Math 142
Joseph C Foster
Summer 2019
Final Exam

Name: Solutions
June 22nd, 2019
Time Limit: 150 minutes

This exam contains 14 pages (including this cover page) and 26 questions. You have 150 minutes to complete the exam.

No calculators are allowed. Turn your phones off and place them on your desk face down. Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero. If you wish to leave the room during the exam you should place your phone at the front of the class before doing so.

Question	Marks	Score	Best
1	10		
2	10		
3	10		
4	10		
5	10		
6	10		
7	10		
8	10		
9	10		
10	10		
11	10		
12	10		
13	10		
14	10		
15	10		
16	10		
17	10		
18	10		
19	10		
20	10		
21	10		
22	10		
23	10		
24	10		
25	10		
26	10		
Total	260		
Best 13	130		
Grade	100		

Target (130)	A	B+	B	C+	C	D+	D

Recitation				

Quizzes			
1		10	
2		11	
3		12	
4		13	
5		14	
6		15	
7		Average	
8			
9			

Midterms	
Exam 1	
Exam 2	
Exam 3	

Final Exam

Total	
Final Grade	

Show **all** *necessary* work to receive full credit. An answer with no work, even if correct, will not receive full marks. Please circle or box your final answer. If you cannot complete a problem but can write down what you want to do, and this is correct, you can still receive partial credit. Don't leave anything blank! The space provided is indicative of the amount of work required.

There are 26 questions in this test. You are expected to answer 13 of them, half from each section. You may answer more, but only your best 13 will count towards your final grade. Instructions as to what to answer are given in each section.

Integration by Parts

For questions 1 − 4, answer any **two**. You may answer more, but only the scores from the best **two** will be counted.

1. (10 marks) Evaluate

$$\int -x^4 e^{-x}\, dx.$$

Differentiate u		Integrate dv
x^4	$+$	$-e^{-x}$
$4x^3$	$-$	e^{-x}
$12x^2$	$+$	$-e^{-x}$
$24x$	$-$	e^{-x}
24	$-$	$-e^{-x}$
0		e^{-x}

$$\int x^4 e^x\, dx = x^4 e^{-x} x + 4x^3 e^{-x} + 12x^2 e^{-x} + 24xe^{-x} + 24e^{-x} + C$$

$$= \boxed{\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right) e^{-x} + C}$$

2. (10 marks) Evaluate

$$\int_1^e \frac{\ln(x)}{x^2}\, dx.$$

$$u = \ln(x) \qquad dv = \frac{1}{x^2}\, dx$$

$$du = \frac{1}{x}\, dx \qquad v = -\frac{1}{x}$$

$$\int_1^e \frac{\ln(x)}{x^2}\, dx = -\frac{\ln(x)}{x}\bigg|_1^e + \int_1^e \frac{1}{x^2}\, dx$$

$$= -\frac{\ln(x)}{x} - \frac{1}{x}\bigg|_1^e$$

$$= -\frac{\ln(e)}{e} - \frac{1}{e} - \left(-\frac{\ln(1)}{1} - \frac{1}{1}\right)$$

$$= -\frac{2}{e} + 1$$

$$= \boxed{\frac{e-2}{e}}$$

3. (10 marks) Evaluate

$$\int e^{3x}\sin(4x)\,dx.$$

$$u_1 = e^{3x} \qquad dv_1 = \sin(4x) \qquad\qquad u_2 = e^{3x} \qquad dv_2 = \cos(4x)$$
$$du_1 = 3e^{3x}\,dx \qquad v_1 = -\frac{1}{4}\cos(4x) \qquad\qquad du_2 = 3e^{3x}\,dx \qquad v_2 = \frac{1}{4}\sin(4x)$$

