Math 142 **Name:** Solutions

Joseph C Foster **June 6th, 2019**

Summer 2019 **Time Limit: 100 minutes**

Exam 2

This exam contains 7 pages (including this cover page) and 7 questions.

The total number of marks is 100. You have 100 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

No calculators are allowed. Turn your phones off and place them on your desk face down. Remove any smartwatches and non-religious head-wear. Cheating of any kind will not be tolerated and will result in a grade of zero.

Question	Marks	Score	Question	Marks	Score
1	40		5	10	
2	10		6	10	
3	10		7	10	
4	10		Total	100	

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1. Fill in the blanks to complete each statement/definition.

(a) (4 marks) n^{th} **Term Test:** For a series $\sum_{n=0}^{\infty} a_n$, if $\lim_{n\to\infty} a_n = 0$ then the series ___may or may not converge___.

If $\lim_{n\to\infty} a_n \neq 0$, then the series ___diverges___.

(b) (6 marks) **Integral Test:** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$ where $f(x)$ is a ___positive___, ___continuous___, ___decreasing___ function of x. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or diverge.

(c) (4 marks) **Direct Comparison Test for Series:** If $0 \leq a_n \leq b_n$ for all $n \geq N$, where N is a positive integer, then,

1. If $\sum_{n=1}^{\infty} a_n$ ___diverges___, then so does $\sum_{n=1}^{\infty} b_n$.
2. If $\sum_{n=1}^{\infty} b_n$ ___converges___, then so does $\sum_{n=1}^{\infty} a_n$.

(d) (2 marks) **Absolute Convergence:** If $\sum_{n=0}^{\infty} |a_n|$ ___converges___, then $\sum_{n=0}^{\infty} a_n$ ___converges___.

(e) (6 marks) **Ratio and Root Tests:** The Ratio and Root Tests state that for any series $\sum a_n$, let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{or} \quad L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then we have the following

- If $L < 1$, then $\sum a_n$ ___converges absolutely___.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ ___diverges___.
- If $L = 1$, then ___the test is inconclusive/there is no conclusion___.

(f) (2 marks) **The Alternating Series Test:** The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \quad (b_n \geq 0)$$

converges if the following two conditions are satisfied:

- $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
- ___$\lim_{n\to\infty} b_n = 0$___.

For parts (g)-(j), choose the best answer. Each part is worth 4 marks. There is only **one correct answer**, but you may choose up to **two answers**. If you choose two and one is the correct answer, you will receive 2 marks.

(g) (4 marks) The sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ is

☐ ∞

☐ $\frac{3}{5}$

√ $\frac{3}{2}$

☐ $\frac{5}{2}$

(h) (4 marks) The limit of the sequence $a_n = \cos(n\pi)$ as n goes to infinity:

☐ is -1

☐ is 1

☐ is 0

√ does not exist

(i) (4 marks) The limit of the sequence $b_n = \dfrac{3n^3 - 2n + 1}{4n^3 + 8n^2 - 3}$ as n goes to infinity:

√ is $\frac{3}{4}$

☐ is $\frac{3}{8}$

☐ is 1

☐ is 0

(j) (4 marks) The limit of the sequence $c_n = \left(1 + \frac{1}{n}\right)^n$ as n goes to infinity:

☐ is 0

√ is e

☐ is 1

☐ is π

For questions 2-7 show **all** *necessary* work to receive full credit. An answer with no work, even if correct, will not receive full marks. Please circle or box your final answer. If you cannot complete a problem but can write down what you want to do, and this is correct, you can still receive partial credit. Don't leave anything blank! The space provided is indicative of the amount of work required.

2. (10 marks) Decide whether the following series diverges or converges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}.$$

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n^2+2n+2} &= \lim_{N\to\infty} \sum_{n=0}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\
&= \lim_{N\to\infty} \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{N+1}} - \frac{1}{N+2}\right) \\
&= \lim_{N\to\infty} 1 - \frac{1}{N+2} = 1
\end{aligned}$$

So, $\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = 1}$

3. (10 marks) Decide whether the following series diverges, converges or absolutely converges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3-3}.$$

$$\frac{n^2}{n^3-3} \geq \frac{n^2}{n^3} = \frac{1}{n}, \qquad \text{so,} \qquad \sum_{n=4}^{\infty} \frac{n^2}{n^3-3} \geq \sum_{n=4}^{\infty} \frac{1}{n}$$

Since $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=4}^{\infty} \frac{n^2}{n^3-3}$ $\boxed{\textit{diverges}}$ by the Direct Comparison Test.

