MATH 142: Calculus II

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Section 8.1: Using Basic Integration Formulas

A Review: The basic integration formulas summarise the forms of indefinite integrals for may of the functions we have studied so far, and the substitution method helps us use the table below to evaluate more complicated functions involving these basic ones. So far, we have seen how to apply the formulas directly and how to make certain u-substitutions. Sometimes we can rewrite an integral to match it to a standard form. More often however, we will need more advanced techniques for solving integrals. First, let's look at some examples of our known methods.

Basic integration formulas			
1. $\displaystyle\int k\,dx = kx + C$	(any number k)	12. $\displaystyle\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	
2. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$(n \neq -1)$	13. $\displaystyle\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$	
3. $\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$		14. $\displaystyle\int \sec(x)\,dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$	
4. $\displaystyle\int e^x\,dx = e^x + C$		15. $\displaystyle\int \csc(x)\,dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$	
5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$	$(a > 0,\ a \neq 1)$	16. $\displaystyle\int \sinh(x)\,dx = \cosh(x) + C$	
6. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$		17. $\displaystyle\int \cosh(x)\,dx = \sinh(x) + C$	
7. $\displaystyle\int \cos(x)\,dx = \sin(x) + C$		18. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
8. $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$		19. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
9. $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$		20. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$	$(a > 0)$
10. $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$		21. $\displaystyle\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$	$(a > 0)$
11. $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$		22. $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$	$(x > a > 0)$

Example 1 - Substitution: Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

Example 2 - Complete the Square: Find

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx.$$

Example 3 - Trig Identities: Calculate

$$\int \cos(x)\sin(2x)+\sin(x)\cos(2x)\,dx.$$

Example 4 - Trig Identities: Find

$$\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin(x)}\,dx.$$

Example 5 - Clever Substitution Evaluate

$$\int \frac{1}{(1+\sqrt{x})^3}\,dx.$$

Example 6 - Properties of Trig Integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\cos(x)\,dx.$$

Section 8.2: Techniques of Integration

A New Technique: **Integration by parts** is a technique used to simplify integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when one of the functions ($f(x)$ or $g(x)$) can be differentiated repeatedly and the other function can be integrated repeatedly without difficulty. The following are two such integrals:

$$\int x\cos(x)\,dx \text{ and } \int x^2e^x\,dx.$$

An Application of the Product Rule: If $f(x)$ and $g(x)$ are differentiable functions of x, the product rule says that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides and rearranging gives us the **Integration by Parts** formula!

In differential form, let $u = f(x)$ and $v = g(x)$. Then,

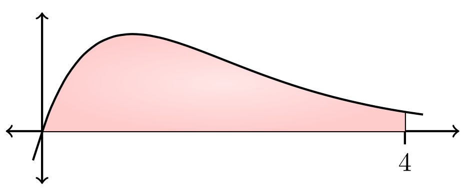
Integration by Parts Formula:

Remember, all of the techniques that we talk about are supposed to make integrating easier! Even though this formula expresses one integral in terms of a second integral, the idea is that the second integral, $\int v\,du$, is easier to evaluate. The key to integration by parts is making the right choice for u and v. Sometimes we may need to try multiple options before we can apply the formula.

Example 1: Find

$$\int x\cos(x)\,dx.$$

Example 3 - Integration by Parts for Definite Integrals: Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



Example 3: Evaluate

$$\int x^2 e^x \, dx.$$

Example 4 - Tabular Method: In Example 2 we have to apply the Integration by Parts Formula multiple times. There is a convenient way to "book-keep" our work. This is done by creating a table. Let's see how by examining Example 2 again.
Evaluate

$$\int x^2 e^x \, dx.$$

Example 5 - Recurring Integrals: Find the integral

$$\int e^x \sin(x)\, dx.$$

This "trick" comes up often when we are dealing with the product of two functions with "non-terminating" derivatives. By this we mean that you can keep differentiating functions like e^x and trig functions indefinitely and never reach 0. Polynomials on the other hand will eventually "terminate" and their n^{th} derivative (where n is the degree of the polynomial) is identically 0.

Example 6 - Challenge: Find the integral

$$\frac{1}{\pi}\int_0^\pi x^3\cos(nx)\,dx,$$

where n is a positive integer.

Section 8.3: Trigonometric Integrals - Worksheet

Goal: By using trig identities combined with u-substitution, we'd like to find antiderivatives of the form

$$\int \sin^m(x)\cos^n(x)\,dx$$

(for integer values of m and n). The goal of this worksheet[1] is for you to work together in groups of 2-3 to discover the techniques that work for these anti-derivatives.

Example 1 - Warm-up: Find

$$\int \cos^4(x)\sin(x)\,dx.$$

Example 2: Find

$$\int \sin^3(x)\,dx.$$

(Hint: Use the identity $\sin^2(x)+\cos^2(x)=1$, then make a substitution.)

[1] Worksheet adapted from BOALA, `math.colorado.edu/activecalc`

Example 3: Find

$$\int \sin^5(x)\cos^2(x)\,dx.$$

(Hint: Write $\sin^5(x)$ as $\left(\sin^2(x)\right)^2 \sin(x)$.)

Example 4: Find

$$\int \sin^7(x)\cos^5(x)\,dx.$$

(The algebra here is long. Only set up the substitution - you do not need to fully evaluate.)

