Asymptotic preserving schemes on kinetic models with singular limits

Changhui Tan

Department of Mathematics
University of South Carolina

DASIV spring school: models and data
March 18, 2019

Outline

1. Introduction
2. Kinetic swarming models and zero-inertia limit
3. Velocity scaling methods
4. Asymptotic-preserving scheme
5. Numerical experiments

Outline

1. Introduction
2. Kinetic swarming models and zero-inertia limit
3. Velocity scaling methods
4. Asymptotic-preserving scheme
5. Numerical experiments

Asymptotic-preserving schemes

$$f_\epsilon \xrightarrow{\epsilon \to 0} f$$

- Consider f_ϵ the solution of an equation with parameter ϵ, and f is the solution of the limiting equation as $\epsilon \to 0$.

Example: Kinetic equations and hydrodynamic limits

Boltzmann equation: $\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon}\mathcal{B}[f_\epsilon, f_\epsilon]$.

$\epsilon \to 0 \quad \downarrow$

Euler limit: $f = \mathcal{M}(\rho, u, \theta)$

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta\mathbb{I}) = 0, \\ \partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta\right) + \nabla_x \cdot \left(\frac{1}{2}\rho|u|^2 u + \frac{D+2}{2}\rho\theta u\right) = 0. \end{cases}$$

Asymptotic-preserving schemes

$$f_\epsilon \xrightarrow{\epsilon \to 0} f$$

- Consider f_ϵ the solution of an equation with parameter ϵ, and f is the solution of the limiting equation as $\epsilon \to 0$.

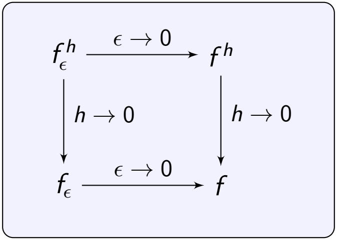
  Example: Kinetic equations and hydrodynamic limits

  Boltzmann equation: $\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \dfrac{1}{\epsilon}\mathcal{B}[f_\epsilon, f_\epsilon]$.

  $\epsilon \to 0 \quad \downarrow$

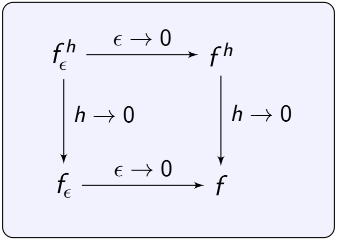
  Euler limit:
  $f = \mathcal{M}(\rho, u, \theta)$
  $$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + \rho\theta\mathbb{I}) = 0, \\ \partial_t \left(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta\right) + \nabla_x \cdot \left(\frac{1}{2}\rho|u|^2 u + \frac{D+2}{2}\rho\theta u\right) = 0. \end{cases}$$

Asymptotic-preserving schemes



- Consider f_ϵ the solution of an equation with parameter ϵ, and f is the solution of the limiting equation as $\epsilon \to 0$.
- f_ϵ^h and f^h are approximations (discretizations) of f^ϵ and f.
- *Asymptotic-preserving property*: h does not depend on ϵ. [Jin '99, ...]

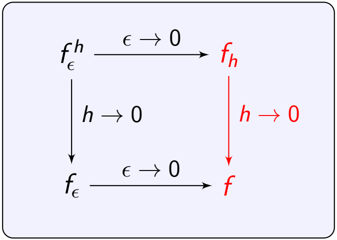
Asymptotic-preserving schemes



- Consider f_ϵ the solution of an equation with parameter ϵ, and f is the solution of the limiting equation as $\epsilon \to 0$.
- f_ϵ^h and f^h are approximations (discretizations) of f^ϵ and f.
- *Asymptotic-preserving property*: h does not depend on ϵ. [Jin '99, ...]

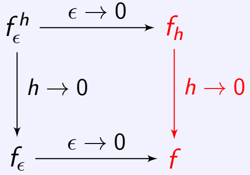
UNIVERSITY OF SOUTH CAROLINA

When the limit is singular



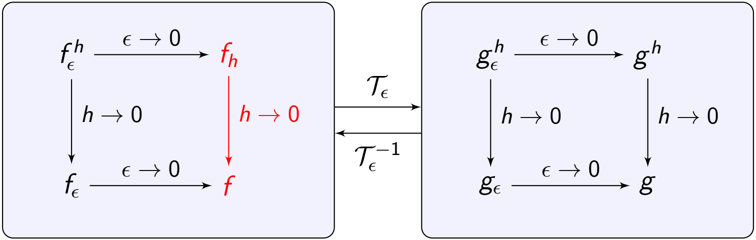
- Consider the case when f is singular, e.g. $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$.
- The discretization f^h can not high accuracy. Therefore, f_ϵ^h is also not accurate when ϵ is small.
- **Idea:** Construct a family of invertible maps $\mathcal{T}_\epsilon$, so that $g_\epsilon = \mathcal{T}_\epsilon f_\epsilon$ converges to a non-singular profile.
- **Main Difficulty:** Find $\mathcal{T}_\epsilon$ that captures the singularity.

When the limit is singular



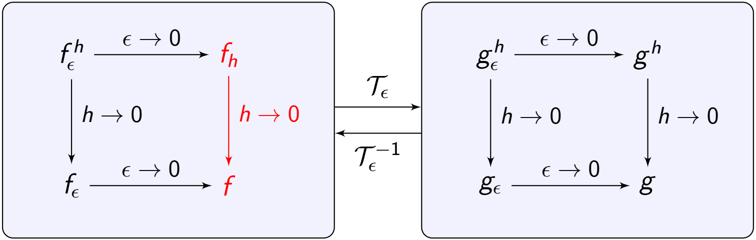
- Consider the case when f is singular, e.g. $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$.
- The discretization f^h can not high accuracy. Therefore, f_ϵ^h is also not accurate when ϵ is small.
- **Idea:** Construct a family of invertible maps $\mathcal{T}_\epsilon$, so that $g_\epsilon = \mathcal{T}_\epsilon f_\epsilon$ converges to a non-singular profile.
- **Main Difficulty:** Find $\mathcal{T}_\epsilon$ that captures the singularity.

When the limit is singular



- Consider the case when f is singular, e.g. $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$.
- The discretization f^h can not high accuracy. Therefore, f_ϵ^h is also not accurate when ϵ is small.
- **Idea:** Construct a family of invertible maps $\mathcal{T}_\epsilon$, so that $g_\epsilon = \mathcal{T}_\epsilon f_\epsilon$ converges to a non-singular profile.
- **Main Difficulty:** Find $\mathcal{T}_\epsilon$ that captures the singularity.

When the limit is singular



- Consider the case when f is singular, e.g. $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$.
- The discretization f^h can not high accuracy. Therefore, f_ϵ^h is also not accurate when ϵ is small.
- **Idea:** Construct a family of invertible maps $\mathcal{T}_\epsilon$, so that $g_\epsilon = \mathcal{T}_\epsilon f_\epsilon$ converges to a non-singular profile.
- **Main Difficulty:** Find $\mathcal{T}_\epsilon$ that captures the singularity.

Outline

1 Introduction

2 Kinetic swarming models and zero-inertia limit

3 Velocity scaling methods

4 Asymptotic-preserving scheme

5 Numerical experiments

Swarming

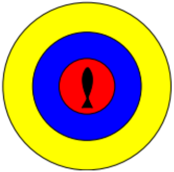
Three-zone models for swarms: [Reynolds '87]

- Long range: Attraction
- Short range: Repulsion
- Middle range: Alignment

Swarming

Three-zone models for swarms: [Reynolds '87]

- Long range: Attraction
- Short range: Repulsion
- Middle range: Alignment



Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t))$.

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i)$.

[**Cucker-Smale '07**, Motsch-Tadmor '11, Shvydkoy-Tadmor '18, Vicsek '95, Krause '97, Kuramoto ...]

