Math 142
Joseph C Foster
Summer 2019
Exam 3

Name: Solutions
June 18th, 2019
Time Limit: 100 minutes

This exam contains 8 pages (including this cover page) and 7 questions.
The total number of marks is 100. You have 100 minutes to complete the exam.
Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.
No calculators are allowed. Turn your phones off and place them on your desk face down. Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero.

Question	Marks	Score	Question	Marks	Score
1	40		5	10	
2	10		6	10	
3	10		7	10	
4	10		Total	100	

<table>
<tr><td>Recitation</td><td colspan="9"></td></tr>
<tr><td rowspan="4">Quizzes</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td colspan="2" rowspan="2">Average</td><td rowspan="2"></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">Midterms</td><td colspan="3">Exam 1</td><td colspan="3">Exam 2</td><td colspan="3">Exam 3</td></tr>
<tr><td colspan="3"></td><td colspan="3"></td><td colspan="3"></td></tr>
<tr><td>Final Exam</td><td colspan="9"></td></tr>
<tr><td>Total</td><td colspan="9"></td></tr>
</table>

1. Fill in the blanks to complete each statement/definition.

(a) (4 marks) **Taylor Series**: Let $f(x)$ be a function with derivatives of all orders throughout some interval containing a. Then the Taylor Series generated by $f(x)$ centred at $x = a$ is given by

$$\underline{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}.$$

(b) (16 marks) **Common Taylor Series**: Some common Taylor Series are

$f(x)$	Taylor Series	Interval of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\lvert x\rvert < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\lvert x\rvert < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\lvert x\rvert < \infty$

(c) (6 marks) **Polar Coordinates**: When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by

$$x = \underline{r\cos(\theta)}, \qquad y = \underline{r\sin(\theta)}, \qquad r^2 = \underline{x^2 + y^2}.$$

(d) (2 marks) **Parametric Equations**: Consider the curve C parametrised by

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

The derivative $\frac{dy}{dx}$ with respect to t is given by $\frac{dy}{dx} = \underline{\frac{dy/dt}{dx/dt}}$.

For parts (e)-(g), choose the best answer. Each part is worth 4 marks. There is only **one correct answer**, but you may choose up to **two answers**. If you choose two and one is the correct answer, you will receive 2 marks.

(e) (4 marks) If a curve C is given by the parametric equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b,$$

where $f(t)$ and $g(t)$ are differentiable functions, then its length is given by

☐ $\displaystyle\int_a^b \sqrt{x^2 + y^2}\, dt.$

☐ $\displaystyle\int_a^b \sqrt{\frac{dx}{dt} + \frac{dy}{dt}}\, dt.$

√ $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

☐ $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right)}\, dt.$

(f) (4 marks) If a curve C is given by the parametric equations

$$x = 2\tan(t), \quad y = 2\cos^2(t), \quad 0 \leq t \leq 2\pi,$$

then an equivalent Cartesian equation is

☐ is $x = \dfrac{y-2}{8}$

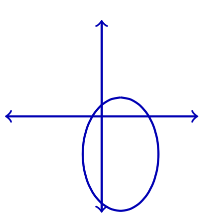
☐ is $x = \dfrac{8}{y-2}$

☐ is $y = \dfrac{x^2+4}{8}$

√ $y = \dfrac{8}{x^2+4}$

(g) (4 marks) Which of the following is a sketch of the curve C given by

$$x = 2\sin(t) + 1, \quad y = 3\cos(t) - 2, \quad 0 \leq t \leq 2\pi,$$

√


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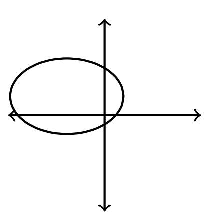

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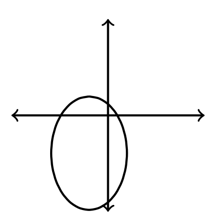

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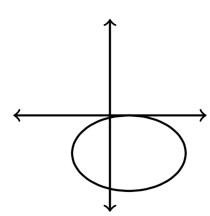


For questions 2-7 show **all** *necessary* work to receive full credit. An answer with no work, even if correct, will not receive full marks. Please circle or box your final answer. If you cannot complete a problem but can write down what you want to do, and this is correct, you can still receive partial credit. Don't leave anything blank! The space provided is indicative of the amount of work required.