Example 5: In general, how would you go about trying to find

$$\int \sin^m(x)\cos^n(x)\,dx,$$

where m is odd? (Hint: consider the previous three problems.)

Example 6: Note that the same kind of trick works when the power on $\cos(x)$ is odd. To check that you understand, what trig identity and what u-substitution would you use to integrate

$$\int \cos^3(x)\sin^2(x)\,dx?$$

Example 7: Now what if the power on $\cos(x)$ and $\sin(x)$ are both even? Find

$$\int \sin^2(x)\,dx,$$

in each of the following two ways:

(a) Use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

(b) Integrate by parts, with $u = \sin(x)$ and $dv = \sin(x)\,dx$.

(c) Show that your answers to parts (a) and (b) above are the same by giving a suitable trig identity.

(d) How would you evaluate the integral

$$\int \sin^2(x)\cos^2(x)\,dx?$$

Example 8: Evaluate the integral in problem (2) above, again, but this time by parts using $u = \sin^2(x)$ and $dv - \sin(x)\,dx$. (After this, you'll probably need to do a substitution.)

Example 9 - For fun: Can you show your answers to problem (2) and (8) above are the same? It's another great trigonometric identity.

Example 10 - Further investigations: (especially for mathematics, physics and engineering majors) We also would like to be able to solve integrals of the form

$$\int \tan^m(x)\sec^n(x)\,dx.$$

These two functions play well with each other, since the derivative of $\tan(x)$ is $\sec^2(x)$, the derivative of $\sec(x)$ is $\sec(x)|tan(x)$ and since there is a Pythagorean identity relating them. It sometimes works to use $u = \tan(x)$ and it sometimes works to use $u = \sec(x)$. Based on the values of m and n, which substitution should you use? Are there cases for which neither substitution works? (See page 472 of the text.)

Section 8.4: Trigonometric Substitution

Motivation: If we want to find the area of a circle or ellipse, we have an integral of the form

$$\int \sqrt{a^2 - x^2}\, dx$$

where $a > 0$. Regular substitution will not work here, observe:

$$\begin{aligned} u &= a^2 - x^2 \\ du &= -2x\, dx \longleftarrow \text{ extra factor of } x \ldots \end{aligned}$$

Solution: Parametrise! We change x to a function of θ by letting $x = a\sin(\theta)$ so,

Generally, we use an injective (one-to-one) function (so it has an inverse) to simplify calculations. Above, we ensure $a\sin(\theta)$ is invertible by restricting the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Common Trig Substitutions: The following is a summary of when to use each trig substitution.

Integral contains:	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\left[0, \frac{\pi}{2}\right)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

If you are worried about remembering the identities, then don't! They can all be derived easily, assuming you know three basic ones (which by now you should):

$$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Example 1: Evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$$

Example 2: Find

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$$

Example 3: Evaluate

$$\int \frac{x^2}{\sqrt{9 - x^2}}\, dx.$$

Section 8.5: Integration by Partial Fractions

Our next technique: We can integrate some rational functions using u-substitution or trigonometric substitution, but these methods do not always work. Our next method of integration allows us to express any rational function as a sum of functions that *can* be integrated using methods with which we are already familiar. That is, we cannot integrate

$$\frac{1}{x^2 - x}$$

as-is, but it is equivalent to

$$\frac{1}{x} - \frac{1}{x-1},$$

each term of which we can integrate.

Example 1: Our goal is to compute

$$\int \frac{x-7}{(x+1)(x-3)}\, dx.$$

(a) $\displaystyle\int \frac{1}{x+1}\, dx =$

(b) $\displaystyle\frac{2}{x+1} - \frac{1}{x-3} =$

(c) $\displaystyle\int \frac{x-7}{(x+1)(x-3)}\, dx =$

Example 2: Compute $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx.$

(a) $\displaystyle\frac{7}{x-1} + \frac{3}{x-4} =$

(b) $\displaystyle\int \frac{10x-31}{(x-1)(x-4)}\, dx =$

The previous two examples were nice since we were given a different expression of our integrand before hand. But what about when we don't? It is clear that the key step is decomposing our integrand into simple pieces, so how do we do it? The next example outlines the method.

Example 3: Goal: Compute $\int \frac{x+14}{(x+5)(x+2)}\,dx$.

Example 4: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

Example 4 - An alternative approach: Find

$$\int \frac{x+15}{(3x-4)(x+1)}\,dx.$$

Example 5: Goal: Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Here, there are not two different linear factors in the denominator. This CANNOT be expressed in the form

$$\frac{5x-2}{(x+3)^2} = \frac{5x-2}{(x+3)(x+3)} \neq \frac{A}{x+3} + \frac{B}{x+3} = \frac{A+B}{x+3}.$$

However, it can be expressed in the form:

$$\frac{5x-2}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

Example 6: What if the denominator is an irreducible quadratic of the form $x^2 + px + q$? That is, it can not be factored (does not have any real roots). In this case, suppose that $\left(x^2 + px + q\right)^n$ is the highest power of this factor that divides the denominator. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2 + px + q)} + \frac{B_1 x + C_1}{(x^2 + px + q)^2} + \frac{B_1 x + C_1}{(x^2 + px + q)^3} + \cdots + \frac{B_1 x + C_1}{(x^2 + px + q)^n}.$$

Compute $\displaystyle\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx$.

