of t at the point $(\sqrt{2}, 1)$. Since $\tan(x)$ is a one-to-one function on the parameter interval we see that

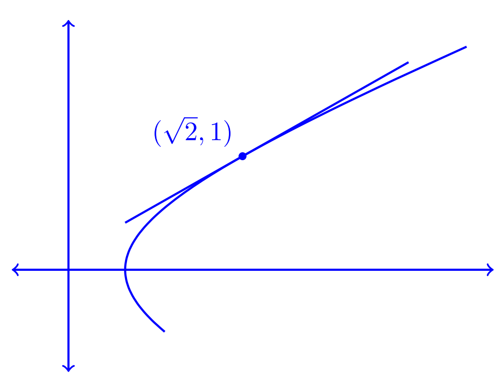
$$t = \tan^{-1}(1) = \frac{\pi}{4}$$

Using this we calculate the slope of the tangent line.

$$m = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = \frac{dy/dt}{dx/dt}\Big|_{t=\frac{\pi}{4}} = \frac{\sec^2(t)}{\sec(t)\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sec(t)}{\tan(t)}\Big|_{t=\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Thus the equation of the tangent line at the point $(\sqrt{2}, 1)$ is

$$\boxed{y = \sqrt{2}(x - \sqrt{2}) + 1}$$



Example 2: Find $\frac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

$$\frac{dx}{dt} = 1 - 2t \qquad \frac{dy}{dt} = 1 - 3t^2$$

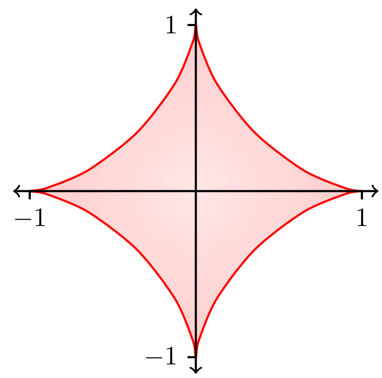
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dt}\right) &= \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)\\ &= \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}\\ &= \frac{-6t+12t^2+2-6t^2}{(1-2t)^2}\\ &= \frac{2-6t+6t^2}{(1-2t)^2}\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt} = \boxed{\frac{2-6t+6t^2}{(1-2t)^3}}$$

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$



The shape we are dealing with is symmetric, so the area we are interested in is four times the area beneath the curve in the first quadrant, corresponding to $0 \le t \le \frac{\pi}{2}$. We will apply the Fundamental Theorem of Calculus using substitution to express the curve y ad the differential dx in terms of t.

$$x = \cos^3(t)$$
$$dx = -3\cos^2(t)\sin(t)\,dt$$

$$u = \sin(2t)$$
$$du = 2\cos(2t)\,dt$$

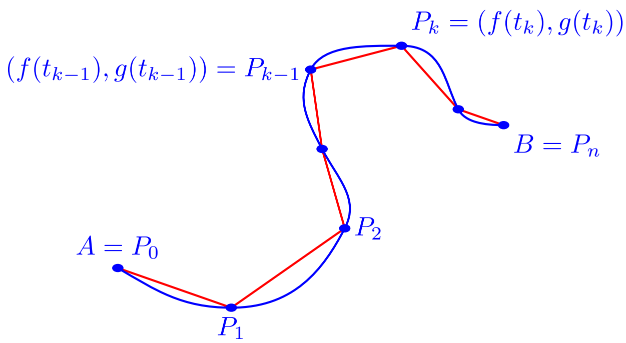
$$\begin{aligned}A = 4\int_0^1 y(x)\,dx &= 4\int_{\frac{\pi}{2}}^0 \sin^3(t)\left(-3\cos^2(t)\sin(t)\right)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \sin^4(t)\cos^2(t)\,dt\\ &= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))^2(1+\cos(2t))\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2t))\left(1-\cos^2(2t)\right)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin^2(2t)-\cos(2t)\sin^2(2t)\,dt\\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4t)}{2}\,dt - \frac{3}{2}\int_{t=0}^{t=\frac{\pi}{2}} \frac{u^2}{2}\,du\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)\right]_0^{\frac{\pi}{2}} - \frac{3}{4}\left[\frac{u^3}{3}\right]_{t=0}^{t=\frac{\pi}{2}}\\ &= \frac{3}{4}\left[t-\frac{1}{4}\sin(4t)-\frac{1}{3}\sin^3(2t)\right]_0^{\frac{\pi}{2}}\\ &= \boxed{\frac{3\pi}{8}}\end{aligned}$$

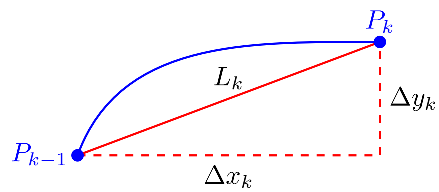
Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

We assume the functions $f(t)$ and $g(t)$ are ___continuously differentiable___ on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a ___smooth curve___.