2. (10 marks) Write out the first five terms of the Taylor series for $f(x) = \sqrt{x}$ centred at $x = 1$. In other words, find the Taylor polynomial of degree 4 for $f(x) = \sqrt{x}$ centred at $x = 1$. Use the table provided to help you.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\left(\frac{1}{2}\right) x^{-1/2}$	$\frac{1}{2}$
2	$\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) x^{-3/2}$	$-\frac{1}{4}$
3	$\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) x^{-5/2}$	$\frac{3}{8}$
4	$\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) x^{-7/2}$	$-\frac{15}{16}$

$$\frac{1}{0!}(x-1)^0 + \frac{1/2}{1!}(x-1)^1 + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 + \frac{-15/16}{4!}(x-1)^4$$

$$\boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4}$$

3. (10 marks) Find the Taylor series for the function $f(x) = \dfrac{2}{x^3}$, centred at $x = 1$. You may use either substitution and manipulation of known series or the table method.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$2x^{-3}$	2
1	$(-3)2x^{-4}$	$(-1) \cdot 3 \cdot 2$
2	$(-4)(-3)2x^{-5}$	$(-1)^2 \cdot 4 \cdot 3 \cdot 2$
3	$(-5)(-4)(-3)2x^{-6}$	$(-1)^3 \cdot 5 \cdot 4 \cdot 3 \cdot 2$
n	$(-n) \cdots (-3)2x^{-(n+3)}$	$(-1)^n (n+2)!$

$$\begin{aligned} \frac{2}{x^3} &= \sum_{n=0}^{n} \frac{(-1)^n (n+2)!}{n!} (x-1)^n \\ &= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)(x-1)^n} \end{aligned}$$

Start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiate twice

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

Replace x with $(1-x)$

$$\begin{aligned} \frac{2}{x^3} &= \sum_{n=2}^{\infty} n(n-1)(1-x)^{n-2} \\ &= \sum_{n=2}^{\infty} (-1)^{n-2} n(n-1)(x-1)^{n-2} \\ &= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+2)(n+1)(x-1)^n} \end{aligned}$$

4. (10 marks) Replace the polar equation with an equivalent Cartesian Equation.

$$r = 1 - \sin(\theta).$$

We know that

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r^2 = x^2 + y^2.$$

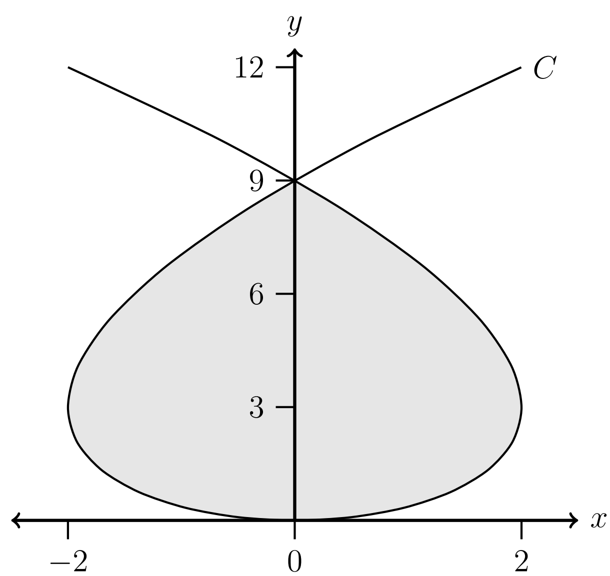
So,

$$\begin{aligned} r = 1 - \sin(\theta) &\Longrightarrow r^2 = r(1 - \sin(\theta)) \\ &\Longrightarrow r^2 = r - r\sin(\theta) \\ &\Longrightarrow r^2 + r\sin(\theta) = r \\ &\Longrightarrow \left(r^2 + r\sin(\theta)\right)^2 = r^2 \\ &\Longrightarrow \boxed{\left(x^2 + y^2 + y\right)^2 = x^2 + y^2} \end{aligned}$$