Summary: Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ is proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(X)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1 x + C_1}{(x^2+px+q)} + \frac{B_1 x + C_1}{(x^2+px+q)^2} + \frac{B_1 x + C_1}{(x^2+px+q)^3} + \cdots + \frac{B_1 x + C_1}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Continue with this process with all irreducible factors, and all powers. The key things to remember are

   (i) One fraction for each power of the irreducible factor that appears

   (ii) The degree of the numerator should be one less than the degree of the denominator

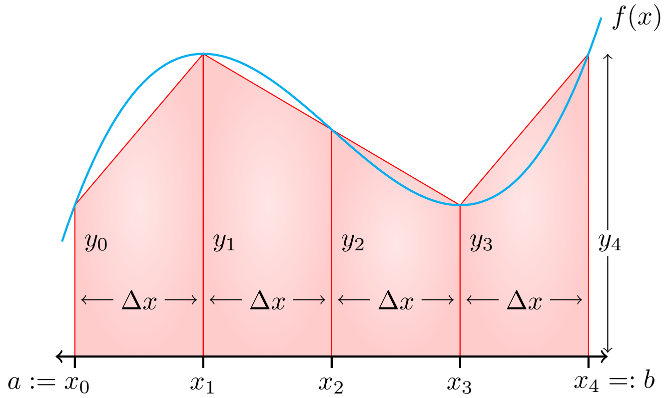
4. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

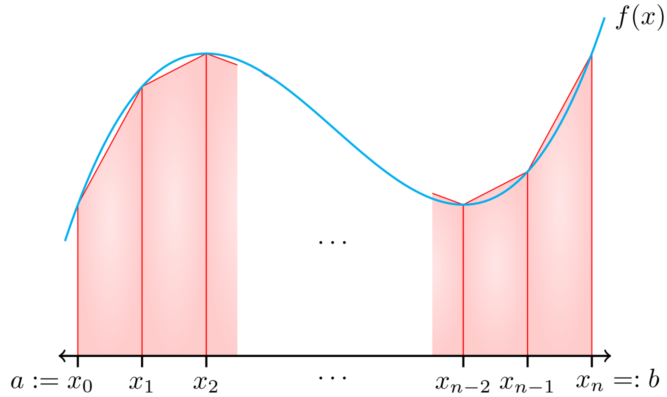
5. Solved for the undetermined coefficients by either strategically plugging in values or comparing coefficients of powers of x.

Section 8.7: Numerical Integration

What to do when there's no nice antiderivative? The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln(x)$ and $\sqrt{1+x^4}$ have no elementary formulas/ When we cannot find a workable antiderivative for a function $f(x)$ that we have to integrate, we can partition the interval of integration, replace $f(x)$ by a closely fitting polynomial on each subinterval, integrate the poynomials and add the results to *approximate* the definite integral of $f(x)$. This is an example of numerical integration. There are many methods of numerical integration but we will study only two: the *Trapezium Rule* and *Simpson's Rule*.

Trapezoidal Approximations: As the name implies, the Trapezium Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapeziums instead of rectangles - which, if you recall, we studied when we looked at Riemann integration in Calculus I.





The Trapezium Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f\left(x_i\right)\right), \end{aligned}$$

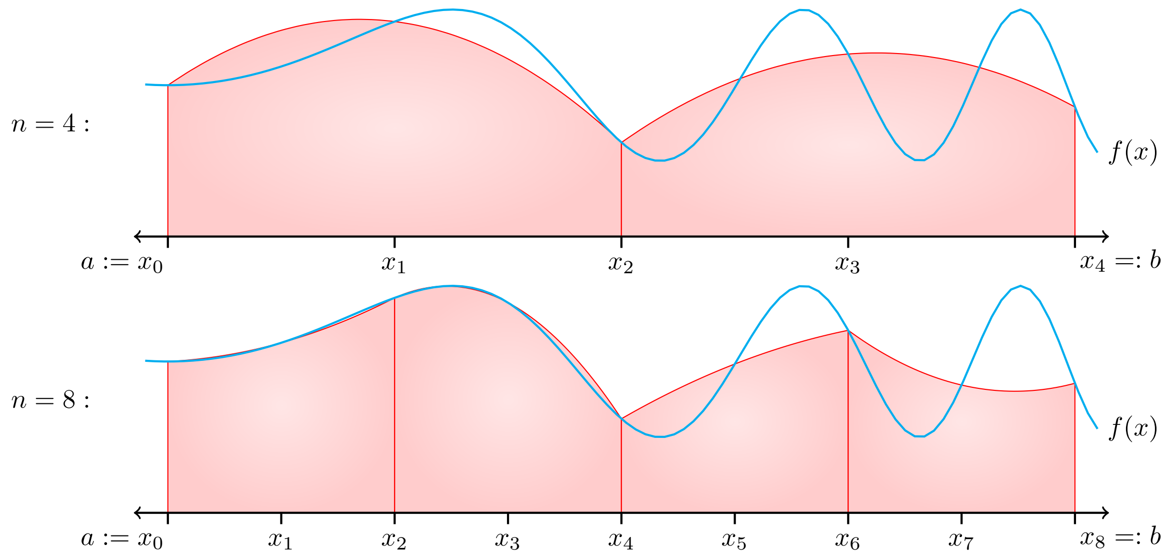
where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$.

Example 1: Use the Trapezium Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Parabolic Approximations: Instead of using the straight-line segments that produced the trapeziums, we can use parabolas to approximate the definite integral of a continuous function. We partition the interval $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ but this time we require n to be an even number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola. A typical parabola passed through three consecutive points: (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) on the curve.