Flocking [Ha-Liu '09]

Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t)).$

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i).$

[**Cucker-Smale '07**, Motsch-Tadmor '11, Shvydkoy-Tadmor '18, Vicsek '95, Krause '97, Kuramoto ...]

Flocking [Ha-Liu '09]

Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t))$.

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^{N} \phi(|x_j - x_i|)(v_j - v_i)$.

[**Cucker-Smale '07**, Motsch-Tadmor '11, Shvydkoy-Tadmor '18, Vicsek '95, Krause '97, Kuramoto ...]

Flocking [Ha-Liu '09]

Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t)).$

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^{N} \phi(|x_j - x_i|)(v_j - v_i).$

[**Cucker-Smale '07**, Motsch-Tadmor '11, Shvydkoy-Tadmor '18, Vicsek '95, Krause '97, Kuramoto ...]

Flocking [Ha-Liu '09]

Agent-based models on swarming

- Agent-based interaction dynamics (based on Newton's second law)

$$\dot{x}_i = v_i, \quad m\dot{v}_i = F_i, \quad i = 1, \cdots, N.$$

The interaction force F_i depends on $\{x_j\}_{j=1}^N$ and $\{v_j\}_{j=1}^N$.

- Attractive/Repulsive force: $F_i(t) = -\frac{1}{N}\sum_{j\neq i} \nabla K(x_j(t) - x_i(t))$.

- Alignment force: $F_i = \frac{1}{N}\sum_{j=1}^N \phi(|x_j - x_i|)(v_j - v_i)$.

[**Cucker-Smale '07**, Motsch-Tadmor '11, Shvydkoy-Tadmor '18, Vicsek '95, Krause '97, Kuramoto ...]

Flocking [Ha-Liu '09]

Kinetic swarming models

- Vlasov-type kinetic equations

$$\boxed{\partial_t f + v \cdot \nabla_x f + \frac{1}{m} \nabla_v \cdot (F(f)f) = 0,}$$

where $f = f(t, x, v)$ is a probability measure in (x, v) space.

- Nonlocal interaction forces:

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v) f(t,y,v_*) dv_* dy$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*) dv_* dy.$$

UNIVERSITY OF SOUTH CAROLINA

Kinetic swarming models

- Vlasov-type kinetic equations

$$\boxed{\partial_t f + v \cdot \nabla_x f + \frac{1}{m} \nabla_v \cdot (F(f)f) = 0,}$$

where $f = f(t, x, v)$ is a probability measure in (x, v) space.

- Nonlocal interaction forces:

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v) f(t,y,v_*) dv_* dy$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*) dv_* dy.$$

UNIVERSITY OF SOUTH CAROLINA

Kinetic flocking models

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{m} \nabla_v \cdot \Big(F^{CS}(f) f \Big) = 0.$$

- Derivation and wellposedness. [Ha-Tadmor '08]
- Flocking: [Carrillo-Fornasier-Rosado-Toscani '10]

$$S(t) := \sup_{(x,v),(y,v^*)\in \mathrm{supp} f(t)} |x-y| \le D < \infty,$$

$$V(t) := \sup_{(x,v),(y,v^*)\in \mathrm{supp} f(t)} |v-v^*| \stackrel{t\to\infty}{\longrightarrow} 0.$$

- Velocity concentration: $\lim_{t\to\infty} f(t,x,v) = \rho_\infty(x)\delta_{v=\bar{v}}$.
- Extensions:
  - Motsch-Tadmor alignment force. [T. '17]
  - Singular influence ϕ: [Mucha-Peszek '17] ... ...

UNIVERSITY OF SOUTH CAROLINA

Kinetic flocking models

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{m}\nabla_v \cdot \left(F^{CS}(f)f\right) = 0.$$

- Derivation and wellposedness. [Ha-Tadmor '08]
- Flocking: [Carrillo-Fornasier-Rosado-Toscani '10]

$$S(t) := \sup_{(x,v),(y,v^*)\in \mathrm{supp} f(t)} |x-y| \le D < \infty,$$

$$V(t) := \sup_{(x,v),(y,v^*)\in \mathrm{supp} f(t)} |v-v^*| \overset{t\to\infty}{\longrightarrow} 0.$$

- Velocity concentration: $\lim_{t\to\infty} f(t,x,v) = \rho_\infty(x)\delta_{v=\bar{v}}$.
- Extensions:

- Motsch-Tadmor alignment force. [T. '17]
- Singular influence ϕ: [Mucha-Peszek '17] ... ...

Vlasov equation with attractive-repulsive potentials

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{m} \nabla_v \cdot \left(F^{AR}(f) f \right) = 0,$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*) dv_* dy.$$

- When $K = \mathcal{N}$ is the Newtonian potential, the system becomes Vlasov-Poisson equations in plasma physics.

  Global wellposedness (3D) [Schaeffer '91]
  Landau damping [Mouhot-Villani '11, Bedrossian-Germain-Masmoudi '17]

- For less singular potential, global wellposedness theory is standard.

  Similar theory can be established for kinetic models with attraction, repulsion and alignment: $F = F^{AR} + F^{CS}$.

Vlasov equation with attractive-repulsive potentials

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{m} \nabla_v \cdot \Big(F^{AR}(f) f \Big) = 0,$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*) dv_* dy.$$

- When $K = \mathcal{N}$ is the Newtonian potential, the system becomes Vlasov-Poisson equations in plasma physics.

  Global wellposedness (3D) [Schaeffer '91]
  Landau damping [Mouhot-Villani '11, Bedrossian-Germain-Masmoudi '17]

- For less singular potential, global wellposedness theory is standard.

  Similar theory can be established for kinetic models with attraction, repulsion and alignment: $F = F^{AR} + F^{CS}$.

UNIVERSITY OF SOUTH CAROLINA

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \cdot P = \rho F. \end{aligned}$$

$$\rho = \int f \; dv, \quad \rho u = \int vf \; dv, \quad P = \int (v-u) \otimes (v-u) f \; dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x) \frac{1}{(2\pi)^{n/2}} e^{-\frac{|v-u(x)|^2}{2}}$.
  2. Mono-kinetic ansatz: $f(x,v) = \rho(x)\delta_{v=u(x)}$.
- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

UNIVERSITY OF SOUTH CAROLINA

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \cdot P = \rho F.$$

$$\rho = \int f \; dv, \quad \rho u = \int v f \; dv, \quad P = \int (v-u) \otimes (v-u) f \; dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x) \frac{1}{(2\pi)^{n/2}} e^{-\frac{|v-u(x)|^2}{2}}$.
  2. Mono-kinetic ansatz: $f(x,v) = \rho(x) \delta_{v=u(x)}$.
- Macroscopic system [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \cdot P = \rho F.$$

$$\rho = \int f \, dv, \quad \rho u = \int v f \, dv, \quad P = \int (v-u) \otimes (v-u) f \, dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x) \frac{1}{(2\pi)^{n/2}} e^{-\frac{|v-u(x)|^2}{2}}$.

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f) f) = \frac{1}{\epsilon} \left[\nabla_v \cdot ((v-u) f) + \Delta_v f \right].$$

  2. Mono-kinetic ansatz: $f(x,v) = \rho(x) \delta_{v=u(x)}$.

- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\begin{aligned}&\partial_t\rho+\nabla\cdot(\rho u)=0,\\&\partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla\cdot P=\rho F.\end{aligned}$$

$$\rho=\int f\ dv,\quad \rho u=\int vf\ dv,\quad P=\int(v-u)\otimes(v-u)f\ dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v)=\rho(x)\frac{1}{(2\pi)^{n/2}}e^{-\frac{|v-u(x)|^2}{2}}$.
     $P=\rho\mathbb{I}.$ [Karper-Mellet-Trivisa '15]
  2. Mono-kinetic ansatz: $f(x,v)=\rho(x)\delta_{v=u(x)}$.
- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\boxed{\begin{aligned}&\partial_t\rho + \nabla\cdot(\rho u) = 0,\\ &\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\cdot P = \rho F.\end{aligned}}$$

$$\rho = \int f\ dv,\quad \rho u = \int vf\ dv,\quad P = \int (v-u)\otimes(v-u)f\ dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x)\frac{1}{(2\pi)^{n/2}}e^{-\frac{|v-u(x)|^2}{2}}$.
     $P = \rho\mathbb{I}$. [Karper-Mellet-Trivisa '15]
  2. Mono-kinetic ansatz: $f(x,v) = \rho(x)\delta_{v=u(x)}$.

