# Review Sheet

A list of stuff that you should know going into the Calc I final

- $\sin^2(\theta) + \cos^2(\theta) = 1$. Note: this only holds when those angles are the same! Divide both sides of this identity by either $\sin^2(\theta)$ or $\cos^2(\theta)$ to get two "other" identities
- $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$
- $\log_b(x^a) = a\log_b(x)$
- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$ Note: this follows as a combination of the previous two log rules.
- Average rate of change of $f$ on $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$
- A left-hand limit is $\lim_{x\to c^-} f(x)$ approaches from the left. A right-hand limit is $\lim_{x\to c^+} f(x)$ approaches from the right. A function $f$ is continuous at $c$ if both the left-hand limit and the right-hand limit $= f(c)$.
- If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then:
  - $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
  - $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
  - $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$
  - If $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
  - If $\frac{p}{q}$ is rational, then $\lim_{x\to c}(f(x))^{p/q} = (\lim_{x\to c} f(x))^{p/q}$
- If $f(x)$ is continuous at $c$, then $\lim_{x\to c} f(x) = f(c)$
- $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$
- $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$
- If $f(x)$ and $g(x)$ are continuous at $c$, then so are: $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$, $kf(x)$, $(f \circ g)(x)$ (which is $f(g(x))$), $f(x)/g(x)$ (provided that g(c) is not 0), and inverse functions $f^{-1}(x)$. (Note that $f^{-1}(x)$ is not necessarily $1/f(x)$).
- Functions which are continuous on their domain:
  - polynomials and rational functions
  - exponential functions
  - logarithms
  - $\sin(x)$ and $\cos(x)$
- Strategies for evaluating limits algebraically:
  - factoring numerator and denominator, canceling common factors (ex. $\frac{x^2-4x-3}{x+1} = \frac{(x-3)(x+1)}{x+1}$)
  - multiply by conjugate (ex. $\frac{\sqrt{x}-3}{x-9} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \frac{x-9}{(x-9)(\sqrt{x}+3)}$)

  - combining terms (ex. $\frac{1}{x-2} - \frac{4}{x^2-4} = \frac{x+2}{x^2-4} - \frac{4}{x^2-4} = \frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)}$)

- **The Squeeze Theorem** If $f(x) \leq g(x) \leq h(x)$ on some interval containing $c$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$

- If $n > 0$, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to\infty} x^{-n} = 0$

- If $n > 0$, $\lim_{x\to-\infty} x^n = \infty$ for even $n$, $\lim_{x\to-\infty} x^n = -\infty$ for odd $n$, and $\lim_{x\to-\infty} x^{-n} = 0$

- If $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$, then $\lim_{x\to\infty} f(x) = \frac{a_n}{b_m} \lim_{x\to\infty} x^{n-m}$

- **The Intermediate Value Theorem** If $f(x)$ is continuous on $[a, b]$ with $f(a) \neq f(b)$, then for every $M$ between $f(a)$ and $f(b)$, there is at least one $c$ with $a < c < b$ such that $f(c) = M$

- **Limit definition of the derivative** The derivative of $f(x)$ is the limit of the difference quotient (if it exists)
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$
Note the striking resemblance to the average rate of change.

- If $f$ is differentiable at $a$ and the equation of the line tangent to $f$ passing though $(a, f(a))$ is given by $y - f(a) = f'(a)(x - a)$.

- **The Power Rule** $\frac{d}{dx}x^n = nx^{n-1}$ whenever $n \neq 0$

- If $f$ and $g$ are differentiable, then $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$, $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$, and $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$.

- $\frac{d}{dx}e^x = e^x$

- $\frac{d}{dx}b^x = \ln(b)b^x$

- $\frac{d}{dx}\sin(x) = \cos(x)$

- $\frac{d}{dx}\cos(x) = -\sin(x)$ REMEMBER THE NEGATIVE!

- $\frac{d}{dx}\tan(x) = \sec^2(x)$

- $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

- $\frac{d}{dx}\cot(x) = -\csc^2(x)$

- $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$

- If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

- **The Product Rule** $\frac{d}{dx}f(x)g(x) = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$ when $f$ and $g$ are differentiable functions.