4. (10 marks) Decide whether the following series diverges, converges or absolutely converges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=0}^{\infty} \left(\frac{n^2+2}{2n^2-n+5} \right)^n .$$

Root Test: $$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n^2+2}{2n^2-n+5} \right)^n} = \frac{n^2+2}{2n^2-n+5}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+2}{2n^2-n+5} = \lim_{n\to\infty} \frac{1+\frac{2}{n^2}}{2-\frac{1}{n}+\frac{5}{n^2}} = \frac{1}{2} < 1$$

So $\sum_{n=0}^{\infty} \left(\frac{n^2+2}{2n^2-n+5} \right)^n$ $\boxed{\textit{converges absolutely}}$ by the Root Test.

5. (10 marks) Decide whether the following series diverges, converges or absolutely converges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=0}^{\infty} \frac{2n^2+3^n}{5^n-3}.$$

Compare $\frac{2n^2+3^n}{5^n-3}$ with $\left(\frac{3}{5}\right)^n$ using the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{(2n^2+3^n)/(5^n-3)}{3^n/5^n} = \lim_{n\to\infty} = \frac{2n^2\cdot 5^n + 15^n}{15^n - 3^{n+1}} = \lim_{n\to\infty} \frac{2n^2\left(\frac{1}{3}\right)^n + 1}{1-3\left(\frac{1}{5}\right)^n} = \frac{0+1}{1-0} = 1 < \infty$$

Since $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ converges, $\sum_{n=0}^{\infty} \frac{2n^2+3^n}{5^n-3}$ converges by the Limit Comparison Test.

Since $\sum_{n=1}^{\infty} \left| \frac{2n^2+3^n}{5^n-3} \right| = \sum_{n=1}^{\infty} \frac{2n^2+3^n}{5^n-3}$, we can conclude that $\sum_{n=0}^{\infty} \frac{2n^2+3^n}{5^n-3}$ $\boxed{\textit{converges absolutely}}$

6. (10 marks) Decide whether the following series diverges, converges or absolutely converges. You may use any of the tests we covered in class, however you **must indicate which test you use** and interpret its result correctly.

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}.$$

Alternating Series Test: $b_n = \dfrac{n^2}{n^3+4}$ is a decreasing sequence and $\displaystyle\lim_{n\to\infty}\frac{n^2}{n^3+4} = 0.$

Thus $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ converges.

$\displaystyle\sum_{n=0}^{\infty}\left|(-1)^{n+1}\frac{n^2}{n^3+4}\right| = \sum_{n=0}^{\infty}\frac{n^2}{n^3+4}.$ Compare $\dfrac{n^2}{n^3+4}$ with $\dfrac{1}{n}$ using the Limit Comparison Test.

$$\lim_{n\to\infty}\frac{n^2/(n^3+4)}{1/n} = \lim_{n\to\infty}\frac{n^3}{n^3+4} = 1 < \infty$$

Since $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$ diverges, $\displaystyle\sum_{n=0}^{\infty}\frac{n^2}{n^3+4}$ diverges by the Limit Comparison Test. Thus, $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$ is

$\boxed{\textit{conditionally convergent}}$

7. (10 marks) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3}} (x+3)^n .$$

Remember to test endpoints.

Ratio Test:
$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}(x+3)^{n+1}/\sqrt{(n+1)^3}}{(-1)^n(x+3)^n/\sqrt{n^3}}\right| \\
&= \left|\frac{(-1)^{\cancel{n+1}}(x+3)^{\cancel{n+1}}}{\sqrt{(n+1)^3}} \cdot \frac{\sqrt{n^3}}{\cancel{(-1)^n(x+3)^n}}\right| \\
&= \left|-(x+3)\left(\frac{n}{n+1}\right)^{3/2}\right| \\
&= |x+3|\left(\frac{n}{n+1}\right)^{3/2}
\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x+3|\left(\frac{n}{n+1}\right)^{3/2} = |x+3| \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = |x+3| < 1 \Longrightarrow -4 < x < -2$$

Endpoints:

$$x=-4: \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3}}((-4)+3)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}(-1)^n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

p-series with $p = 3/2 > 1$, thus *converges*

$$x=-2: \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^3}}((-2)+3)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}(1)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$$

converges either by the alternating series test or p-series, with $p = 3/2 > 1$

Thus the interval of convergence is

$$\boxed{-4 \le x \le -2}$$