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(F(f)f) = \frac{1}{\epsilon}\left[\nabla_v\cdot((v-u)f)\right].$$

- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\begin{aligned}\partial_t\rho + \nabla\cdot(\rho u) &= 0,\\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\cdot P &= \rho F.\end{aligned}$$

$$\rho = \int f\ dv,\quad \rho u = \int vf\ dv,\quad P = \int (v-u)\otimes(v-u)f\ dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x)\frac{1}{(2\pi)^{n/2}}e^{-\frac{|v-u(x)|^2}{2}}$.
     $P = \rho\mathbb{I}$. [Karper-Mellet-Trivisa '15]
  2. Mono-kinetic ansatz: $f(x,v) = \rho(x)\delta_{v=u(x)}$.
     $P = 0$. (Pressureless) [Figalli-Kang '17]
- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

UNIVERSITY OF SOUTH CAROLINA

Hydrodynamic limits

- Macroscopic system by taking moments in v.

$$\partial_t \rho + \nabla\cdot(\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\cdot P = \rho F.$$

$$\rho = \int f\ dv,\quad \rho u = \int vf\ dv,\quad P = \int (v-u)\otimes(v-u) f\ dv.$$

- Rigorous derivation by imposing a closure on the pressure.
  1. Isothermal ansatz: $f(x,v) = \rho(x)\frac{1}{(2\pi)^{n/2}} e^{-\frac{|v-u(x)|^2}{2}}$.
     $P = \rho\mathbb{I}$. [Karper-Mellet-Trivisa '15]
  2. Mono-kinetic ansatz: $f(x,v) = \rho(x)\delta_{v=u(x)}$.
     $P = 0$. (Pressureless) [Figalli-Kang '17]
- Macroscopic system
  [Tadmor-T. '14, Carrillo-Choi-Tadmor-T. '16, Carrillo-Choi-Zatorska '16, Shvydkoy-Tadmor '17, Do-Kiselev-Ryzhik-T. '18, Kiselev-T. '18, Tadmor-He '18, Choi '18, T. '19... ...]

Zero inertia limit

- Consider the limit when total mass $m = \epsilon \to 0$.

$$\boxed{\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,}$$

- Two systems that we concern:
  1. [ARR] Attraction-Repulsion-Relaxation: $F = F^{AR} - v$.
  2. [ARA] Attraction-Repulsion-Alignment(3 zones): $F = F^{AR} + F^{CS}$.

$$F^{CS}(f)(t,x,v) = \iint \phi(|x-y|)(v_* - v) f(t,y,v_*) dv_* dy$$

$$F^{AR}(f)(t,x,v) = \iint -\nabla_x K(x-y) f(t,y,v_*) dv_* dy.$$

Zero inertia limit

- Consider the limit when total mass $m = \epsilon \to 0$.

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- Two systems that we concern:
  1. [ARR] Attraction-Repulsion-Relaxation: $F = F^{AR} - v$.
  2. [ARA] Attraction-Repulsion-Alignment(3 zones): $F = F^{AR} + F^{CS}$.

$$F^{CS}(f)(t, x, v) = \iint \phi(|x - y|)(v_* - v) f(t, y, v_*) dv_* dy$$

$$F^{AR}(f)(t, x, v) = \iint -\nabla_x K(x - y) f(t, y, v_*) dv_* dy.$$

UNIVERSITY OF SOUTH CAROLINA

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit $(f_\epsilon \to f)$:

$$\nabla_v \cdot (F(f) f) = 0$$

$\varphi(v) = 1$: $\quad \partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: $\quad$ [ARR] $\; u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\; \int \phi(|x-y|)(u(x) - u(y))\rho(y) dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: $\quad$ [ARR] $\; \int |v-u|^2 f(x,v) dv = 0,$

[ARA] $\; (\phi * \rho)(x) \int |v-u|^2 f(x,v) dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x)\; \delta_{v=u(t,x)}.$$

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\int \varphi(v) \nabla_v \cdot (F(f)f)\, dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x) - u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: [ARR] $\int |v-u|^2 f(x,v) dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v) dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\int \nabla_v \varphi(v) \cdot F(f) f \, dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x) - u(y))\rho(y) dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: [ARR] $\int |v-u|^2 f(x,v) dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v) dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x) \; \delta_{v=u(t,x)}.$$

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\int \nabla_v \varphi(v) \cdot F(f) f \; dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x) - u(y))\rho(y) dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: [ARR] $\int |v-u|^2 f(x,v) dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v) dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x) \; \delta_{v=u(t,x)}.$$

UNIVERSITY OF SOUTH CAROLINA

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\int \nabla_v \varphi(v) \cdot F(f) f \; dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x - y|)(u(x) - u(y))\rho(y) dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v - u|^2$: [ARR] $\int |v - u|^2 f(x, v) dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v - u|^2 f(x, v) dv = 0.$

$$\Rightarrow \quad f(t, x, v) = \rho(t, x) \; \delta_{v = u(t,x)}.$$

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon} \nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit $(f_\epsilon \to f)$:

$$\int \nabla_v \varphi(v) \cdot F(f) f \; dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x) - u(y))\rho(y) dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: [ARR] $\int |v-u|^2 f(x,v) dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v) dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x) \; \delta_{v=u(t,x)}.$$

A formal derivation

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon)f_\epsilon) = 0,$$

- A formal derivation of the $\epsilon \to 0$ limit ($f_\epsilon \to f$):

$$\int \nabla_v \varphi(v) \cdot F(f)f \; dv = 0.$$

$\varphi(v) = 1$: $\partial_t \rho + \nabla_x \cdot (\rho u) = 0.$

$\varphi(v) = v$: [ARR] $u(x) = -(\nabla_x K * \rho)(x),$

[ARA] $\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$

$\varphi(v) = \frac{1}{2}|v-u|^2$: [ARR] $\int |v-u|^2 f(x,v)dv = 0,$

[ARA] $(\phi * \rho)(x) \int |v-u|^2 f(x,v)dv = 0.$

$$\Rightarrow \quad f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x) - u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

$$\int \phi(|x-y|)(u(x) - u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

UNIVERSITY OF SOUTH CAROLINA

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x)-u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x) - u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

Limiting system

$$f(t,x,v) = \rho(t,x)\ \delta_{v=u(t,x)}.$$

- For [ARR], the limiting system is the *aggregation equation*

$$\partial_t \rho + \nabla_x \cdot ((-\nabla_x K * \rho)\rho) = 0.$$

Wellposedness: [Laurent '07, Bertozzi-Carrillo-Laurent '09, ...]
Rigorous passage to the limit: [Jabin '99, Fetecau-Sun '15]

- For [ARA], the limiting system has an implicitly defined velocity u.

