# Data and Models - Introductory Comments

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# Outline



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# Guiding Theme

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- What does ”optimality” mean?

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  - identify relevant model - benchmark scenarios

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- Discrete $\leftrightarrow$ continuous ($\infty$-dimensional) settings;
- nonlinear and adaptive concepts;
- optimization

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Quantify information/prediction accuracy

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Examples: $\rightsquigarrow$ ”Small-Data Problem”

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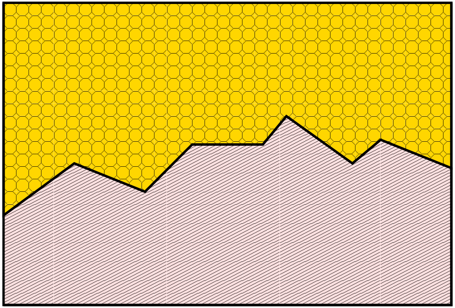

Model:

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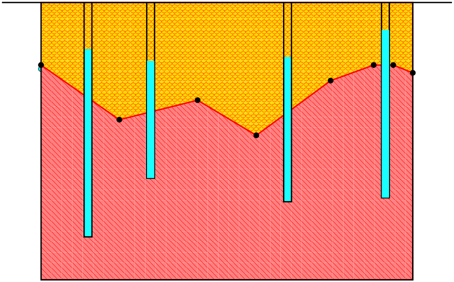

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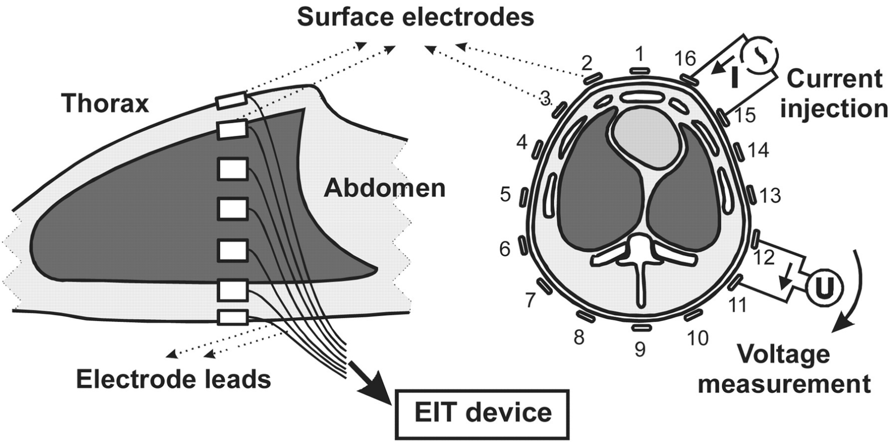


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# Outline

# Data Driven Approaches

Big Data Problem

- Given data/observations/measurements
  $\mathcal{Z}_n = \{(z_i = (x_i, y_i) : i = 1, \ldots, n\} \subset \mathbb{X} \times \mathbb{Y}$

  learn ”functional law”: $f : \mathbb{X} \to \mathbb{Y}$, $f(x_i) \approx y_i$, explaining the data
- Stochastic model: $z_i$ i.i.d. with respect to (often) unknown probability density $\rho$ on $\mathbb{X} \times \mathbb{Y}$
- Typical goal: find an estimator $\hat{f}_{\mathcal{Z}_n} : \mathbb{X} \to \mathbb{Y}$ that approximates $f$ in a certain sense
- $\mathbb{Y} =$ continuum $\rightsquigarrow$ regression
- $\#\mathbb{Y} < \infty \rightsquigarrow$ classification

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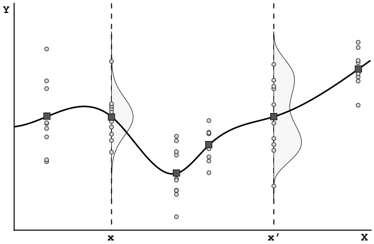
# Regression



# Regression

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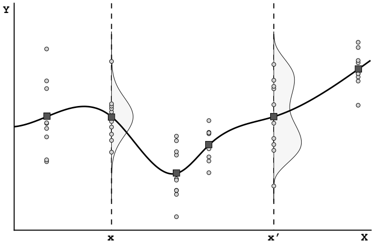

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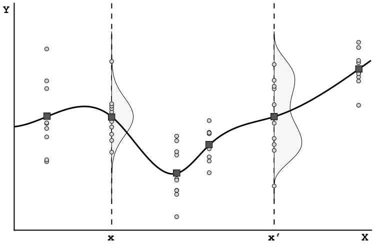


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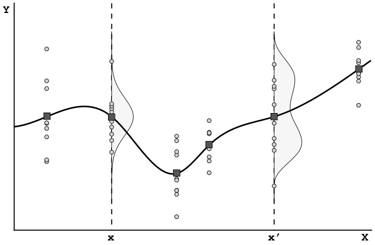