Simpson's Rule: To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{\Delta x}{3}\left(f\left(x_0\right) + f\left(x_n\right) + 2\left(\sum_{i=1}^{\frac{n-1}{2}} f\left(x_{2i-1}\right) + 2f\left(x_{2i}\right)\right)\right), \end{aligned}$$

where the y's are the values of f at the partition points

$$x_0 := a,\ x_1 := a + \Delta x,\ x_2 := a + 2\Delta x,\ \ \ldots,\ x_{n-1} := a + (n-1)\Delta x,\ x_n := a + n\Delta x = b,$$

and $\Delta x = \dfrac{b-a}{n}$ with n an *even* number.

Example 2: Use the Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$. Compare the estimate with the exact value.

Error Estimates in the Trapezium and Simpson's Rules If $f''(x)$ is continuous and M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, then the error E_T in the Trapezium Rule for approximating the definite integral of $f(x)$ over the interval $[a, b]$ using n trapeziums satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

If $f^{(4)}(x)$ is continuous and M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a, b]$, then the error E_S in Simpson's Rule for approximating the definite integral of $f(x)$ over the interval $[a, b]$ using $\frac{n}{2}$ parabolas satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

Example 3: Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$. What value of n should we pick so that the error is within 0.001 of the true value?

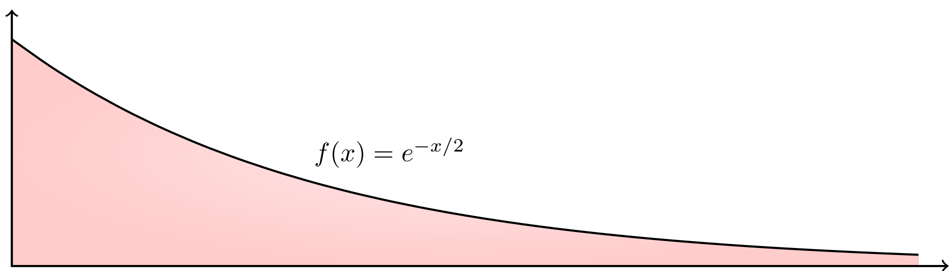
Section 8.8: Improper Integrals

Switching up the Limits of Integration: Up until now, we have required two properties of *definite* integral:

1. the domain of integration, $[a, b]$, is finite
2. the range of the integrand is finite on this domain.

We will now see what happens if we allow the domain or range to be infinite!

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



Definition: Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

   where c is any real number.

In each case, if the limit is finite we sat that the improper integral ____________________ and that the limit is the

__________________ of the improper integral. If the limit fails to exist, the improper integral ____________________

Any of the integrals in the above definition can be interpreted as an area if $f(x) \geq 0$ on the interval of integration. If $f(x) \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

Example 1: Evaluate

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx.$$

Example 2: Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

A Special Example: For what values of p does the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converge? When the integral does converge, what is its value?

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration. We apply a similar technique as in the previous examples of integrating over an altered interval before obtaining the integral we want by taking limits.

Example 4: Investigate the convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Definition: Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^a f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we sat that the improper integral ____________________ and that the limit is the

_________________ of the improper integral. If the limit fails to exist, the improper integral ____________________

Example 5: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges of diverges. If the integral diverges, we are done. If it converges we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $(a, \infty]$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

Example 7: Determine if the following integral is convergent of divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Example 8: Show that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges.

Example 9: Show that

$$\int_1^\infty \frac{1-e^{-x}}{x}\,dx$$

dinverges.

Section 10.1: Sequences

Definition: A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The order is important here, for example 2, 4, 6, 8, ... is *not* the same as 4, 2, 6, 8, ...

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

Definition: (**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$\text{for all } n \geq N \qquad |a_n - L| < \varepsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences:

If $a_n \longrightarrow L$, $b_n \longrightarrow M$, then:

1. Sum Rule: $\lim\limits_{n\to\infty} (a_n + b_n) =$,
2. Constant Rule: $\lim\limits_{n\to\infty} c =$,
3. Product Rule: $\lim\limits_{n\to\infty} a_n \cdot b_n =$,
4. Quotient Rule: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} =$
5. Power Rule: $\lim\limits_{n\to\infty} a_n^p =$

Squeeze Theorem: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim\limits_{n\to\infty} b_n = L$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

2. $\left\{\dfrac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

3. $\left\{\dfrac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

4. $\left\{\dfrac{(-1)^n}{n}\right\}$

Examples of Divergent Sequences:

1. $\{(-1)^n\}_{n=1}^{\infty}$

2. $\{(-1)^n n\}_{n=1}^{\infty}$

3. $\{\sin(n)\}_{n=1}^{\infty}$

Definition: The product of the first n positive integers,

$$n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is denoted by $n!$ (read n **factorial**.)

Example 1: Find the limit of the sequence $\left\{ \dfrac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is ________________________________ if there exists a number M such that $a_n \leq M$ for all n.

The number M is an _________________________ for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the

____ of $\{a_n\}_{n=1}\infty$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is ________________________________ if there exists a number m such that $a_n \geq m$ for all n.

The number m is a _________________________ for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the

____ of $\{a_n\}_{n=1}\infty$.

(c) **Completeness Axiom**: If S is any non-empty set of real numbers that has an upper bound M, then S has a least upper bound b. Similarly for least upper bound.