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\int \phi(|x-y|)(u(x) - u(y))\rho(y)dy = -(\nabla_x K * \rho)(x).$$

Wellposedness: [Fetecau-Sun-T. '16]

Additional restriction: $\displaystyle\int \rho(t,x)u(t,x)dx = \int \rho_0(x)u_0(x)dx.$

Rigorous passage to the limit: [Fetecau-Sun-T. '16]

Outline

1 Introduction

2 Kinetic swarming models and zero-inertia limit

3 Velocity scaling methods

4 Asymptotic-preserving scheme

5 Numerical experiments

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v - u_\epsilon(t,x)}{\omega_\epsilon}.$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$\boxed{f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v - u_\epsilon(t,x)}{\omega_\epsilon}.}$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$\boxed{f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v - u_\epsilon(t,x)}{\omega_\epsilon}.}$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$\boxed{f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v - u_\epsilon(t,x)}{\omega_\epsilon}.}$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$\boxed{f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v - u_\epsilon(t,x)}{\omega_\epsilon}.}$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: framework

$$f_\epsilon(t,x,v) \to \rho(t,x)\ \delta_{v=u(t,x)}.$$

- The transformation $\mathcal{T}_\epsilon$: rescale $f_\epsilon \leftrightarrow (g_\epsilon, u_\epsilon, \omega_\epsilon)$:

$$\boxed{f_\epsilon(t,x,v) = \frac{1}{\omega_\epsilon^d} g_\epsilon(t,x,\xi), \quad \xi = \frac{v-u_\epsilon(t,x)}{\omega_\epsilon}.}$$

- u_ϵ is the macroscopic velocity: $u_\epsilon(t,x) = \dfrac{\int v f_\epsilon(t,x,v)dv}{\int f_\epsilon(t,x,v)dv}$.
- ω_ϵ is the *scaling factor*.

Goal: choose ω_ϵ appropriately so that $g_\epsilon \to g$ and g is not singular.

Velocity scaling: history

- Kinetic system with singular equilibrium.

$$f(t,x,v) \to \rho^\infty(x)\delta_{v=v^\infty}, \quad \text{as } t\to\infty.$$

- Rescale $f \leftrightarrow (g,u,\omega)$:

$$f(t,x,v) = \frac{1}{\omega(t,x)^d} g(t,x,\xi), \quad \xi = \frac{v-u(t,x)}{\omega}.$$

- Linear Fokker-Planck [Filbet-Russo '04], Granular gas [Filbet-Rey '13]:

$$\omega = \sqrt{\text{Temperature}}.$$

- Kinetic flocking models [Rey-T. '16]:
  Propose a new way to learn the scaling ω dynamically.
  The learned ω is **exact** for spatially homogenous system.

Velocity scaling: history

- Kinetic system with singular equilibrium.

$$f(t,x,v) \to \rho^\infty(x)\delta_{v=v^\infty}, \quad \text{as } t \to \infty.$$

- Rescale $f \leftrightarrow (g, u, \omega)$:

$$\boxed{f(t,x,v) = \frac{1}{\omega(t,x)^d} g(t,x,\xi), \quad \xi = \frac{v - u(t,x)}{\omega}.}$$

- Linear Fokker-Planck [Filbet-Russo '04], Granular gas [Filbet-Rey '13]:

$$\omega = \sqrt{\text{Temperature}}.$$

- Kinetic flocking models [Rey-T. '16]:
  Propose a new way to learn the scaling ω dynamically.
  The learned ω is **exact** for spatially homogenous system.

Velocity scaling: history

- Kinetic system with singular equilibrium.

$$f(t,x,v) \to \rho^\infty(x)\delta_{v=v^\infty}, \quad \text{as } t \to \infty.$$

- Rescale $f \leftrightarrow (g, u, \omega)$:

$$\boxed{f(t,x,v) = \frac{1}{\omega(t,x)^d} g(t,x,\xi), \quad \xi = \frac{v - u(t,x)}{\omega}.}$$

- Linear Fokker-Planck [Filbet-Russo '04], Granular gas [Filbet-Rey '13]:

$$\omega = \sqrt{\text{Temperature}}.$$

- Kinetic flocking models [Rey-T. '16]:
  Propose a new way to learn the scaling ω dynamically.
  The learned ω is **exact** for spatially homogenous system.

Velocity scaling: history

- Kinetic system with singular equilibrium.

$$f(t,x,v) \to \rho^\infty(x)\delta_{v=v^\infty}, \quad \text{as } t \to \infty.$$

- Rescale $f \leftrightarrow (g, u, \omega)$:

$$\boxed{f(t,x,v) = \frac{1}{\omega(t,x)^d} g(t,x,\xi), \quad \xi = \frac{v - u(t,x)}{\omega}.}$$

- Linear Fokker-Planck [Filbet-Russo '04], Granular gas [Filbet-Rey '13]:

$$\omega = \sqrt{\text{Temperature}}.$$

- Kinetic flocking models [Rey-T. '16]:
  Propose a new way to learn the scaling ω dynamically.
  The learned ω is **exact** for spatially homogenous system.

UNIVERSITY OF SOUTH CAROLINA

Spatially "Homogenous" system

$$\boxed{\partial_t f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0.}$$

- Rewrite the system in terms of g_ϵ

$$\partial_t g_\epsilon = \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon}\mathcal{A}_\epsilon\right)\nabla_\xi \cdot (\xi g_\epsilon) + \frac{1}{\omega_\epsilon}\left(\partial_t u_\epsilon - \frac{1}{\epsilon}\mathcal{B}_\epsilon\right)\cdot \nabla_\xi g_\epsilon.$$

[ARR]: $\mathcal{A}_\epsilon(t,x) = 1, \quad \mathcal{B}_\epsilon(t,x) = -u_\epsilon(t,x) - \int \nabla_x K(x-y)\rho_\epsilon(y)dy,$
[ARA]: $\mathcal{A}_\epsilon(t,x) = \int \phi(|x-y|)\rho_\epsilon(t,y)dy,$

$\mathcal{B}_\epsilon(t,x) = \int \phi(|x-y|)(u_\epsilon(t,y) - u_\epsilon(t,x))\rho_\epsilon(y)dy - \int \nabla_x K(x-y)\rho_\epsilon(y)dy.$

- It is easy to check $\partial_t u_\epsilon = \frac{1}{\epsilon}\mathcal{B}_\epsilon(t,x)$.
- Take $\omega_\epsilon(t,x) = \exp\left(-\frac{1}{\epsilon}\int_0^t \mathcal{A}_\epsilon(s,x)ds\right)$. Then $\partial_t g_\epsilon = 0$!!
  The *exact* scaling is valid for any initial configurations.

Spatially "Homogenous" system

$$\boxed{\partial_t f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon)f_\epsilon) = 0.}$$

- Rewrite the system in terms of g_ϵ

$$\partial_t g_\epsilon = \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon}\mathcal{A}_\epsilon\right)\nabla_\xi \cdot (\xi g_\epsilon) + \frac{1}{\omega_\epsilon}\left(\partial_t u_\epsilon - \frac{1}{\epsilon}\mathcal{B}_\epsilon\right)\cdot \nabla_\xi g_\epsilon.$$

[ARR]: $\mathcal{A}_\epsilon(t,x) = 1, \quad \mathcal{B}_\epsilon(t,x) = -u_\epsilon(t,x) - \int \nabla_x K(x-y)\rho_\epsilon(y)dy,$

[ARA]: $\mathcal{A}_\epsilon(t,x) = \int \phi(|x-y|)\rho_\epsilon(t,y)dy,$

$\mathcal{B}_\epsilon(t,x) = \int \phi(|x-y|)(u_\epsilon(t,y) - u_\epsilon(t,x))\rho_\epsilon(y)dy - \int \nabla_x K(x-y)\rho_\epsilon(y)dy.$

- It is easy to check $\partial_t u_\epsilon = \frac{1}{\epsilon}\mathcal{B}_\epsilon(t,x)$.
- Take $\omega_\epsilon(t,x) = \exp\left(-\frac{1}{\epsilon}\int_0^t \mathcal{A}_\epsilon(s,x)ds\right)$. Then $\partial_t g_\epsilon = 0$!!

  The *exact* scaling is valid for any initial configurations.