The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \le t \le 2\pi.$$

$$\frac{dx}{dt} = -r\sin(t) \qquad \frac{dy}{dt} = r\cos(t)$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\left(\sin^2(t) + \cos^2(t)\right)}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2}\, dt \\ &= \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = \boxed{2\pi r} \end{aligned}$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$

As in Example 3, the perimeter of the astroid is 4 times the length of the curve in the first quadrant.

$$\frac{dx}{dt} = 3\cos^2(t)\sin(t) \qquad \frac{dy}{dt} = -3\cos^2(t)\sin(t)$$

$$u = \sin(t) \qquad du = \cos(t)\, dt$$

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)}\, dt \\ &= 4\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t)\, dt \\ &= 12\int_{t=0}^{t=\frac{\pi}{2}} u\, du \\ &= 12\left[\frac{u^2}{2}\right]_{t=0}^{t=\frac{\pi}{2}} \\ &= 12\left[\frac{\sin^2(t)}{2}\right]_0^{\frac{\pi}{2}} \\ &= 12\left[\frac{1}{2} - 0\right] \\ &= \boxed{6} \end{aligned}$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.

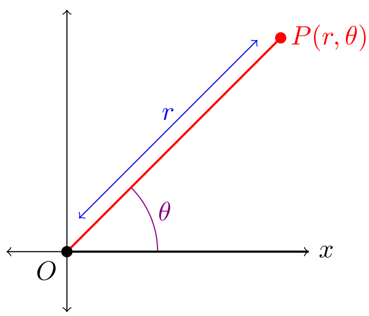
$$\frac{dx}{dt} = -\sin(t)$$

$$\frac{dy}{dt} = \cos(t)$$

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi \int_0^{2\pi} (2 + \sin(t)) \sqrt{(-\sin(t))^2 + (\cos(t))}\, dt \\ &= 2\pi \int_0^{2\pi} 2 + \sin(t)\, dt \\ &= 2\pi \left[2t - \cos(t)\right]_0^{2\pi} \\ &= 2\pi \left[(4\pi - 1) - (0 - 1)\right] \\ &= \boxed{8\pi^2} \end{aligned}$$

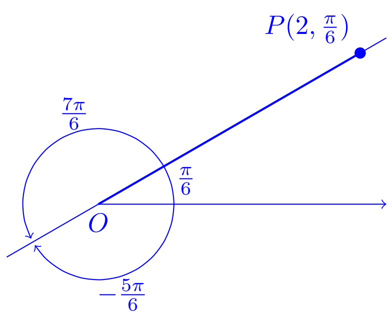
Section 11.3: Polar Coordinates

Definition: To define polar coordinates, we first fix an origin O (called the pole) and an initial ray from O (usually the positive x-axis). Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray OP.



Just like trigonometry, θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is not unique.
In some cases, we allow r to be negative. For instance, the point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians anticlockwise from the initial ray and going forward 2 units, or we could turn $\pi/6$ radians clockwise and go backwards 2 units; corresponding to $P(-2, \pi/6)$.

Example 1: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.



For $r = 2$,

$$\theta = \frac{\pi}{6}, \ \frac{\pi}{6} \pm 2\pi, \ \frac{\pi}{6} \pm 4\pi, \ \frac{\pi}{6} \pm 6\pi, \ \ldots$$

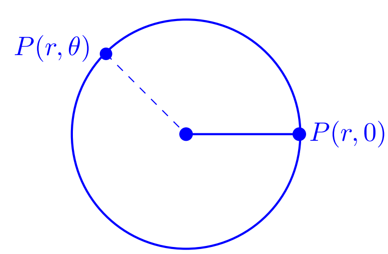
For $r = -2$,

$$\theta = -\frac{5\pi}{6}, \ -\frac{5\pi}{6} \pm 2\pi, \ -\frac{5\pi}{6} \pm 4\pi, \ -\frac{5\pi}{6} \pm 6\pi, \ \ldots$$