(d) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is ______________________.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an ________________________ sequence.

(e) Every ________________________ sequence is ______________________ but **not** every ______________________________

______________________. (consider ________________________).

(f) A sequence $\{a_n\}_{n=1}^{\infty}$ is ___________________________ if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ___________________________ if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is ________________________ if it is either non-decreasing or non-increasing.

The Monotone Convergence Theorem: Every bounded, monotonic sequence converges.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$

Section 10.2: Infinite Series

Sum of an Infinite Sequence: An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead, we look at the result of summing the first n terms of the sequences,

$$S_n := a_1 + a_2 + a_3 + \cdots + a_n.$$

S_n is called the n^{th} **partial sum**. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense as the terms of a sequence approach a limit.

Example 1: To assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning to look for a pattern in how these partial sums grow:

Partial Sum		**Value**
First:	$S_1 = 1$	1
Second:	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$
Third:	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$
$\vdots$	$\vdots$	$\vdots$
n^{th}:	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ________________________. The number a_n is the ____________________ of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the _________________________________ of the series, the number S_n being the __________________________.

If the sequence of partial sums converges to a limit L, we say that the series _____________________ and that the ________________ is L. In this case we write

If the sequence of partial sums of the series does not converge, we say that the series ____________________.

Notation: Sometimes it is nicer, or even more beneficial, to consider sums starting at $n = 0$ instead. For example, we can rewrite the series in Example 1 as

At times it may also be nicer to start indexing at some number other than $n = 0$ or $n = 1$. This idea is called **re-indexing** the series (or sequence). So don't be alarmed if you come across series that do not start at $n = 1$.

Geometric Series: A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The _________________ r can be positive (as in Example 1) or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}.$$

If $r = 1$, the n^{th} partial sum of the geometric series is

If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

Convergence of Geometric Series: If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots ar^{n-1} + \cdots$ converges:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Example 2: Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}.$$

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

Example 4: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The n^{th} Term Test for Divergence: The series $\sum_{n=1}^{\infty} a_n$ *diverges* if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series: If $\sum a_n = A$ and $b_n = B$, then

1) Sum Rule : $\sum_{n=1}^{\infty} (a_n + b_n)$ 2) Constant Multiple Rule : $\sum_{n=1}^{\infty} ca_n$

Some True Facts:

1. Every non-zero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n \pm b_n)$ diverges.

Adding/Deleting Terms: Adding/deleting a finite number of terms will not alter the convergence or divergence of a series.

Section 10.3: The Integral Test

Tests for Convergence: The most basic question we can ask about a series is whether or not it converges. In the next few sections we will build the tools necessary to answer that question. If we establish that a series does converge, we generally do not have a formula for its sum (unlike the case for Geometric Series). So, for a convergent series we need to investigate the error involved when using a partial sum to approximate its total sum.

Non-decreasing Partial Sums: Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor since $S_{n+1} = S_n + a_{n+1}$, so

Since the partial sums form a non-decreasing sequence, the Monotone Convergence Theorem give us the following result:

Corollary Of MCT: A series $\displaystyle\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Example 1: Consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We now introduce the Integral Test with a series that is related to the harmonic series, but whose n^{th} term is $1/n^2$ instead of $1/n$.

Example 2: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a ____________________, ____________________, ____________________ function of x. Then the series $\displaystyle\sum_{n=\_\_\_\_}^{\infty} a_n$ and the integral $\displaystyle\int_{\_\_\_\_}^{\infty} f(x)\,dx$ both converge or diverge.

Example 3: Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

(where p is a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Example 4: Determine the convergence of divergence of the series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Bound for the Remainder in the Integral Test: Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms with $a_k = f(k)$, where $f(x)$ is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum_{k=1}^{\infty} a_k$ converges to S. Then the remainder $R_n = R - S_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Example 5: Estimate the sum, S, of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.

Section 10.4: Comparison Tests for Series - Worksheet

Goal: In Section 8.8 we saw that a given improper integral converges if its integrand is less than the integrand of another integral known to converge. Similarly, a given improper integral diverges if its integrand is greater than the integrand of another integral known to diverge. In Problems $1-8$, you'll apply a similar strategy to determine if certain series converge or diverge.

Problem 1: For each of the following situations, determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or if one cannot tell without more information.

(a) If $0 \le a_n \le \frac{1}{n}$ for all n, we can conclude ____________________.

(b) If $\frac{1}{n} \le a_n$ for all n, we can conclude ____________________.

(c) If $0 \le a_n \le \frac{1}{n^2}$ for all n, we can conclude ____________________.

(d) If $\frac{1}{n^2} \le a_n$ for all n, we can conclude ____________________.

(e) If $\frac{1}{n^2} \le a_n \le \frac{1}{n}$ for all n, we can conclude ____________________.

Problem 2: For each of the cases in Problem 1 where you needed more information to determine the convergence of the series, give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

Direct Comparison Test for Series: If $0 \leq a_n \leq b_n$ for all $n \geq N$, where $N \in \mathbb{N}$, then,

1. If $\sum_{n=1}^{\infty} b_n$ ________________________, then so does $\sum_{n=1}^{\infty} a_n$.
2. If $\sum_{n=1}^{\infty} a_n$ ________________________, then so does $\sum_{n=1}^{\infty} b_n$.

Now we'll practice using the Direct Comparison Test:

Problem 3: Let $a_n = \dfrac{1}{2^n + n}$ and let $b_n = \left(\dfrac{1}{2}\right)^n$.