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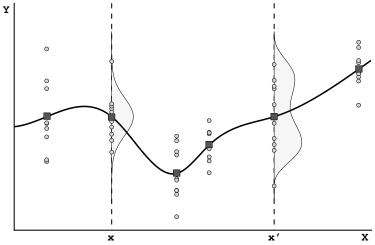


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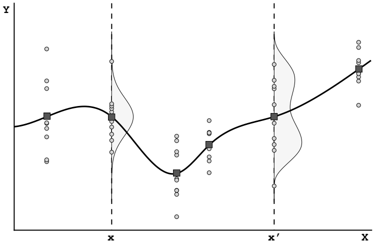


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Ideal desirable performance bound:

$$\mathbb{P}_{\rho^n}\left\{\mathcal{Z} : \|f_\rho - \hat{f}_{\mathcal{Z}}\| \geq c(d)\left(\frac{\log n}{n}\right)^{\frac{s}{2s+1}}\right\} \leq Cn^{-\beta}$$

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Applications: imaging, spectral imaging, learning noise models

# Model Based Approaches

Modeling: Physical, technological process described by $u$: find "balance law" describing $u$

$$F(u, f) = 0$$

where $F(u, f) = G(u, \partial_t u, \partial_x^\alpha u) - f$ is a PDE, integral equation, etc.

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Forward problem: given $f$ (and $y$), find $u$ s.t.

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Issues:

- fixed discretization: efficient solvers, preconditioning

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Issues:

- optimize discretization, adaptive solvers

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Forward problem: given $f \in \mathbb{F} = ???$ (and $y$), find $u \in \mathbb{U} = ???$ s.t.

$$F(u, f) = 0 \qquad \text{is well conditioned}$$

Discretization: Contrive a discrete model

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Issues:

- choice of $\mathbb{U}, \mathbb{F}$ finite $\leftrightarrow$ infinite-dimensional problem

# Ideal Stability

$F(u, f) = 0$ : $\bar{u}$ approximation to $u$ - would like to have:

$$\|u - \bar{u}\|_{\mathbb{U}} \eqsim \|F(\bar{u}, f)\|_{???}$$

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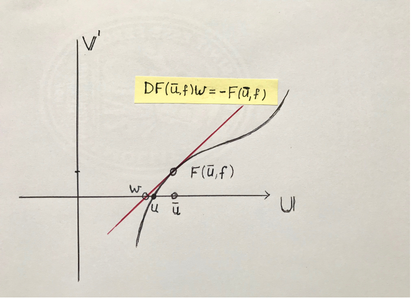

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- $F(u,f)=\mathcal{B}u-f$ linear $\rightsquigarrow$ Babuska-Banach-Necas Theorem
  inf-sup condition $\rightsquigarrow$ DPG framework

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  inf-sup condition $\rightsquigarrow$ DPG framework
- A posteriori error indicators that drive adaptive refinements
- Residuals $\rightsquigarrow$ surrogates for greedy constructions of
  reduced bases

# Example: Flooded City

Courtesy of Nils Gerhard and Siegfried Müller, RWTH Aachen, Germany

| | |
|---|---|
| Reference: | Experiment in a Channel |
| Computational Domain: | $36 \times 3.6\ m^2$ |
| Size of the Buildings: | $0.3 \times 0.3\ m^2$ |
| Width of Streets between Buildings: | $0.1\ m$ |
| $\#$ Buildings: | 25 |

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|---|---|
| # mesh cells: | $13,271,040$ |
| # of D.o.Fs in Uniform Discretization: | $79,626,240$ |
| maximal # of cells in Adaptive Mesh: | $623,025$ |
| | ($4,7\%$ of Reference Method) |
| # of Time Steps: | ca $1,000,000$ |
| CPU Time: | 48 Hours on 320 CPUs |
| | (Intel Xeon Cluster) |

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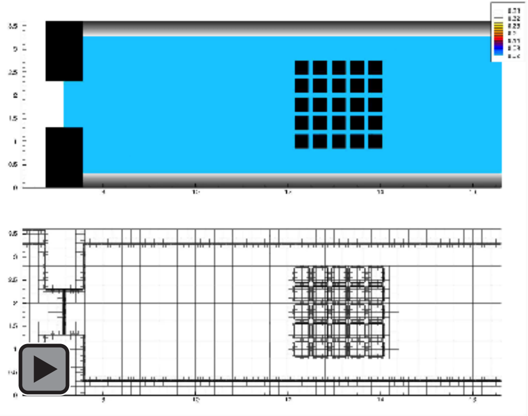
(click video to play)



# Outline

1. What is this about?
   - Prelude
   - Examples
2. Two Worlds
3. How Should the Two Worlds Talk to Each Other?

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# The Role of Reduced Models

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# The Role of Reduced Models

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- Frequent parameter queries (design tasks, optimal control, calibration, parameter estimation, etc.)
- Given $\epsilon > 0$, find possibly small space $\mathbb{U}_\epsilon \subset \mathbb{U}$ such that

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This workshop:
- foundational aspects of reduced models, data assimilation, state estimation
- structural imaging, optimization
- DPG - a framework for well conditioned variational formulations