- **The Quotient Rule** If $f$ and $g$ are differentiable functions, then for all $x$ such that $g(x) \neq 0$, the derivative of the quotient is given by $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

- **The Chain Rule** If the functions $f$, $g$, and $f \circ g$ are differentiable, then $(f(g(x)))' = f'(g(x))g'(x)$.

- If $f(x)$ and $g(x)$ are inverse functions (that is, $f(g(x)) = x$), then $g'(x) = \frac{1}{f'(g(x))}$ for any $x$ such that $f'(g(x)) \neq 0$

- **Implicit differentiation** Use this when you cannot solve explicitly for $y$. That is, when you can't manipulate your equation to be $y =$ (*stuff with no y's in it*).

  In this case, take the derivative (with respect to x) of both sides of the equation, remembering that $y$ is actually a function of $x$, so you must use the chain rule. Then isolate all of the $\frac{dy}{dx}$'s on one side.

  [ex: $(y = \ln(y) + x) \Rightarrow (\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} + 1) \Rightarrow (\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 1) \Rightarrow (\frac{dy}{dx}(1 - \frac{1}{y}) = 1) \Rightarrow (\frac{dy}{dx} = \frac{1}{1-(1/y)})$]

- **Related Rates** When you want to figure out how much one thing is changing when you know how much a related thing is changing. (See examples in sec 3.11 of the text).

  Draw a picture and label all appropriate parts. Then find an equation that relates some of the things that you have information about. Do not plug anything in! Leave it general. Then differentiate both sides of this equation, taking not of which things are change and using the chain rule. Now plug in information for the specific instant in which you are interested.

- **Linear Approximation** Used to approximate the value of a function at some weird point when you know the value of the function at a nearby point. (Tangent lines lie really close to a curve when you don't stray too far from the point to tangency. So this is just a fancy version of the equation of the tangent line).

  $L(x) = f'(a)(x - a) + f(a)$ where $a$ is the point whose function value you know and $x$ is the point whose function value you want to approximate.

- Percent error $= |\frac{\text{approximated value} - \text{actual value}}{\text{actual value}}| * 100\%$

- **Extrema/Optimization** Finding maxima and minima of some differentiable function $f(x)$

  Any point where $f'(x) = 0$ or where $f'(x)$ is not defined is a *critical point*. These, along with the endpoints of the interval (DON'T FORGET THE ENDPOINTS) are possible extrema. You can find out which are actually extreme values by applying the first or second derivative tests.

  To find absolute maxima, plug all your critical points (and endpoints!) which correspond to maxima into your original function. Choose the point with the highest output. Similarly for minima, but choose the smallest output.

  If there was some type of constraint, use that to express the quantity you want to maximize/minimize a function of ONE variable.

- **First Derivative Test** If $c$ is a critical point of $f(x)$, then $f(c)$ is a local maximum if $f'(x)$ changes from positive to negative at $c$. Similarly, $f(c)$ is a local minimum if $f'(x)$ changes from negative to positive at $c$.

- **Second Derivative Test** If $c$ is a critical point of $f(x)$, then $f(c)$ is a local maximum if $f''(c) < 0$ and $f(c)$ is a local minimum if $f''(c) > 0$

- A *point of inflection* is a point $c$ at which $f''$ changes sign (goes from $+$ to $-$ or from $-$ to $+$). Also, $f''(c) = 0$

- If $f'(c)$ is positive, then $f$ is increasing at $c$. If $f'(c)$ is negative, then $f$ is decreasing at $c$.
- If $f''(c)$ is positive, then $f$ is concave up at $c$. If $f''(c)$ is negative, then $f$ is concave down at $c$.

  Remember that concave up looks like a cup and concave down looks like a frown!
- **Mean Value Theorem** If $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is some $c$ with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ (That is, there's some point in the interval whose derivative is the average rate of change)
- **Curve sketching**
  - Find the domain of your function (Ignore all $x$-values that would give a denominator of 0, a logarithm of a negative number, or a square root of a negative number).
  - Find $f'(x)$ and $f''(x)$.
  - Find all places where $f'(x) = 0$ or is undefined. This divides the number line up into a bunch of sections. Figure out which sections have $f'$ positive and which have $f'$ negative.
  - Find all places where $f''(x) = 0$. This divides the number line up into a bunch of sections. Figure out which sections have $f''$ positive and which have $f''$ negative.
  - Plug critical points and inflection points into $f(x)$.
  - Draw any horizontal asymptotes (ie, take $\lim_{x\to\pm\infty} f(x)$).
  - Finally, connect the dots with the appropriate shape (ex. increasing and concave up)
- **Newton's Method** Used to approximate a root of $f(x)$.