(a) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge? Why?

(b) How do the sizes of the terms a_n and b_n compare?

(c) What can you conclude about $\sum_{n=1}^{\infty} \dfrac{1}{2^n + n}$?

Problem 4: Let $a_n = \dfrac{1}{n^2 + n + 1}$.

(a) By considering the rate of growth of the denominator of a_n, what choice would you make for b_n?

(b) Does $\sum_{n=1}^{\infty} b_n$ converge or diverge?

(c) How do the sizes of the terms a_n and b_n compare?

(d) What can you conclude about $\sum_{n=1}^{\infty} a_n$?

Problem 5: Use the Direct Comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4-1}}{n^5+3}$ converges or diverges. (Hint: What are the *dominant* terms of a_n?)

Problem 6: Use the Direct Comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3+n}}$ converges or diverges.

Problem 7: Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. Let $a_n = \dfrac{1}{n^2}$ and $b_n = \dfrac{1}{n^2-1}$ for $n \geq 2$.

(a) By comparing the relative sizes of the terms of the two sequences, do we have enough information to determine if $\displaystyle\sum_{n=2}^{\infty} b_n$ converges or diverges?

(b) Show that $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = 1$.

(c) Using part (b), explain carefully why, for all n large enough (more precisely, for all n larger than some integer N), $b_n \leq 2a_n$. Now can you determine if $\sum_{n=N}^{\infty} b_n$ converges or diverges?

The Limit Comparison Test: Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is finite and $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both ______________________ or both _____________________.

Problem 8: Using either the Limit or Direct Comparison Test, determine if the series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.

Problem 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ converges or diverges.

Section 10.5: Absolute Convergence & the Ratio and Root Tests

When the terms of a series are positive *and* negative, the series may or may not converge.

Example 1: Consider the series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n.$$

Example 2: Now consider

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(-\frac{5}{4}\right)^n.$$

The Absolute Convergence Test:

$$\text{If } \sum_{n=0}^{\infty} |a_n| \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 3: Consider $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 4: Use the Ratio Test to decide whether the series

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

converges absolutely, is conditionally convergent or diverges.

Example 5: Use the Ratio Test to decide whether the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

converges absolutely, is conditionally convergent or diverges.

The Root Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** about the series using this test.

Example 6: Use the Root Test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges absolutely, is conditionally convergent, or diverges.

Section 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where $b_n = |a_n|$ is a positive number.

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- $\lim_{n\to\infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the requirements with $N = 1$ and therefore converges.

Instead of verifying $b_n \geq b_{n+1}$, we can follow the steps we did in the integral test to verify the sequence is decreasing. Define a differentiable function $f(x)$ satisfying $f(n) = b_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is non-increasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $b_n \geq b_{n+1}$ for all N.

Example 2: Consider the sequence where $b_n == \dfrac{10n}{n^2+16}$. Define $f(x) = \dfrac{10x}{x^2+16}$. Then $f'(x) = \dfrac{10(16-x^2)}{(x^2+16)} \geq 0$ when $x \geq 4$. It follows that $b_n \geq b_{n+1}$ for $n \geq 4$.

The Alternating Series Test Estimation Theorem: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ satisfies the conditions of the AST, then for $n \geq N$,

$$S_n = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1} b_n$$

approximates the sum L of the series with an error whose absolute value is less than b_{n+1}, the absolute value of the first unused term.

Furthermore, the sum L lies between any two successive partial sums S_n and S_{n+1}, and the remainder, $L - S_n$, has the same sign as the first unused term.

Example 3: Let's apply the Estimation Theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}+\frac{1}{256}-\cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}.$$

Example 4 - Conditional Convergence: We have seen that in absolute value, the Alternating Harmonic Series diverges. The presence of infinitely many negative terms is essential to its convergence. We say the Alternating Harmonic Series if **conditionally convergent**. We can extend this idea to the alternating p-series.
If p is a positive constant, the sequence $\dfrac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p} = 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots, \qquad p>0$$

converges.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \ldots, b_n \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum b_n = \sum a_n.$$

Example 5: We know $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to some number L.

Section 10.7: Power Series

Definition: A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **centre** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1 - Geometric Power Series: Taking all the coefficients to be 1 in the power series centred at $x = 0$ gives the geometric power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

This is the geometric series with first term 1 and ratio x.

Instead of focussing on finding a formula for the sum of a power series, we are now going to think of the partial sums of the series as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need only take a few terms of the series to get a good approximation. As we move toward $x = 1$ or $x = -1$, we need more terms.

One of the most important questions we can ask about a power series is "for what values of x will the series converge?" Since power series are functions, what we are really asking here is "what is the **domain** of the power series?"

Example 2: Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

Example 3: For what values of x do the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Fact: There is always at least one point for which a power series converges: the point $x = a$ at which the series is centred.

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Convergence Theorem and the previous examples lead to the conclusion that a power series $\sum c_n(x-a)^n$ behaves in one of three possible ways;

- If might converge on some interval of *radius* R.
- It might converge everywhere.
- It might converge only at $x = a$.

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x $(R = \infty)$

3. The series converges only at $x = a$ and diverges elsewhere $(R = 0)$

R is called the **radius of convergence** of the power series, and the interval of radius R centred at $x = 1$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series.

How to test a Power Series for Convergence:

1. Use the Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test fo convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the n^{th} term does not approach zero for those values of x.