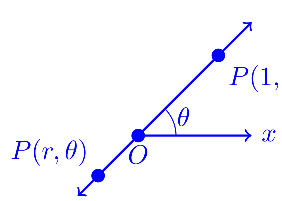
So,

$$\left\{ \left(2, \frac{\pi}{6} + 2n\pi\right), \left(-2, -\frac{5\pi}{6} + 2n\pi\right) \mid n \in \mathbb{Z} \right\}$$

Polar Equations and Graphs: If we fix r at a constant value (not equal to zero), the point $P(r, \theta)$ will lie $|r|$ unites from the origin O. As θ varies over any interval of length 2π, P traces a what? A circle!



If we fix θ at a constant value and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces a what? A line!



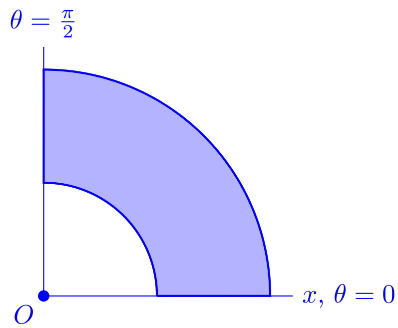
Example 2: A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for a circle of radius 1 centred at the origin.

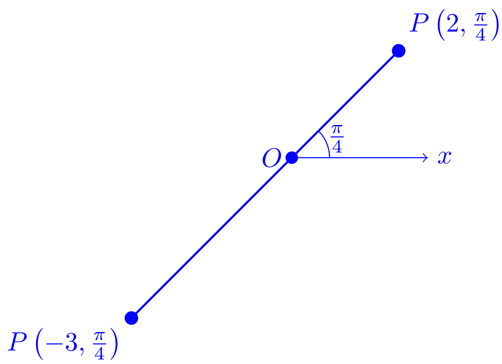
(b) $\theta = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, \ldots$ are all equations for the line passing through the Cartesian points $(0, 0)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 3: Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments and rays. Graph the sets of points whose polar coordinates satisfy the given conditions:

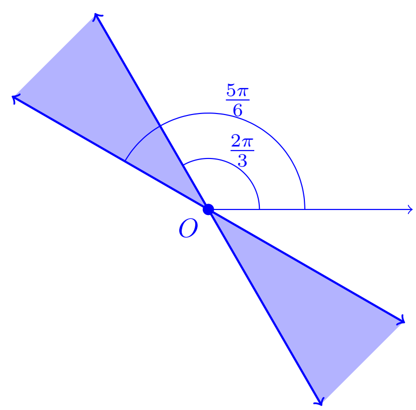
(a) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$



(b) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$



(c) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$



Relating Polar and Cartesian Coordinates: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r^2 = x^2 + y^2.$$

Example 4: Given the polar equation, find the Cartesian equivalent:

(a) $r\cos(\theta) = 2$

$$\boxed{x = 2}$$

(b) $r^2\cos(\theta)\sin(\theta) = 4$

$$r\cos(\theta)\cdot r\sin(\theta) = 4 \Longrightarrow \boxed{xy = 4}$$

(c) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$

$$\left(r\cos(\theta)\right)^2 - \left(r\sin(\theta)\right)^2 = 1 \Longrightarrow \boxed{x^2 - y^2 = 1}$$

(d) $r = 1 + 2r\cos(\theta)$

$$r^2 = \left(1 + 2r\cos(\theta)\right)^2 = 1 + 4r\cos(\theta) + 4\left(r\cos(\theta)\right)^2 \Longrightarrow \boxed{x^2 + y^2 = 1 + 4x + 4x^2}$$

(e) $r = 1 - \cos(\theta)$

$$\begin{aligned} r^2 = \left(1 - \cos(\theta)\right)r = r - r\cos(\theta) &\Longrightarrow r^2 + r\cos(\theta) = r \\ &\Longrightarrow \left(r^2 + r\cos(\theta)\right)^2 = r^2 \\ &\Longrightarrow \boxed{\left(x^2 + y^2 + x\right)^2 = x^2 + y^2} \end{aligned}$$

Example 5: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

$$x^2 + y^2 - 6y + 9 = 9 \Longrightarrow \left(x^2 + y^2\right) - 6y = 0 \Longrightarrow \boxed{r^2 - 6r\sin(\theta) = 0}$$