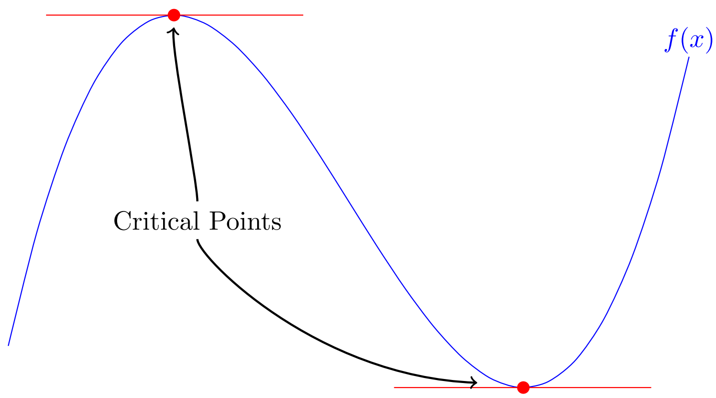
Work through the examples and questions on this worksheet in groups, or on your own. Focus on understanding when and why you look at the derivative of a function for these new concepts.

A **critical point** (or **stationary point**) of $f(x)$ is a point $(a, f(a))$ such that $f'(a) = 0$.

Recall that, geometrically, these are points on the graph of $f(x)$ who have a “flat” tangent line, i.e. a *constant* tangent line.



Example 1:

Find all critical points of $f(x) = x^3 - 3x^2 - 9x + 5$.

We see that the derivative is $f'(x) = 3x^2 - 6x - 9$. We need to solve $f'(x) = 0$.

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3) = 0 \Longrightarrow x = -1, x = 3$$

Thus the critical points of $f(x)$ are $(-1, f(-1)) = (-1, 10)$ and $(3, f(3)) = (3, -22)$.

Example 2:

Find all critical points of $f(t) = e^{-3t} + 2t$.

Differentiating yields $f'(t) = -3e^{-3t} + 2$. Now we solve $f'(t) = 0$.

$$f'(t) = -3e^{-3t} + 2 = 0 \Longrightarrow 2 = 3e^{-3t} \Longrightarrow \frac{2}{3} \Longrightarrow \ln\left(\frac{2}{3}\right) = -3t \Longrightarrow \frac{1}{3}\ln\left(\frac{3}{2}\right) = t$$

Thus the only critical point of $f(t)$ is $\left(\frac{1}{3}\ln(1.5), f\left(\frac{1}{3}\ln(1.5)\right)\right) = \left(\frac{1}{3}\ln(1.5), \frac{2}{3}(1 + \ln(1.5))\right) \approx (0.135, 0.937)$.

Problems

Find all critical points of the given functions.

1. $f(x) = x^3 - 6x + 1$
2. $f(x) = x^3 + 6x + 1$
3. $f(x) = 3x^5 - 5x^3$
4. $f(x) = e^x - 10x$
5. $y = x\ln(x)$, $(x > 0)$
6. $y = xe^{-3x}$
7. $f(x) = x + \frac{1}{x}$
8. $f(x) = 3x^4 - 4x^3 + 6$
9. $y = (x^2 - 4)^7$
10. $y = (x^3 - 8)^7$
11. $f(x) = x^2\ln(x)$
12. $y = (x+1)^5$
13. $y = \frac{x}{x^2+1}$
14. $y = \sqrt{x^2 + 1}$
15. $g(x) = (4x^2 + 1)^7$

You may notice, particularly from the graph on page 1, that the critical points seem to coincide with the peaks of the graph. These is *almost* true. In fact we have the following definition:

Suppose $(a, f(a))$ is a critical point of $f(x)$. Then,

$$(a, f(a)) \text{ is a } \textbf{local minimum} \iff f''(a) > 0$$
$$(a, f(a)) \text{ is a } \textbf{local maximum} \iff f''(a) < 0$$
$$(a, f(a)) \text{ is a } \textbf{point of inflection} \iff f''(a) = 0$$

We can think of this definition as a test to identify the local maximum and local minimum points of a function $f(x)$. If $f''(a) < 0$ or $f''(a) > 0$ then we have a local maximum or minimum, respectively, and if $f''(a) = 0$ then we know nothing. Different cases of $f''(a) = 0$ will be explored later.