Example 4: Find the interval and radius of convergence for

$$\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}.$$

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants. They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product.

The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B_n(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$. For example:

Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

Term-by-Term Differentiation Theorem: If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on the interval $a - R < x < a + R$. This function $f(x)$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.
Note: When we differentiate we may have to start our index at one more than it was before. This is because we lose the constant term (if it exists) when we differentiate.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=0}^{\infty} \frac{n!\cos(n!x)}{n^2}$$

which *diverges* for all x. This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find a series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

converges for $a - R < x < a + R$ for $R > 0$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$.

Example 6: Given $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$

Section 10.8: Taylor and Maclaurin Series

Series Representations: We've seen that geometric series can be used to generate a power series for functions having a special form, such as $f(x) = \dfrac{1}{1-x}$ or $g(x) = \dfrac{3}{x-2}$. Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n.

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by $f(x)$ at $x = a$** is

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin Series generated by** $f(x)$ is the Taylor series generated by $f(x)$ at $a = 0$.

Example 1: Find the Taylor series generated by $f(x) = \dfrac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial** of order n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Just as the linearisation of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $a = 0$.

Example 3: Find the Maclaurin Series generated by $f(x) = \sin(x)$.

Example 4: Find the Taylor Series generated by $f(x) = e^x$.

Section 10.9: Convergence of Taylor Series

Taylors Theorem: In the last section, we asked when a Taylor Series for a function can be expected to that (generating) function. That question is answered by the following theorem:

If $f(x)$ and its first n derivatives $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$ are continuous on the closed interval between a and b, and $f^{(n)}(x)$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Interesting Fact: Taylor's Theorem is a generalisation of the Mean Value Theorem!

Taylor's Formula: If $f(x)$ has derivatives of all orders in a n open interval I containing a, then for each positive integer n and for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Stating Taylor's Theorem in this way says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where the function $R_n(x)$ is determined by the value of the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ at a point c that depends on both a and x, and that it lies somewhere between them.

Definitions: The second equation is called **Taylor's formula**. The function $R_n(x)$ is called the ________________________

________________________or the ________________________ for the approximation of $f(x)$ by $P_n(x)$ over I.

If $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in I$, we say that the Taylor Series generated by $f(x)$ at $x = a$ **converges** to $f(x)$ on I, and we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Often we can estimate $R_n(x)$ without knowing the value of c.

Example 1: Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

The Remainder Estimation Theorem: If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Example 2: Show that the Taylor Series generated by $f(x) = \sin(x)$ at $a = 0$ converges to $\sin(x)$ for all x.

Using Taylor Series: Since every Taylor series is a power series, the operations of adding, subtracting and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 3: Using known series, find the first few terms of the Taylor series for

$$\frac{1}{3}(2x + x\cos(x))$$

using power series operations.

Example 4: For what values of x can we replace $\sin(x)$ by the polynomial $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?

Section 10.10: Applications of Taylor Series

Evaluating Non-elementary Integrals: Taylor series can be used to express non-elementary integrals in terms of series. Integrals like the one in the next example arise in the study of the diffraction of light.

Example 1: Express

$$\int \sin(x^2)\,dx$$

as a power series.

Example 2: Estimate

$$\int_0^1 \sin(x^2)\,dx$$

with an error of less than 0.001.

If we extend this to 5 terms, we obtain

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} \approx 0.310268303.$$

This gives an error of about 1.08×10^{-9}. To guarantee this accuracy (using the error formula) for the Trapezium Rule, we would need to use about 8000 subintervals!

Euler's Identity: A complex number is a number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$. So then

$$i=\sqrt{-1} \qquad i^2=-1 \qquad i^3=-\sqrt{-1} \qquad i^4=1$$

If we substitute $x=i\theta$ into the Taylor series for e^x ans use the relations above, we obtain

This identity is actually amazing. You can use this identity to derive all of the angle sum formulas, so you never need to remember them all! Also we see that $e^{i\pi}=-1$, which we can rewrite to obtain

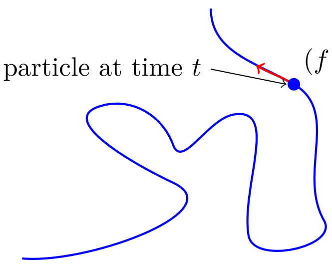
$$\boxed{e^{i\pi}+1=0}$$

which combines 5 of the most important constants in mathematics; e, π, i, 1 and 0.

Common Taylor Series			
1. $\dfrac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\dfrac{1}{1+x}$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!}x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots$	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}x^n$	$-1 < x \leq 1$
7. $\tan^{-1}(x)$	$x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1}x^{2n+1}$	$\lvert x\rvert \leq 1$

Section 11.1: Parametrisations of Plane Curves

Parametric Equations: Below we have the path of a moving particle on the xy-plane. We can sometimes describe such a path by a pair of equations, $x = f(t)$ and $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions. Equations like these describe more general curves than those described by a single function, and they provide not only the graph of the path traced out but also the location of the particle $(x, y) = (f(t), g(t))$ at any time t.



Definitions: If x and y are given as functions

$$x = f(t) \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ________________________.

The equations are ________________________________ for the curve.

The variable t is the _______________________ for the curve and its domain I is the ______________________________.