$$\begin{aligned}
\int e^{3x}\sin(4x)\,dx &= -\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{4}\int e^{3x}\cos(4x)\,dx \\
&= -\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{4}\left[\frac{1}{4}e^{3x}\sin(4x) - \frac{3}{4}\int e^{3x}\sin(4x)\,dx\right] \\
&= -\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{16}e^{3x}\sin(4x) - \frac{9}{16}\int e^{3x}\sin(4x)\,dx \\
\implies \frac{25}{16}\int e^{3x}\sin(4x)\,dx &= -\frac{1}{4}e^{3x}\cos(4x) + \frac{3}{16}e^{3x}\sin(4x) + C \\
\implies \int e^{2x}\sin(3x)\,dx &= -\frac{4}{25}e^{3x}\cos(4x) + \frac{3}{25}e^{3x}\sin(4x) + C \\
&= \boxed{\frac{1}{25}e^{3x}\left(3\sin(4x) - 4\cos(4x)\right) + C}
\end{aligned}$$

4. (10 marks) Evaluate

$$\int_0^1 4\arctan(x)\,dx.$$

$$u = \arctan(x) \qquad dv = 1\,dx$$
$$du = \frac{1}{x^2+1}\,dx \qquad v = x$$

$$w = x^2 + 1$$
$$dw = 2x\,dx$$

$$\begin{aligned}
\int_0^1 4\arctan(x)\,dx &= 4x\arctan(x)\Big|_0^1 - 4\int_0^1 \frac{x}{x^2+1}\,dx \\
&= 4x\arctan(x)\Big|_0^1 - 2\int_{x=0}^{x=1}\frac{1}{u}\,dw \\
&= 4x\arctan(x) - 2\ln|w|\Big|_{x=0}^{x=1} \\
&= 4x\arctan(x) - 2\ln\left(x^2+1\right)\Big|_0^1 \\
&= \pi - 2\ln(2) - (0 - 2\ln(1)) \\
&= \boxed{\pi - \ln(4)}
\end{aligned}$$

Trigonometric Integrals

For questions $5-6$, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

5. (10 marks) Calculate

$$\int 24\sin^5(x)\cos^3(x)\ dx.$$

$u = \sin(x)$

$du = \cos(x)\ dx$

$$\begin{aligned}\int 24\sin^5(x)\cos^3(x)\ dx &= 24\int \sin^5(x)\cos^2(x)\cos(x)\ dx\\ &= 24\int \sin^5(x)\left(1-\sin^2(x)\right)\cos(x)\ dx\\ &= 24\int u^5\left(1-u^2\right)\ du\\ &= 24\int u^5 - u^7\,du\\ &= 4u^6 - 3u^8 + C\\ &= 4\sin^6(x) - 3\sin^8(x) + C\\ &= \boxed{\sin^6(x)\left(4-3\sin^2(x)\right)+C}\end{aligned}$$

6. (10 marks) Calculate

$$\int_0^{2\pi} \cos^2(2x)\ dx.$$

$$\begin{aligned}\int_0^{2\pi}\cos^2(2x)\ dx &= \frac{1}{2}\int_0^{2\pi} 1+\cos(4x)\ dx\\ &= \frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\Bigg|_0^{2\pi}\\ &= \frac{1}{2}\left(2\pi+\frac{1}{4}\sin(8\pi)-\left(0+\frac{1}{4}\sin(0)\right)\right)\\ &= \boxed{\pi}\end{aligned}$$

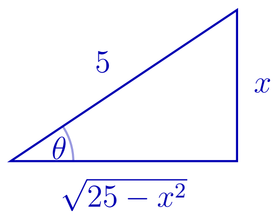
Trigonometric Substitution

For questions $7-8$, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

7. (10 marks) Evaluate

$$\int \frac{25}{x^2\sqrt{25-x^2}}\,dx.$$

$$x = 5\sin(\theta)$$
$$dx = 5\cos(\theta)\,d\theta$$



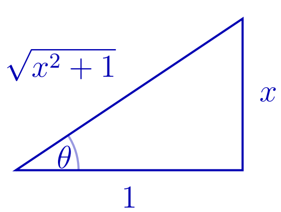
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj.}}{\text{opp.}}$$

$$\int \frac{25}{x^2\sqrt{25-x^2}}\,dx = \int \frac{25\cdot 5\cos(\theta)}{25\sin^2(\theta)\sqrt{25-\sin^2(\theta)}}\,d\theta$$
$$= \int \frac{\cancel{25}\cdot\cancel{5\cos(\theta)}}{\cancel{25}\sin^2(\theta)\cdot\cancel{5\cos(\theta)}}\,d\theta$$
$$= \int \csc^2(\theta)\,d\theta$$
$$= -\cot(\theta) + C$$
$$= \boxed{-\frac{\sqrt{25-x^2}}{x} + C}$$