Spatially "Homogenous" system

$$\boxed{\partial_t f_\epsilon + \frac{1}{\epsilon}\nabla_v \cdot (F(f_\epsilon) f_\epsilon) = 0.}$$

- Rewrite the system in terms of g_ϵ

$$\partial_t g_\epsilon = \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon}\mathcal{A}_\epsilon\right) \nabla_\xi \cdot (\xi g_\epsilon) + \frac{1}{\omega_\epsilon}\left(\partial_t u_\epsilon - \frac{1}{\epsilon}\mathcal{B}_\epsilon\right) \cdot \nabla_\xi g_\epsilon.$$

[ARR]: $\mathcal{A}_\epsilon(t,x) = 1, \quad \mathcal{B}_\epsilon(t,x) = -u_\epsilon(t,x) - \int \nabla_x K(x-y)\rho_\epsilon(y)dy,$
[ARA]: $\mathcal{A}_\epsilon(t,x) = \int \phi(|x-y|)\rho_\epsilon(t,y)dy,$
$\mathcal{B}_\epsilon(t,x) = \int \phi(|x-y|)(u_\epsilon(t,y) - u_\epsilon(t,x))\rho_\epsilon(y)dy - \int \nabla_x K(x-y)\rho_\epsilon(y)dy.$

- It is easy to check $\partial_t u_\epsilon = \frac{1}{\epsilon}\mathcal{B}_\epsilon(t,x)$.
- Take $\omega_\epsilon(t,x) = \exp\left(-\frac{1}{\epsilon}\int_0^t \mathcal{A}_\epsilon(s,x)ds\right)$. Then $\partial_t g_\epsilon = 0$!!
  The *exact* scaling is valid for any initial configurations.

Scaling on the full system

- With free transport, the full system in terms of g_ϵ reads

$$\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi)\cdot\nabla_x g_\epsilon$$
$$= \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + (u_\epsilon + \omega_\epsilon\xi)\cdot\frac{\nabla_x \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon}\mathcal{A}_\epsilon\right)\nabla_\xi\cdot(\xi g_\epsilon)$$
$$+ \frac{1}{\omega_\epsilon}\left(\partial_t u_\epsilon + (u_\epsilon + \omega_\epsilon\xi)\cdot\nabla_x u_\epsilon - \frac{1}{\epsilon}\mathcal{B}_\epsilon\right)\cdot\nabla_\xi g_\epsilon.$$

- Exact scaling can not be expected:
  1. The dynamics of u_ϵ:

$$\partial_t u_\epsilon + u_\epsilon\cdot\nabla_x u_\epsilon + \frac{1}{\rho_\epsilon}\nabla_x\cdot(\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon}\mathcal{B}_\epsilon, \quad P_\epsilon = \int \xi\otimes\xi g_\epsilon(\xi)d\xi.$$

  2. The choice of ω_ϵ:

$$\partial_t \omega_\epsilon + u_\epsilon\cdot\nabla_x\omega_\epsilon + \frac{1}{\epsilon}\mathcal{A}_\epsilon\omega_\epsilon = 0.$$

Scaling on the full system

- With free transport, the full system in terms of g_ϵ reads

$$\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon = \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + (u_\epsilon + \omega_\epsilon \xi) \cdot \frac{\nabla_x \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon} \mathcal{A}_\epsilon \right) \nabla_\xi \cdot (\xi g_\epsilon) + \frac{1}{\omega_\epsilon} \left(\partial_t u_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x u_\epsilon - \frac{1}{\epsilon} \mathcal{B}_\epsilon \right) \cdot \nabla_\xi g_\epsilon.$$

- Exact scaling can not be expected:
  1. The dynamics of u_ϵ:

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon, \quad P_\epsilon = \int \xi \otimes \xi g_\epsilon(\xi) d\xi.$$

  2. The choice of ω_ϵ:

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

UNIVERSITY OF SOUTH CAROLINA

Scaling on the full system

- With free transport, the full system in terms of g_ϵ reads

$$\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon$$
$$= \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + (u_\epsilon + \omega_\epsilon \xi) \cdot \frac{\nabla_x \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon} \mathcal{A}_\epsilon \right) \nabla_\xi \cdot (\xi g_\epsilon)$$
$$+ \frac{1}{\omega_\epsilon} \left(\partial_t u_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x u_\epsilon - \frac{1}{\epsilon} \mathcal{B}_\epsilon \right) \cdot \nabla_\xi g_\epsilon.$$

- Exact scaling can not be expected:
  1. The dynamics of u_ϵ:

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon, \quad P_\epsilon = \int \xi \otimes \xi g_\epsilon(\xi) d\xi.$$

  2. The choice of ω_ϵ:

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

Scaling on the full system

- With free transport, the full system in terms of g_ϵ reads

$$\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon$$
$$= \left(\frac{\partial_t \omega_\epsilon}{\omega_\epsilon} + (u_\epsilon + \omega_\epsilon \xi) \cdot \frac{\nabla_x \omega_\epsilon}{\omega_\epsilon} + \frac{1}{\epsilon} \mathcal{A}_\epsilon \right) \nabla_\xi \cdot (\xi g_\epsilon)$$
$$+ \frac{1}{\omega_\epsilon} \left(\partial_t u_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x u_\epsilon - \frac{1}{\epsilon} \mathcal{B}_\epsilon \right) \cdot \nabla_\xi g_\epsilon.$$

- Exact scaling can not be expected:
  1. The dynamics of u_ϵ:

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon, \quad P_\epsilon = \int \xi \otimes \xi g_\epsilon(\xi) d\xi.$$

  2. The choice of ω_ϵ:

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

Scaling on the full system

- With free transport, the full system in terms of g_ϵ reads

$$\begin{aligned}\partial_t g_\epsilon &+ (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon \\ &= (\xi \cdot \nabla_x \omega_\epsilon) \nabla_\xi \cdot (\xi g_\epsilon) \\ &+ ((\xi \cdot \nabla_x) u_\epsilon) \cdot \nabla_\xi g_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon)\right) \cdot \nabla_\xi g_\epsilon,\end{aligned}$$

- Exact scaling can not be expected:
  1. The dynamics of u_ϵ:

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon, \quad P_\epsilon = \int \xi \otimes \xi g_\epsilon(\xi) d\xi.$$

  2. The choice of ω_ϵ:

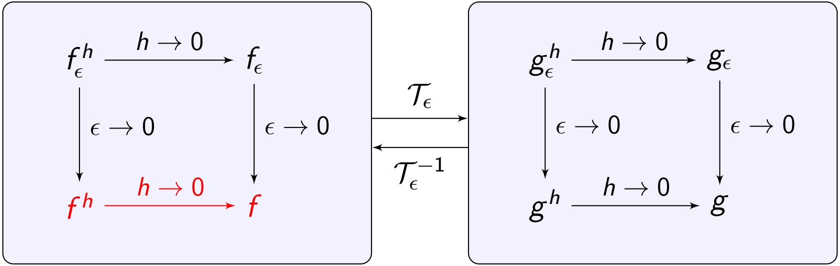
$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

Outline

1. Introduction
2. Kinetic swarming models and zero-inertia limit
3. Velocity scaling methods
4. Asymptotic-preserving scheme
5. Numerical experiments

Design asymptotic-preserving scheme

Recall the main idea to overcome singular limit

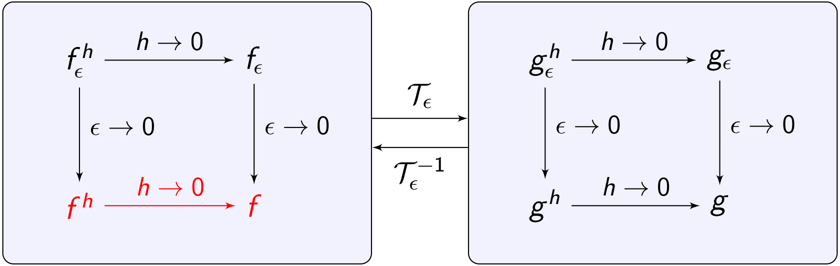


Two ingredients for the scheme to be asymptotic-preserving:

1. g_ϵ does not become singular as $\epsilon \to 0$.
2. An asymptotic-preserving scheme on $(g_\epsilon, u_\epsilon, \omega_\epsilon)$.