If I is a closed interval, $a \leq t \leq b$, the ________________________ of the curve is the point $(f(a), g(a))$ and the

__________________________ of the curve is $(f(b), g(b))$.

Example 1: Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

Example 2: Identify geometrically the curve in Example 1 by eliminating the parameter t and obtaining an algebraic equation in x and y.

Example 3: Graph the parametric curves

(a) $x = \cos(t), \qquad y = \sin(t), \qquad 0 \leq t \leq 2\pi,$

(b) $x = a\cos(t), \quad y = a\sin(t), \quad 0 \leq t \leq 2\pi, \quad a \in \mathbb{R}.$

Example 4: The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

Example 5 - Natural Parametrisation: A parametrisation of the function $f(x) = x^2$ is given by

Example 6: Find a parametrisation for the line through the point (a, b) having slope m.

Example 7: Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$$

Section 11.2: Calculus with Parametric Equations

Tangents and Areas: A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If all three derivatives exist and $\dfrac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Further we also have

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}/dt}{dx/dt}.$$

Example 1: Find the tangent to the curve

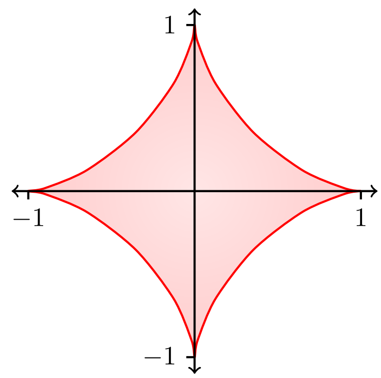
$$x = \sec(t), \quad y = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$.

Example 2: Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi.$$

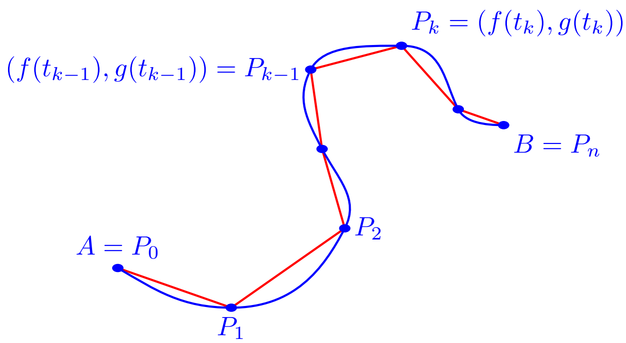


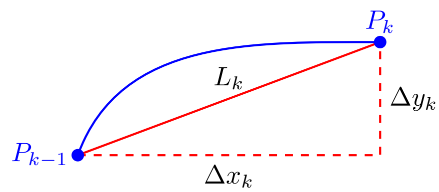
Length of a Parametrically Defined Curve: Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

We assume the functions $f(t)$ and $g(t)$ are ______________________________ on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps.

Such a curve is called a _____________________.





The smooth curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment shown on the right, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \text{ and } g'(t_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes

$$L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\Delta t_k.$$

In an surprising turn of events, we obtain the exact value of L by taking a limit of this sum, resulting in a definite integral. To summarise:

Definition: If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Example 4: Using the definition, find the length of the circle of radius r defined parametrically by

$$x = r\cos(t), \quad y = r\sin(t), \quad 0 \leq t \leq 2\pi.$$

Example 5: Find the length of the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi.$$

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \le t \le 2\pi.$$

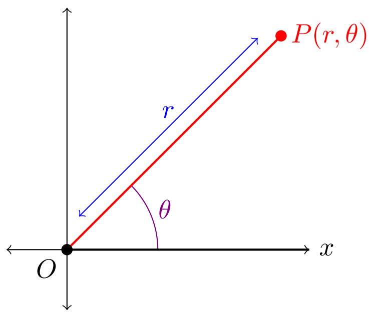
Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.

Section 11.3: Polar Coordinates

Definition: To define polar coordinates, we first fix an ________________ O (called the ________________) and an

____________________ from O (usually the positive x-axis). Then each point P can be located by assigning to it a

_____________________________ (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to the ray OP.



Just like trigonometry, θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is not unique.
In some cases, we allow r to be negative. For instance, the point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians anticlockwise from the initial ray and going forward 2 units, or we could turn $\pi/6$ radians clockwise and go backwards 2 units; corresponding to $P(-2, \pi/6)$.

Example 1: Find all the polar coordinates of the point $P(2, \frac{\pi}{6})$.

Polar Equations and Graphs: If we fix r at a constant value (not equal to zero), the point $P(r, \theta)$ will lie $|r|$ unites from the origin O. As θ varies over any interval of length 2π, P traces a what?

If we fix θ at a constant value and let r vary between $-\infty$ and ∞, then the point $P(r, \theta)$ traces a what?

Example 2: A circle or line can have more than one polar equation.

Example 3: Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments and rays. Graph the sets of points whose polar coordinates satisfy the given conditions:

(a) $1 \leq r \leq 2$ and $0 \leq \theta \leq \dfrac{\pi}{2}$

(b) $-3 \leq r \leq 2$ and $\theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \leq \theta \leq \dfrac{5\pi}{6}$

Relating Polar and Cartesian Coordinates: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following:

Example 4: Given the polar equation, find the Cartesian equivalent:

(a) $r\cos(\theta) = 2$

(b) $r^2\cos(\theta)\sin(\theta) = 4$

(c) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$

(d) $r = 1 + 2r\cos(\theta)$

(e) $r = 1 - \cos(\theta)$

Example 5: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.