8. (10 marks) Evaluate

$$\int \frac{1}{(x^2+1)^{3/2}}\,dx.$$

$$x = \tan(\theta)$$
$$dx = \sec^2(\theta)\,d\theta$$



$$\sin(\theta) = \frac{\text{opp.}}{\text{hyp.}}$$

$$\int \frac{1}{(x^2+1)^{3/2}}\,dx = \int \frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^{3/2}}\,d\theta$$
$$= \int \frac{\cancel{\sec^2(\theta)}}{\sec^{\cancel{3}}(\theta)}\,d\theta$$
$$= \int \cos(\theta)\,d\theta$$
$$= \sin(\theta) + C$$
$$= \boxed{\frac{x}{\sqrt{x^2+1}} + C}$$

Integration of Rational Functions

For questions $9-10$, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

9. (10 marks) Calculate

$$\int \frac{x^2+2x-1}{x^3-x}\,dx.$$

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x-1)(x+1)} \Longrightarrow x^2+2x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$x=0: \quad -1=-A \Longrightarrow A=1, \qquad x=1: \quad 2 = 2B \Longrightarrow B=1, \qquad x=-1: \quad -2=2C \Longrightarrow C=-1.$$

$$\int \frac{x^2+2x-1}{x^3-x}\,dx = \int \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}\,dx = \boxed{\ln|x| + \ln|x-1| - \ln|x+1| + C}$$

10. (10 marks) Calculate

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx.$$

$$\frac{5x^2+3x-2}{x^3+2x^2} = \frac{x^2+2x-1}{x^2(x+2)} \Longrightarrow 5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$$

$$x=0: \quad -2=2B \Longrightarrow B=-1, \qquad x=-2: \quad 12 \qquad = 4C \Longrightarrow C=3,$$

$$5x^2+3x-2 = Ax(x+2) - (x+2) + 3x^2 \Longrightarrow 2x^2+4x = Ax(x+2) \Longrightarrow A=2$$

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\,dx = \boxed{2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C}$$

Improper Integrals

For questions 11 − 12, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

11. (10 marks) Decide whether

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

converges or diverges. If it converges, find the value of the integral.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{b\to 3^-} -2\sqrt{3-x}\Big|_0^b\\ &= \lim_{b\to 3^-} -2\sqrt{3-b}+2\sqrt{3}\\ &= 0+2\sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

So this integral $\boxed{\textit{converges} \text{ to } 2\sqrt{3}}$

12. (10 marks) Decide whether

$$\int_4^\infty e^{-x/2}\,dx$$

converges or diverges. If it converges, find the value of the integral.

$$\begin{aligned}\int_4^\infty e^{-x/2}\,dx &= \lim_{b\to\infty}\int_4^b e^{-x/2}\,dx\\ &= \lim_{b\to\infty} -2e^{-x/2}\Big|_4^b\\ &= \lim_{b\to\infty} -2e^{-b/2}+2e^{-4/2}\\ &= 0+2e^{-2}\\ &= 2e^{-2}\end{aligned}$$

So this integral $\boxed{\textit{converges} \text{ to } 2e^{-2}}$

Infinite Series

For questions 13 – 16, answer any **two**. You may answer more, but only the scores from the best **two** will be counted.