Design asymptotic-preserving scheme

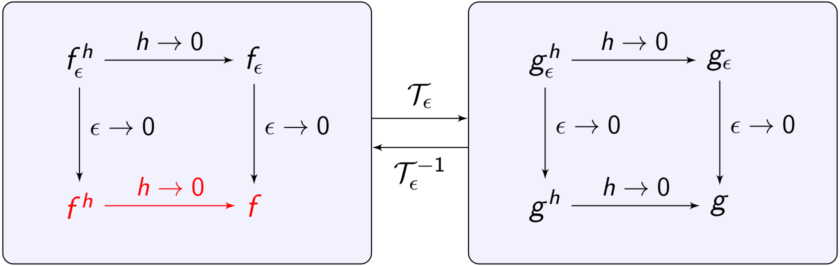
Recall the main idea to overcome singular limit



Two ingredients for the scheme to be asymptotic-preserving:

1. g_ϵ does not become singular as $\epsilon \to 0$.
2. An asymptotic-preserving scheme on $(g_\epsilon, u_\epsilon, \omega_\epsilon)$.

Recall the main idea to overcome singular limit



Two ingredients for the scheme to be asymptotic-preserving:

1. g_ϵ does not become singular as $\epsilon \to 0$.
2. An asymptotic-preserving scheme on $(g_\epsilon, u_\epsilon, \omega_\epsilon)$.

Criterion for *non-singular* $\{g_\epsilon\}$

- We call $\{g_\epsilon\}$ is non-singular if g_ϵ neither concentrate nor spread out in v, as ϵ approaches 0.

$$\max_\xi |g_\epsilon(t,x,\xi)| \le G, \quad \text{and} \quad \operatorname{supp}_\xi g_\epsilon(t,x,\xi) \subset B_R(0).$$

  for all (t,x). G, R are independent with respect to ϵ.

- Goal: Prove that under our choice of transformation $\mathcal{T}_\epsilon$, the rescaled family of solutions $\{g_\epsilon\}$ is non-singular.

UNIVERSITY OF SOUTH CAROLINA

Criterion for *non-singular* $\{g_\epsilon\}$

- We call $\{g_\epsilon\}$ is non-singular if g_ϵ neither concentrate nor spread out in v, as ϵ approaches 0.

$$\max_\xi |g_\epsilon(t,x,\xi)| \le G, \quad \text{and} \quad \underset{\xi}{\text{supp}} g_\epsilon(t,x,\xi) \subset B_R(0).$$

  for all (t,x). G, R are independent with respect to ϵ.

- Goal: Prove that under our choice of transformation $\mathcal{T}_\epsilon$, the rescaled family of solutions $\{g_\epsilon\}$ is non-singular.

Spatial oscillation

- Recall the dynamics of g_ϵ:

$$\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon$$
$$= (\xi \cdot \nabla_x \omega_\epsilon) \nabla_\xi \cdot (\xi g_\epsilon)$$
$$+ ((\xi \cdot \nabla_x) u_\epsilon) \cdot \nabla_\xi g_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon)\right) \cdot \nabla_\xi g_\epsilon,$$

One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.

- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon.$$

Spatial oscillation

- Recall the dynamics of g_ϵ:

$$\begin{aligned}\partial_t g_\epsilon + (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon \\ = (\xi \cdot \nabla_x \omega_\epsilon) \nabla_\xi \cdot (\xi g_\epsilon) \\ + ((\xi \cdot \nabla_x) u_\epsilon) \cdot \nabla_\xi g_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon)\right) \cdot \nabla_\xi g_\epsilon,\end{aligned}$$

One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.

- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon.$$

UNIVERSITY OF SOUTH CAROLINA

Spatial oscillation

- One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.
- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon .$$

1. Without pressure ($P_\epsilon \equiv 0$): $\sup_{0 \le \epsilon \le \epsilon_0} \|\nabla_x u_\epsilon\|_{L^\infty} \le C.$ [Tadmor-T. '14]
2. Limiting system ($u_\epsilon \to u$): $\|\nabla_x u\|_{L^\infty} \le C.$ [Fetecau-Sun-T. '16]
3. Note that $u_\epsilon \to u$ weak-$\star$ in measure. Therefore, the bound on the limiting system does not imply uniform bound on $\|\nabla_x u_\epsilon\|_{L^\infty}$.

UNIVERSITY OF SOUTH CAROLINA

Spatial oscillation

- One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.
- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon.$$

1. Without pressure ($P_\epsilon \equiv 0$): $\sup_{0 \leq \epsilon \leq \epsilon_0} \|\nabla_x u_\epsilon\|_{L^\infty} \leq C.$ [Tadmor-T. '14]
2. Limiting system ($u_\epsilon \to u$): $\|\nabla_x u\|_{L^\infty} \leq C.$ [Fetecau-Sun-T. '16]
3. Note that $u_\epsilon \to u$ weak-$\star$ in measure. Therefore, the bound on the limiting system does not imply uniform bound on $\|\nabla_x u_\epsilon\|_{L^\infty}$.

Spatial oscillation

- One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.
- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon.$$

1. Without pressure ($P_\epsilon \equiv 0$): $\sup_{0 \le \epsilon \le \epsilon_0} \|\nabla_x u_\epsilon\|_{L^\infty} \le C.$ [Tadmor-T. '14]
2. Limiting system ($u_\epsilon \to u$): $\|\nabla_x u\|_{L^\infty} \le C.$ [Fetecau-Sun-T. '16]
3. Note that $u_\epsilon \to u$ weak-$\star$ in measure. Therefore, the bound on the limiting system does not imply uniform bound on $\|\nabla_x u_\epsilon\|_{L^\infty}$.

Spatial oscillation

- One major **difficulty** is to control the spacial derivatives $\nabla_x g_\epsilon, \nabla_x \omega_\epsilon, \nabla_x u_\epsilon$ and $\nabla_x P_\epsilon$ uniformly in ϵ.
- Take u_ϵ as an example. Recall its dynamics

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon.$$

1. Without pressure ($P_\epsilon \equiv 0$): $\sup_{0 \le \epsilon \le \epsilon_0} \|\nabla_x u_\epsilon\|_{L^\infty} \le C$. [Tadmor-T. '14]
2. Limiting system ($u_\epsilon \to u$): $\|\nabla_x u\|_{L^\infty} \le C$. [Fetecau-Sun-T. '16]
3. Note that $u_\epsilon \to u$ weak-$\star$ in measure. Therefore, the bound on the limiting system does not imply uniform bound on $\|\nabla_x u_\epsilon\|_{L^\infty}$.

Non-oscillatory assumptions

- We assume that the solution does not have spatial oscillations:

$$|\nabla_x g_\epsilon(t,x,\xi)| \leq C_1 g_\epsilon(t,x,\xi),$$
$$|\nabla_x u_\epsilon(t,x)| \leq C_2.$$

- The assumptions imply non-oscillatory bounds for other macroscopic quantities:

$$|\nabla_x \rho_\epsilon(t,x)| \leq C_1 \rho_\epsilon(t,x),$$
$$|\nabla_x P_\epsilon(t,x)| \leq C_1 P_\epsilon(t,x),$$
$$\|\nabla_x \omega_\epsilon(t,\cdot)\|_{L^\infty} \leq \frac{C_1(e^{C_2 t}-1)}{C_2 \epsilon} \exp\left(-\frac{c}{\epsilon}t\right).$$

Non-oscillatory assumptions

- We assume that the solution does not have spatial oscillations:

$$|\nabla_x g_\epsilon(t,x,\xi)| \leq C_1 g_\epsilon(t,x,\xi),$$
$$|\nabla_x u_\epsilon(t,x)| \leq C_2.$$

- The assumptions imply non-oscillatory bounds for other macroscopic quantities:

$$|\nabla_x \rho_\epsilon(t,x)| \leq C_1 \rho_\epsilon(t,x),$$
$$|\nabla_x P_\epsilon(t,x)| \leq C_1 P_\epsilon(t,x),$$
$$\|\nabla_x \omega_\epsilon(t,\cdot)\|_{L^\infty} \leq \frac{C_1(e^{C_2 t}-1)}{C_2 \epsilon} \exp\left(-\frac{c}{\epsilon} t\right).$$

Non-oscillatory implies non-singular

Theorem ([Chertock-T.-Yan '18])

Let $(g_\epsilon, u_\epsilon, \omega_\epsilon)$ *be the solution of the rescaled dynamics.*
Assume the solution satisfies the non-oscillatory conditions.