Example 1:

Find all local extrema of $f(x) = x^3 - 3x^2 - 9x + 5$.

In Example 1 on the previous page we found $f'(x) = 3x^2 - 6x - 9$ and that the critical points occur at $(-1, 10)$ and $(3, -22)$. To apply the second derivative test we must first compute the second derivative;

$$f''(x) = 6x - 6$$

Then we simply take the points $x = -1$ and $x = 3$ and plug them into $f''(x)$.

$x = -1$:

$$f''(-1) = 6(-1) - 6 = -6 - 6 = -12 < 0$$
$$\Longrightarrow \text{Local Maximum}$$

$x = 3$:

$$f''(3) = 6(3) - 6 = 18 - 6 = 12 > 0$$
$$\Longrightarrow \text{Local Minimum}$$

So the function $f(x)$ has a local maximum at the point $(-1, 10)$ and a local minimum at the point $(3, -22)$.

Example 2:

Find all local extrema of $f(t) = e^{-3t} + 2$.

In Example 2 on the previous page we found $f'(t) = -3e^{-3t} + 2$ and that the only critical point occurs at $t = \frac{1}{3}\ln(1.5)$. To apply the second derivative test we must first compute the second derivative;

$$f''(t) = 9e^{-3t}$$

Now we simply plug in $t = \frac{1}{3}\ln(1.5)$ into $f''(t)$.

$$f''\left(\tfrac{1}{3}\ln(1.5)\right) = 9e^{-3(\ln(1.5)/3)} = 9e^{-\ln(1.5)} = 9e^{\ln(2/3)} = 9(2/3) = 6 > 0 \Longrightarrow \text{Local Minimum.}$$

So the function $f(t)$ has a local minimum at the point $t = \frac{1}{3}\ln(1.5)$.

Example 3:

Find all local extrema of $y = x + \dfrac{1}{x}$.

First we calculate the critical point(s) of y. Differentiating gives $y' = 1 - x^{-2}$. Solving $y' = 0$ gives

$$y' = 1 - x^{-2} = 0 \Longrightarrow 1 = x^{-2} \Longrightarrow 1 = \frac{1}{x^2} \Longrightarrow 1 = x^2 \Longrightarrow x = -1 \text{ or } x = 1$$

Next we find the second derivative,

$$y'' = 2x^{-3} = \frac{2}{x^3}$$

Finally we plug both $x = -1$ and $x = 1$ into y''.

Cont.

$x = -1$:

$$y''(-1) = \frac{2}{(-1)^3} = \frac{2}{-1} = -2 < 0 \Longrightarrow \text{Local Maximum}$$

$x = 1$:

$$y''(1) = \frac{2}{1^3} = \frac{2}{1} = 2 > 0 \Longrightarrow \text{Local Minimum.}$$

Thus y has a local maximum at the point $(-1, y(-1)) = (-1, -2)$ and a local minimum at the point $(1, y(1)) = (1, 2)$.

Problems

Go back and classify the critical points in questions $1 - 15$ on page 1 as maximum, minimum or neither (points of inflection).

Answers

Critical Points

1. $f'(x) = 3x^2 - 6$
   C. Pts:
   $(\sqrt{2}, 1 - 4\sqrt{2}) \approx (1.414, -4.657)$
   $(-\sqrt{2}, 1 + 4\sqrt{2}) \approx (-1.414, 6.657)$

2. $f'(x) = 3x^2 + 6$
   C. Pts:
   None.