13. (10 marks) Decide whether the following series converges or diverges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}.$$

1.
- $\frac{n}{5^n} \geq \frac{n+1}{5^{n+1}}$ for all $n \in \mathbb{N}$
- $\lim_{n\to\infty} \frac{n}{5^n} = 0$

$$\Bigg\} \Longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n} \quad \boxed{\textit{converges} \text{ by the Alternating Series Test}}$$

2.
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{5} = \frac{1}{5} < 1$$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$ $\boxed{\textit{converges absolutely} \text{ by the Root Test}}$

3.
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{5} = \frac{1}{5} < 1$$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$ $\boxed{\textit{converges absolutely} \text{ by the Root Test}}$

14. (10 marks) Decide whether the following series converges or diverges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}.$$

1.
$$f(x) = \frac{\ln(x)}{x} \text{ is positive, continuous and decreasing for } x \geq e > 2.$$

$$\int_3^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_3^b \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_{\ln(3)}^{\ln(b)} u\,du = \frac{1}{2} \lim_{b\to\infty} u^2 \Big|_{\ln(3)}^{\ln(b)} = \frac{1}{2}\left(-\ln(3)^2 + \lim_{b\to\infty} \ln(b)^2\right) = \infty$$

Thus, since $\int_3^{\infty} \frac{\ln(x)}{x}\,dx$ diverges, the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ $\boxed{\textit{diverges} \text{ by the Integral Test}}$.

2. $\sum_{n=3}^{\infty} \frac{\ln(n)}{n} \geq \sum_{n=3}^{\infty} \frac{1}{n} \overset{n\to\infty}{\longrightarrow} \infty$ since its a p-series with $p = 1 \Longrightarrow$ $\boxed{\textit{Diverges} \text{ by the Direct Comparison Test}}$

15. (10 marks) Decide whether the following series converges or diverges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}.$$

1. Compare with $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$.

$$0 < \lim_{n\to\infty} \frac{(n+1)/(n^2\sqrt{n})}{1/(n\sqrt{n})} = \lim_{n\to\infty} \frac{n+1}{n} = 1 < \infty.$$

Thus, since $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges (it's a p-series with $p = 1.5 > 1$), the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges by the Limit Comparison Test

2.

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{2n}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} < \infty \quad \text{since it is a } p\text{-series with } p = 1.5$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} \quad \boxed{\textit{converges} \text{ by the Direct Comparison Test}}$$

16. (10 marks) Find **all** real numbers r such that

$$\sum_{n=2}^{\infty} r^n = \frac{1}{12}.$$

$$\sum_{n=2}^{\infty} r^n = r^2 \sum_{n=2}^{\infty} r^{n-2} = r^2 \sum_{n=0}^{\infty} r^n = r^2\left(\frac{1}{1-r}\right) = \frac{r^2}{1-r} = \frac{1}{12}$$

$$\Longrightarrow 12r^2 = 1 - r \Longrightarrow 12r^2 + r - 1 = 0 \Longrightarrow (3x+1)(4x-1) = 0$$

So,

$$\boxed{r = -\frac{1}{3} \text{ or } \frac{1}{4}}$$

Power Series

For questions 17 − 18, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

17. (10 marks) Determine the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x-3)^n .$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{2n+1}(x-3)^n\right|} = |x-3| \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{2n+1}} = |x-3| < 1 \Longrightarrow 2 < x < 4$$

$x = 2:$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(2-3)^n = \sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges by Comparison Test with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$

$x = 4:$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(4-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by Alternating Series Test

Thus the interval of convergence is $\boxed{2 < x \leq 4}$

18. (10 marks) Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}(x-4)^n .$$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{n^3+1}(x-4)^n\right|} = |x-4| \cdot \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{n^3+1}} = |x-4| < 1 \Longrightarrow 3 < x < 5$$

$x = 3:$ $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^3+1}(3-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{n^3+1}$ converges by Alternating Series Test

$x = 5:$ $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^3+1}(5-4)^n = \sum_{n=0}^{\infty} \frac{n}{n^3+1}$ converges by Comparison Test with $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$

Thus the interval of convergence is $\boxed{3 \leq x \leq 5}$

Taylor Series

For questions 19 – 20, answer any **one**. You may answer both, but only the score from the best **one** will be counted. You may use either substitution and manipulation of known series or the table method.