Then, $g_\epsilon(t)$ *is non-singular uniformly in* $\epsilon \in [0, \epsilon_0]$ *for all* $t \geq 0$.

- If the solution is not oscillatory in spatial variable, the proposed transformation based on velocity scaling resolves the singularity in the original limit.
- The discrete version of the non-oscillatory conditions can be verified a posteriorly numerically.

Non-oscillatory implies non-singular

Theorem ([Chertock-T.-Yan '18])

Let $(g_\epsilon, u_\epsilon, \omega_\epsilon)$ be the solution of the rescaled dynamics.
Assume the solution satisfies the non-oscillatory conditions.

Then, $g_\epsilon(t)$ is non-singular uniformly in $\epsilon \in [0, \epsilon_0]$ for all $t \geq 0$.

- If the solution is not oscillatory in spatial variable, the proposed transformation based on velocity scaling resolves the singularity in the original limit.
- The discrete version of the non-oscillatory conditions can be verified a posteriorly numerically.

Non-oscillatory implies non-singular

Theorem ([Chertock-T.-Yan '18])

Let $(g_\epsilon, u_\epsilon, \omega_\epsilon)$ be the solution of the rescaled dynamics.
Assume the solution satisfies the non-oscillatory conditions.

Then, $g_\epsilon(t)$ is non-singular uniformly in $\epsilon \in [0, \epsilon_0]$ for all $t \geq 0$.

- If the solution is not oscillatory in spatial variable, the proposed transformation based on velocity scaling resolves the singularity in the original limit.
- The discrete version of the non-oscillatory conditions can be verified a posteriorly numerically.

UNIVERSITY OF SOUTH CAROLINA

Asymptotic-preserving scheme for the rescaled system

- For $(u_\epsilon, \omega_\epsilon)$, the stiff term is *linear*. Use standard IMEX scheme.

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon,$$
$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

- For g_ϵ, there is no explicit dependence on ϵ. Use explicit schemes.

$$\begin{aligned}\partial_t g_\epsilon &+ (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon \\ &= (\xi \cdot \nabla_x \omega_\epsilon) \nabla_\xi \cdot (\xi g_\epsilon) \\ &+ ((\xi \cdot \nabla_x) u_\epsilon) \cdot \nabla_\xi g_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) \right) \cdot \nabla_\xi g_\epsilon,\end{aligned}$$

We use finite volume method, e.g. upwind.
Some corrections are introduced to ensure $\int v g_\epsilon(t,x,v) dv = 0$.
(Follow from [Rey-T. '16])

UNIVERSITY OF SOUTH CAROLINA

Asymptotic-preserving scheme for the rescaled system

- For $(u_\epsilon, \omega_\epsilon)$, the stiff term is *linear*. Use standard IMEX scheme.

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon,$$
$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

- For g_ϵ, there is no explicit dependence on ϵ. Use explicit schemes.

$$\begin{aligned}\partial_t g_\epsilon &+ (u_\epsilon + \omega_\epsilon \xi) \cdot \nabla_x g_\epsilon \\ &= (\xi \cdot \nabla_x \omega_\epsilon) \nabla_\xi \cdot (\xi g_\epsilon) \\ &+ ((\xi \cdot \nabla_x) u_\epsilon) \cdot \nabla_\xi g_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) \right) \cdot \nabla_\xi g_\epsilon,\end{aligned}$$

We use finite volume method, e.g. upwind.
Some corrections are introduced to ensure $\int v g_\epsilon(t,x,v)dv = 0$.
(Follow from [Rey-T. '16])

Asymptotic-preserving scheme for the rescaled system

- For $(u_\epsilon, \omega_\epsilon)$, the stiff term is *linear*. Use standard IMEX scheme.

$$\partial_t u_\epsilon + u_\epsilon \cdot \nabla_x u_\epsilon + \frac{1}{\rho_\epsilon} \nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) = \frac{1}{\epsilon} \mathcal{B}_\epsilon,$$

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla_x \omega_\epsilon + \frac{1}{\epsilon} \mathcal{A}_\epsilon \omega_\epsilon = 0.$$

- For g_ϵ, there is no explicit dependence on ϵ. Use explicit schemes.

$$\partial_t g_\epsilon + \nabla_x \cdot ((u_\epsilon + \omega_\epsilon \xi) g_\epsilon) = \nabla_\xi \cdot \left[\left((\xi \cdot \nabla_x \omega_\epsilon)\xi + (\xi \cdot \nabla_x) u_\epsilon - \frac{1}{\rho_\epsilon \omega_\epsilon} \left(\nabla_x \cdot (\omega_\epsilon^2 P_\epsilon) \right) \right) g_\epsilon \right].$$

We use finite volume method, e.g. upwind.
Some corrections are introduced to ensure $\int v g_\epsilon(t, x, v) dv = 0$.
(Follow from [Rey-T. '16])

Outline

1 Introduction

2 Kinetic swarming models and zero-inertia limit

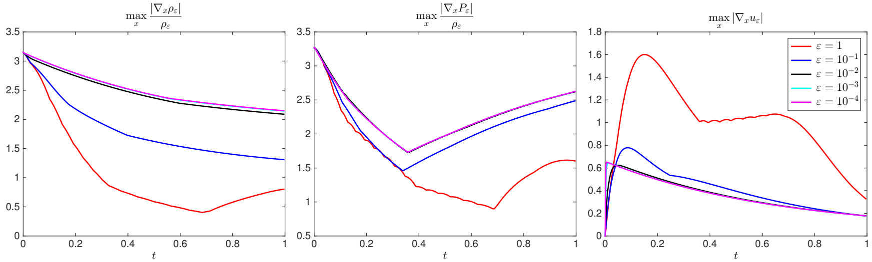
3 Velocity scaling methods

4 Asymptotic-preserving scheme

5 Numerical experiments

Validation of non-oscillatory assumptions

Plots of $\max_x |\nabla_x \rho_\epsilon(t,x)/\rho_\epsilon(t,x)|$, $\max_x |\nabla_x P_\epsilon(t,x)/\rho_\epsilon(t,x)|$ and $\max_x |\nabla_x u_\epsilon(t,x)|$, for $t \in [0,1]$ and different choices of ϵ.

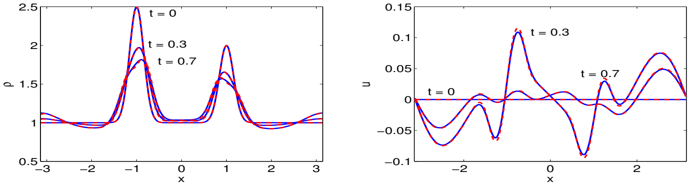


Initial condition:

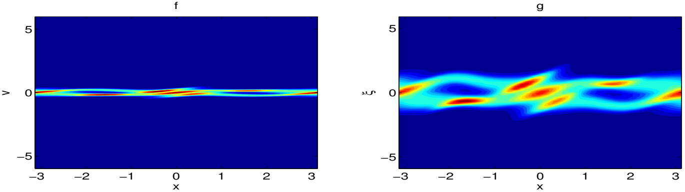
$$g^0(x,\xi) = \rho^0(x) M(\xi), \quad M(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2},$$

$$\rho^0(x) = 1 + e^{-20(x-1)^2} + e^{-20(x+1)^2}, \quad u^0(x) = 0, \quad \omega^0(x) = 1.$$

Comparison between solving f_ϵ and $(g_\epsilon, u_\epsilon, \omega_\epsilon)$ for $\epsilon = 1$.
Snapshots of (ρ, u) at $t = 0, 0.3, 0.7$.