Questions 5, 6 and 7 will deal with the curve C defined parametrically by

$$x = t^3 - 3t, \quad y = 3t^2.$$

A sketch of the curve is given on the right.



5. (10 marks) Find the equation of the tangent line to C at the point where $t = \sqrt{3}$.

$$m = \left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\sqrt{3}} = \left.\frac{6t}{3t^2 - 3}\right|_{t=\sqrt{3}} = \left.\frac{2t}{t^2 - 1}\right|_{t=\sqrt{3}} = \frac{2\sqrt{3}}{3-1} = \sqrt{3}$$

$$y - y_0 = m(x - x_0) \Longrightarrow y - 9 = \sqrt{3}(x - 0) \Longrightarrow \boxed{y = \sqrt{3}x + 9}$$

6. (10 marks) Find the length of C over the interval $-\sqrt{3} \leq t \leq \sqrt{3}$.

$$\begin{aligned}
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(3t^2-3)^2 + (6t)^2}\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9t^4 - 18t^2 + 9 + 36t^2}\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9t^4 + 18t^2 + 9}\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9\left(t^4 + 2t^2 + 1\right)}\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9\left(t^2 + 1\right)^2}\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} 3t^2 + 3\, dt \\
&= t^3 + 3t \Big|_{-\sqrt{3}}^{\sqrt{3}} \\
&= \left(3\sqrt{3} + 3\sqrt{3}\right) - \left(-3\sqrt{3} - 3\sqrt{3}\right) \\
&= 2\,(3+3)\sqrt{3} \\
&= \boxed{12\sqrt{3}}
\end{aligned}$$

7. (10 marks) Find the area of the shaded region.

$$\begin{aligned}
A = \int_{t=a}^{t=b} y\, dx &= \int_{-\sqrt{3}}^{\sqrt{3}} 3t^2\left(3t^2 - 3\right)\, dt \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} 9t^4 - 9t^2\, dt \\
&= \frac{9}{5}t^5 - 3t^3 \Big|_{-\sqrt{3}}^{\sqrt{3}} \\
&= \left(\frac{81}{5}\sqrt{3} - 9\sqrt{3}\right) - \left(-\frac{81}{5}\sqrt{3} + 9\sqrt{3}\right) \\
&= 2\left(\frac{81}{5} - 9\right)\sqrt{3} \\
&= \boxed{\frac{72}{5}\sqrt{3}}
\end{aligned}$$

Which of the following sections do you want to take Quiz 15 on? Only select one choice. If you select more than one I will choose the first one.

- ☐ Section 8.2 - Integration by Parts
- ☐ Section 8.3 - Trigonometric Integrals
- ☐ Section 8.4 - Trigonometric Substitution
- ☐ Section 8.5 - Integration by Partial Fractions
- ☐ Section 8.7 - Numerical Integration
- ☐ Section 8.8 - Infinite Sequences
- ☐ Section 10.1 - Infinite Sequences
- ☐ Section 10.2 - Infinite Series
- ☐ Section 10.3 - Integral Test
- ☐ Section 10.5 - Ratio/Root Test
- ☐ Section 10.6 - Alternating Series Test
- ☐ Section 10.7 - Power Series
- ☐ Section 10.8 - Taylor Series
- ☐ Section 10.9 - More Taylor Series
- ☐ Section 11.1 - Parametric Equations
- ☐ Section 11.2 - Calculus with Parametric Equations
- ☐ Section 11.3 - Polar Coordinates