19. (10 marks) Find the Taylor series for the function $f(x) = \dfrac{2}{x^3}$, centred at $x = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$2x^{-3}$	2
1	$(-3)2x^{-4}$	$(-1)\cdot 3\cdot 2$
2	$(-4)(-3)2x^{-5}$	$(-1)^2\cdot 4\cdot 3\cdot 2$
3	$(-5)(-4)(-3)2x^{-6}$	$(-1)^3\cdot 5\cdot 4\cdot 3\cdot 2$
n	$(-n)\cdots(-3)2x^{-(n+3)}$	$(-1)^n(n+2)!$

$$\frac{2}{x^3} = \sum_{n=0}^{n}\frac{(-1)^n(n+2)!}{n!}(x-1)^n$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)(x-1)^n}$$

Start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiate twice

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

Replace x with $(1-x)$

$$\frac{2}{x^3} = \sum_{n=2}^{\infty} n(n-1)(1-x)^{n-2}$$

$$= \sum_{n=2}^{\infty}(-1)^{n-2}n(n-1)(x-1)^{n-2}$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)(x-1)^n}$$

20. (10 marks) Find the Taylor series for the function $f(x) = x^4 - 2x^3 + 3x^2 + 3$, centred at $x = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 2x^3 + 3x^2 + 3$	5
1	$4x^3 - 6x^2 + 6x$	4
2	$12x^2 - 12x + 6$	6
3	$24x - 12$	12
4	24	24

$$x^4 - 2x^3 + 3x^2 + 3 = \boxed{(x-1)^4 + 2(x-1)^3 + 3(x-1)^2 + 4(x-1) + 5}$$

Parametric Equations

For questions $21 - 24$, answer any **two**. You may answer more, but only the scores from the best **two** will be counted.

21. (10 marks) Compute the tangent line to the curve given by the parametric equations

$$x = \sec(t) - 1, \qquad y = \cos(t), \qquad , t = \frac{\pi}{4}.$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{4}} = \left.\frac{-\sin(t)}{\sec(t)\tan(t)}\right|_{t=\frac{\pi}{4}} = \left.-\cos^2(t)\right|_{t=\frac{\pi}{4}} = -\frac{1}{2}$$

$$y - y_0 = m(x - x_0) \implies \boxed{y - \frac{\sqrt{2}}{2} = -\frac{1}{2}(x-1)}$$

22. (10 marks) Find the length of the curve given by

$$x = \ln(\sec(t) + \tan(t)) - \sin(t), \qquad y = \cos(t),$$

over the interval $0 \le t \le \frac{\pi}{4}$.

$$L = \int_a^b = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\frac{\pi}{4}} \sqrt{(\sec(t) - \cos(t))^2 + (-\sin(t))^2}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(t) - 2\sec(t)\cos(t) + \cos^2(t) + \sin^2(t)}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(t) - 1}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \tan(t)\, dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin(t)}{\cos(t)}\, dt$$

$$= \int_{t=0}^{t=\frac{\pi}{4}} -\frac{1}{u}\, du \qquad (\text{Let } u = \cos(t), \quad du = -\sin(t)\, dt)$$

$$= -\ln|u|\Big|_{t=0}^{t=\frac{\pi}{4}}$$

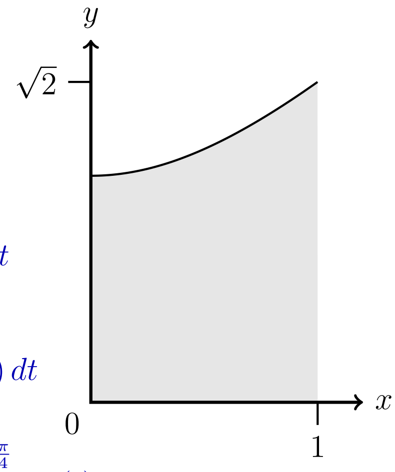
$$= -\ln|\cos(t)|\Big|_{t=0}^{t=\frac{\pi}{4}}$$

$$= -\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1) = -\ln\left(\sqrt{\frac{1}{2}}\right) = \boxed{\frac{\ln(2)}{2}}$$

