The Discontinuous Petrov Galerkin (DPG) Method with Optimal Test Functions: Lecture 3

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Outline of Lecture 3

1. The importance of Y
2. "Broken" forms for Laplace & Maxwell equations
3. Verification of **[U+I]**
4. Verification of **[F]**

Designing DPG methods

The game is to reformulate boundary value problems into operator equations $Bx = \ell$ where $B : X \to Y^*$ is a continuous linear operator and

$$\| \cdot \|_{Y^*} \text{ is locally and easily approximable.}$$

Recall one of the definitions of the DPG method

Exact problem: Given Hilbert spaces X and Y, a continuous linear operator $B : X \to Y^*$ and an $\ell \in Y^*$, solve for x in X satisfying

$$Bx = \ell.$$

Discretization: Pick finite dimensional subspaces $X_h \subset X$ and $Y_h \subset Y$ and compute

$$x_h = \arg \min_{z_h \in X_h} \|\ell - Bz_h\|_{Y_h^*}.$$

When Y_h admits functions without interelement continuity, we call this the **DPG method**. [G+Demkowicz 2011]

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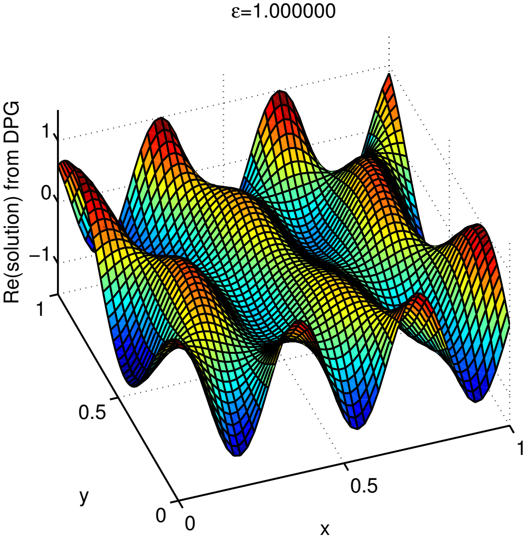
Relatives:

- FOSLS ($Y = L^2$) [Cai+Lazarov+Manteuffel+McCormick 1994]
- Negative-norm least-squares ($Y = H_0^1$) [Bramble+Lazarov+Pasciak 1997]

In what norm will you minimize?



Experiment: Simulate a plane wave propagating at $\theta = \pi/8$.

Apply DPG minimization in a relaxed graph norm where L^2 terms are scaled by ε.

Dissipation$\to 0$ as $\varepsilon \to 0$.
[Nicole Olivares 2016] dissertation.

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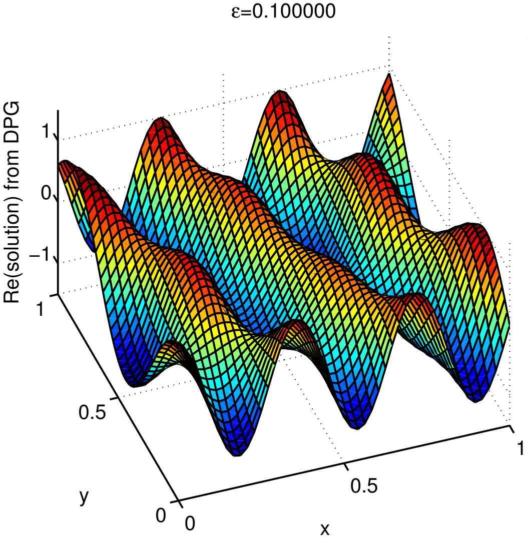

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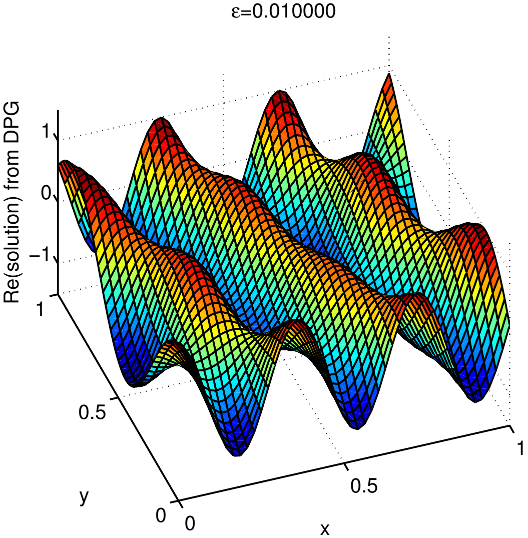

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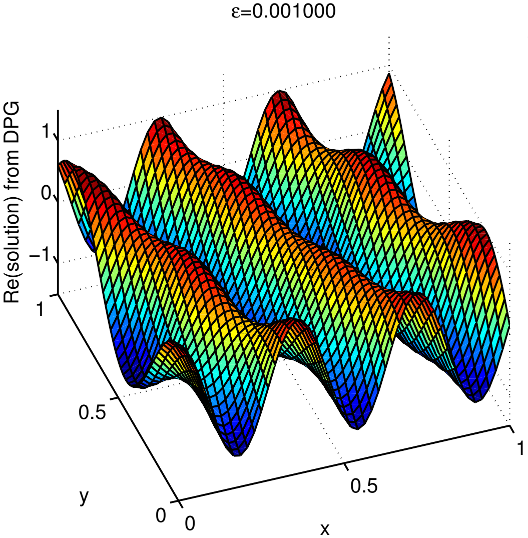