  Pick an initial guess $x_0$. Then use your previous guess to get a point even closer to a root: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. (Note the negative sign!)
- If $F'(x) = f(x)$, then $\int f(x)\mathrm{d}x = F(x) + C$. Notice that since there are not bounds (ie, this is an *indefinite integral*), we need a $+C$. If you forget the $+C$, you will suffer...
- $\int (f(x) + g(x))\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$ and $\int kf(x)\mathrm{d}x = k\int f(x)\mathrm{d}x$
- **Power Rule (integrals)** $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$.
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$
- $\int e^x \mathrm{d}x = e^x + C$
- Displacement (which is net change in position) is $\int v(t)\mathrm{d}t$
- Distance traveled is $\int |v(t)|\mathrm{d}t$
- **Approximating integrals with rectangles**
  - Right endpoint: $R_N = \Delta x \sum_{j=1}^{N} f(a + j\Delta x)$
  - Left endpoint: $L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)$
  - Midpoint: $M_N = \Delta x \sum_{j=1}^{N} f(a + (j - 0.5)\Delta x)$
- **The Fundamental Theorem of Calculus** If $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$ on [a,b], then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.

- If $f(x)$ is continuous on an open interval which includes $a$, and $A(x) = \int_a^b f(t)\mathrm{d}t$, then $\frac{d}{dx}A(x) = f(x)$

  That is, $\frac{d}{dx}\int_a^x f(t)\mathrm{d}t = f(x)$

- **u-substitution** Used when you see some function $u$ shoved inside some other function and multiplied by $u'$. If $F'(x) = f(x)$, then:

  $\int f(u(x))\mathrm{d}x = F(u(x)) + C$ and $\int_a^b f(u(x))\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$

  When making the substitution, use that $du = u'(x)dx$.

- The area between two functions $= \int_a^b$ (top function) - (bottom function) d$x$

  The area between two functions $= \int_c^d$ (right function) - (left function) d$y$ (remember to solve for $x$ to make these functions of $y$)

- If $A(x)$ gives the crossectional area of a solid body on $[a, b]$, then the volume of the solid is given by $\int_a^b A(x)\mathrm{d}x$

- The average value of a function $f(x)$ on $[a, b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$.

- **Mean Value Theorem (integrals)** If $f(x)$ is continuous on $[a, b]$, then there exist some $c$ with $a < c < b$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$

- **Disk method** Used when spinning one function around a horizontal axis

  $Volume = \int_a^b \pi(R)^2\mathrm{d}x = \int_a^b \pi(f(x) - (\text{y-value of axis you're rotating about}))^2\mathrm{d}x$

  When spinning around the $x$ axis in particular, $Volume = \int_a^b \pi(f(x))^2\mathrm{d}x$

- **Washer method** Used when spinning the area between two curves around a horizontal axis

  $Vol = \int_a^b \pi(R_{top}^2 - R_{bottom}^2)\mathrm{d}x = \int_a^b \pi(f(x) - g(x) - (\text{y-value of axis you're rotating about}))^2\mathrm{d}x$

- **Cylinder method** Used when spinning some function around a vertical line

  $Volume = \int_a^b 2\pi rh\mathrm{d}x = \int_a^b 2\pi(x - (\text{x-value of axis you're rotating about}) f(x - (\text{x-value of axis you're rotating about}))\mathrm{d}x$

  When spinning about the y-axis in particular: $Volume = \int_a^b 2\pi x f(x)\mathrm{d}x$

  When spinning the area between two functions around the $y$-axis, assuming that $f(x) \geq g(x)$

  $Volume = \int_a^b 2\pi x(f(x) - g(x))\mathrm{d}x$