23. (10 marks) Find the area bounded by the curve

$$x = \tan(t), \qquad y = \sec(t), \qquad 0 \le t \le \frac{\pi}{4},$$

and the x-axis. A sketch of the curve has been provided.



$$\int_0^1 y\,dx = \int_0^{\frac{\pi}{4}} \sec^3(t)\ dt = \sec(t)\tan(t)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec(t)\tan^2(t)\,dt$$

$u = \sec(t)$ $\qquad dv = \sec^2(t)\ dt$

$du = \sec(t)\tan(t)\ dt$ $\qquad v = \tan(t)$

$$= \sec(t)\tan(t)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3(t) - \sec(t)\,dt$$

$$= \sec(t)\tan(t)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3(t)\,dt + \int_0^{\frac{\pi}{4}} \sec(t)\,dt$$

$$= \sec(t)\tan(t) + \ln|\sec(t)+\tan(t)|\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3(t)\,dt + \int_0^{\frac{\pi}{4}} \sec(t)\,dt$$

$$\Longrightarrow 2\int_0^{\frac{\pi}{4}} \sec^3(t)\ dt = \sec(t)\tan(t) + \ln|\sec(t)+\tan(t)|\Big|_0^{\frac{\pi}{4}}$$

$$= \sqrt{2} + \ln\left(\sqrt{2}+1\right) - (0+0)$$

$$\Longrightarrow \int_0^{\frac{\pi}{4}} \sec^3(t)\ dt = \boxed{\frac{1}{2}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right)}$$

Or if you want to be a little fancy:

$$\int_0^1 y\,dx = xy\Big|_0^1 - \int_0^1 x\,dy = xy\Big|_0^1 - \int_0^{\frac{\pi}{4}} x^2 y\,dt = xy\Big|_0^1 - \int_0^{\frac{\pi}{4}} y^3 - y\,dt = xy\Big|_0^1 - \int_0^1 y\,dx + \int_0^{\frac{\pi}{4}} y\,dt = xy + \ln|x+y|\Big|_0^1 - \int_0^1 y\,dx = \sqrt{2} + \ln\left(1+\sqrt{2}\right) - (0+0) - \int_0^1 y\,dx$$

$$\Longrightarrow 2\int_0^1 y\,dx = \sqrt{2} + \ln\left(1+\sqrt{2}\right) \Longrightarrow \int_0^1 y\,dx = \boxed{\frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)}$$

24. (10 marks) Find a Cartesian equation for the path traced by the parametric equations

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

$$\begin{cases} x + y = 2t, \\ x - y = \dfrac{2}{t}, \end{cases} \Longrightarrow \boxed{x^2 - y^2 = 4}$$

Polar Coordinates

For questions $25-26$, answer any **one**. You may answer both, but only the score from the best **one** will be counted.

25. (10 marks) Replace the polar equation with an equivalent Cartesian Equation.

$$r = 1 - 2\sin(\theta).$$

We know that

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r^2 = x^2 + y^2.$$

So,

$$\begin{aligned} r = 1 - 2\sin(\theta) &\implies r^2 = r(1 - 2\sin(\theta)) \\ &\implies r^2 = r - 2r\sin(\theta) \\ &\implies r^2 + 2r\sin(\theta) = r \\ &\implies \left(r^2 + 2r\sin(\theta)\right)^2 = r^2 \\ &\implies \boxed{\left(x^2 + y^2 + 2y\right)^2 = x^2 + y^2} \end{aligned}$$

26. (10 marks) Replace the polar equation with an equivalent Cartesian Equation.

$$r^2\sin(2\theta) = 2.$$

We know that

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r^2 = x^2 + y^2, \qquad \sin(2\theta) = 2\cos(\theta)\sin(\theta).$$

So,

$$\begin{aligned} r^2\sin(2\theta) = 2 &\implies 2r^2\cos(\theta)\sin(\theta) = 2 \\ &\implies r\cos(\theta)\cdot r\sin(\theta) = 1 \\ &\implies \boxed{xy = 1} \end{aligned}$$