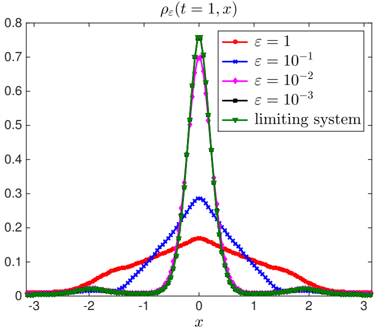
Snapshots of g at $t = 0.7$.

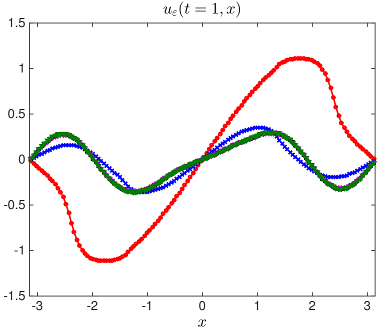


For t large or ϵ small, $f_\epsilon(t)$ is singular and the direct scheme fails.

Asymptotic-preserving test

Snapshots of $(\rho_\epsilon, u_\epsilon)$ at $t = 1$ for different ϵ. When ϵ becomes small, the profile approaches the limiting system.





An application

Aggregation system *(ARR)*

Rescaled Morse potential:

$$K(x) = -e^{-|x|} + e^{-2|x|}.$$

Initial configuration:

$$g^0(x,\xi) = \frac{\rho^0(x)}{2\sqrt{0.4\pi}}\left[e^{-\frac{(\xi+2)^2}{0.4}} + e^{-\frac{(\xi-2)^2}{0.4}}\right],$$

$$\rho^0(x) = 10^{-8} + e^{-40x^2},$$

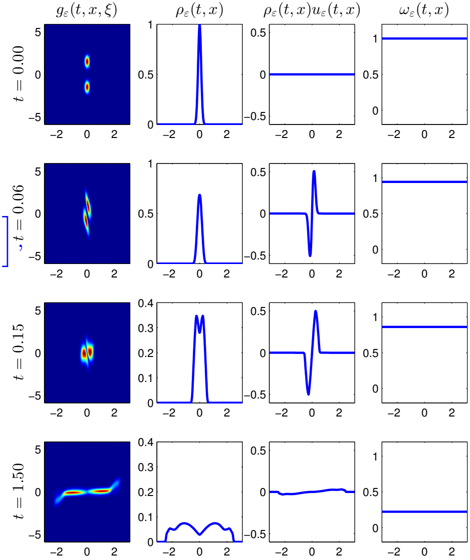
$$u^0(x) = 0, \quad \omega^0(x) = 1.$$

Two groups, same location (near 0), opposite velocity (around ± 2).

Kinetic regime: $\epsilon = 1$

g_ϵ stays regular in all time.

Long time behavior: alignment.



An application

Aggregation system *(ARR)*

Rescaled Morse potential:

$$K(x) = -e^{-|x|} + e^{-2|x|}.$$

Initial configuration:

$$g^0(x,\xi) = \frac{\rho^0(x)}{2\sqrt{0.4\pi}}\left[e^{-\frac{(\xi+2)^2}{0.4}} + e^{-\frac{(\xi-2)^2}{0.4}}\right],$$

$$\rho^0(x) = 10^{-8} + e^{-40x^2},$$

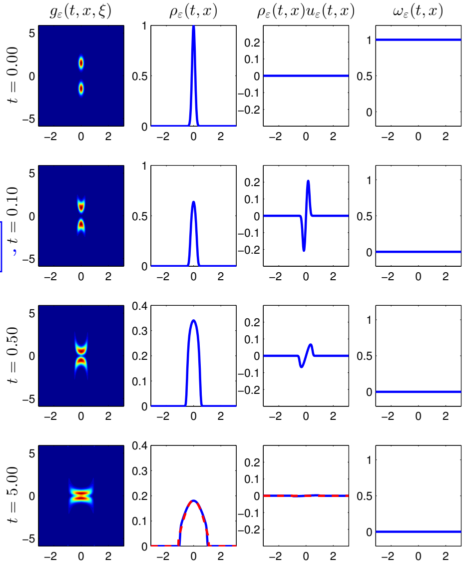
$$u^0(x) = 0, \quad \omega^0(x) = 1.$$

Two groups, same location (near 0), opposite velocity (around ± 2).

Hydrodynamic regime: $\epsilon = 10^{-4}$

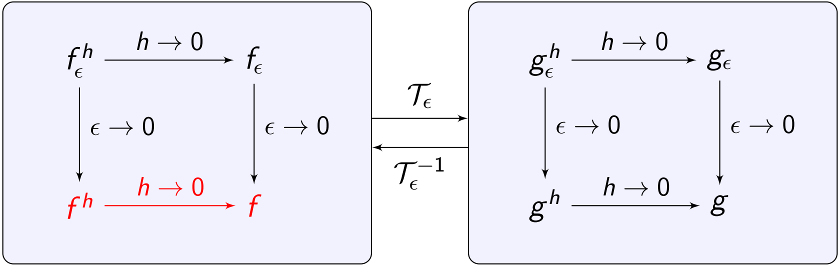
g_ϵ also stays regular in all time.

Long time behavior: aggregation.



Conclusion

Asymptotic preserving schemes on kinetic models with singular limits



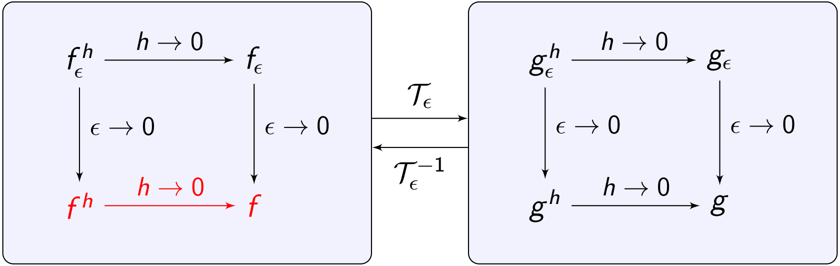
Extensions:

- Hydrodynamic limits for kinetic swarming models with singular alignment [Potayo-Soler '17]
- Other systems with singular or near-singular limits (Boltzmann, granular gas, ...)
- Data

Thanks for your attention!

Conclusion

Asymptotic preserving schemes on kinetic models with singular limits



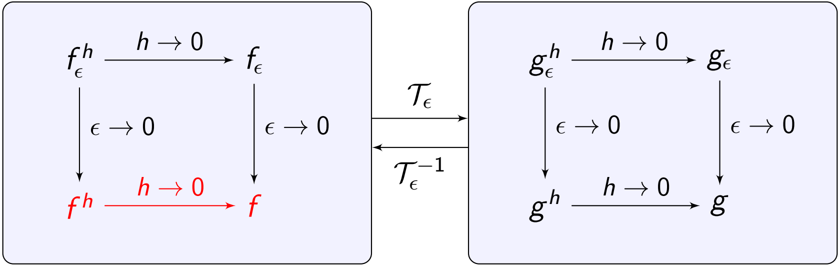
Extensions:

- Hydrodynamic limits for kinetic swarming models with singular alignment [Potayo-Soler '17]
- Other systems with singular or near-singular limits (Boltzmann, granular gas, ...)
- Data

Thanks for your attention!

Conclusion

Asymptotic preserving schemes on kinetic models with singular limits



Extensions:

- Hydrodynamic limits for kinetic swarming models with singular alignment [Potayo-Soler '17]
- Other systems with singular or near-singular limits (Boltzmann, granular gas, ...)
- Data

Thanks for your attention!