3. $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$
   C. Pts:
   $(0, 0)$
   $(1, -2)$
   $(-1, 2)$

4. $f'(x) = e^x - 10$
   C. Pts:
   $(\ln(10), 10(1 - \ln(10))) \approx (2.302, -13.026)$

5. $y' = \ln(x) + 1$
   C. Pts:
   $\left(\frac{1}{e}, -\frac{1}{e}\right) \approx (0.368, -0.368)$

6. $y' = e^{-3x} - 3xe^{-3x}$
   C. Pts:
   $\left(\frac{1}{3}, \frac{1}{3e}\right)$

7. $f'(x) = 1 - x^{-2}$
   C. Pts:
   $(-1, -2)$
   $(1, 2)$

8. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$
   C. Pts:
   $(0, 6)$
   $(1, 5)$

9. $y' = 14x\left(x^2 - 4\right)^6$
   C. Pts:
   $(0, -16384)$
   $(2, 0)$
   $(-2, 0)$

10. $y' = 21x^2(x^3 - 8)^6$
    C. Pt.s
    $(0, -2097152)$
    $(2, 0)$

11. $f'(x) = 2x\ln(x) + x$
    C. Pts:
    $\left(\frac{1}{\sqrt{e}}, 1\frac{1}{2e}\right)$

12. $y' = 5(x + 1)^4$
    C.Pts:
    $(-1, 0)$

13. $y' = (x^2 + 1)^{-1} - 2x^2(x^2 + 1)^{-2}$
    C. Pts:
    $\left(-1, -\frac{1}{2}\right)$
    $\left(1, \frac{1}{2}\right)$

14. $y' = x\sqrt{x^2 + 1}^{-1/2}$
    C. Pts:
    $(0, 1)$

15. $g'(x) = 56x(4x^2 + 1)^6$
    C. Pts:
    $(0, 1)$

Local Extrema

1. $f''(x) = 6x$
   $(\sqrt{2}, 1 - 4\sqrt{2})$ Min.
   $(-\sqrt{2}, 1 + 4\sqrt{2})$ Max.

2. $f''(x) = 6x$
   None.

3. $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$
   $(0, 0)$ Neither
   $(1, -2)$ Min.
   $(-1, 2)$ Max.

4. $f''(x) = e^x$
   $(\ln(10), 10(1 - \ln(10)))$ Min.

5. $y'' = x^{-1}$
   $\left(\frac{1}{e}, -\frac{1}{e}\right)$ Min.

6. $y'' = -6e^{-3x} + 9xe^{-3x}$
   $\left(\frac{1}{3}, \frac{1}{3e}\right)$ Max.

7. $f''(x) = 2x^{-3}$
   $(-1, -2)$ Min.
   $(1, 2)$ Max.

8. $f''(x) = 36x^2 - 24x = 12x(3x - 2)$
   $(0, 6)$ Neither
   $(1, 5)$ Min.

9. $y'' = 14\left(x^2 - 4\right)^6 + 168x^2(x^2 - 4)^5$
   $(0, -16384)$ Min.
   $(2, 0)$ Neither
   $(-2, 0)$ Neither

10. $y'' = 42x(x^3 - 8)^6 + 378x^4(x^3 - 8)^5$
    $(0, -2097152)$ Neither
    $(2, 0)$ Neither

11. $f''(x) = 2\ln(x) + 3$
    $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$ Min.

12. $y'' = 20(x + 1)^3$
    $(-1, 0)$ Neither

13. $y'' = -2x(x^2 + 1)^{-2} - 4x(x^2 + 1)^{-2} + 8x^3(x^2 + 1)^{-3}$
    $\left(-1, -\frac{1}{2}\right)$ Min.
    $\left(1, \frac{1}{2}\right)$ Max.

14. $y'' = \sqrt{x^2 + 1}^{-1/2} - x^2\sqrt{x^2 + 1}^{-1/2}$
    $(0, 1)$ Min.

15. $g''(x) = 56(4x^2 + 1)^6 + 3088x^2(4x^2 + 1)^5$
    $(0, 1)$ Min.

The End.