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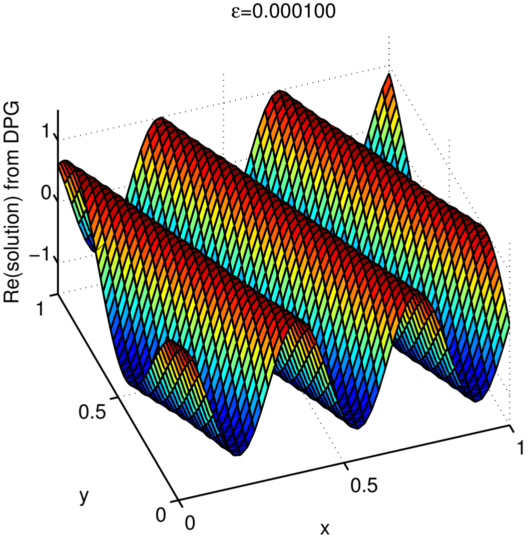

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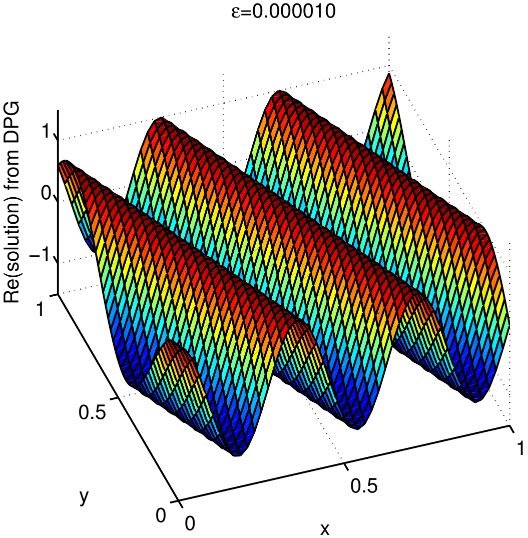

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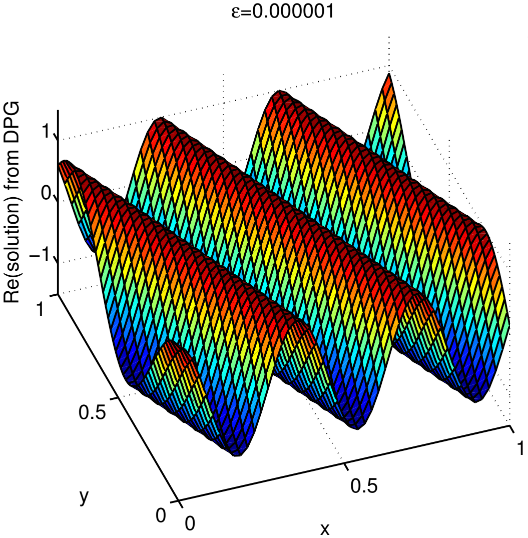

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Computational feasibility

Interesting DPG methods arise when
Y_h has a basis whose Gram matrix is easy to invert.

- $\| \cdot \|_{Y_h^*}$ is easily computable.

$$x_h = \arg \min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*}.$$

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- $\| \cdot \|_{Y_h^*}$ is easily computable.
- T_h is easily computable.

$$x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*}.$$

$$b(x_h, y) = \ell(y), \qquad y \in T_h(X_h).$$

Test space $T_h(X_h)$ is determined by solving $(T_h z, y)_Y = b(z, y)$ for all $y \in Y_h$ and $z \in X_h$.

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Y_h **has a basis whose Gram matrix is easy to invert**.

- $\| \cdot \|_{Y_h^*}$ is easily computable.
- T_h is easily computable.
- e_h is easily condensed out.

$$x_h = \arg \min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*}.$$

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Test space $T_h(X_h)$ is determined by solving $(T_h z, y)_Y = b(z, y)$ for all $y \in Y_h$ and $z \in X_h$.

$$\begin{aligned} (e_h, y)_Y + b(x_h, y) &= \ell(y), & \forall y \in Y_h, \\ b(z_h, e_h) &= 0, & \forall z_h \in X_h. \end{aligned}$$

Recall the 3 assumptions

Let $b(x,y) = (Bx)(y)$, the sesquilinear form on $X \times Y$ generated by B.

Assumption [U] Uniqueness

$$\{y \in Y : \ b(x,y) = 0 \text{ for all } x \in X\} = \{0\}.$$

Assumption [I] Inf-Sup

$$\exists c_1 > 0 : \quad \forall x \in X, \qquad c_1 \|x\|_X \le \sup_{0 \ne y \in Y} \frac{|b(x,y)|}{\|y\|_Y} \le \|b\| \, \|x\|_X.$$

Assumption [F] Fortin Operator

$\exists$ continuous linear $\Pi : Y \to Y_h$ such that

$$b(z_h, y - \Pi y) = 0 \qquad \text{for all } z_h \in X_h, \ y \in Y.$$

Example: A new weak form for the old Laplacian

$$\text{Find } u: \quad \begin{cases} -\Delta u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Let Ω_h be a mesh of Ω and $K \in \Omega_h$ be a mesh element. Then

$$\int_K \operatorname{grad} u \cdot \operatorname{grad} v - \int_{\partial K} (n \cdot \operatorname{grad} u) v = \int_K f\, v.$$

This allows test function $v \in Y$ to be in a "broken" Sobolev space

$$Y = H^1(\Omega_h) := \prod_{K \in \Omega_h} H^1(K).$$

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$$\sum_{K \in \Omega_h} \left[\int_K \operatorname{grad} u \cdot \operatorname{grad} v - \int_{\partial K} n \cdot \hat{q}\, v \right] = \int_\Omega f v.$$

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Primal DPG formulation for Dirichlet problem

$$b(\,(u, \hat{q}\cdot n), v) = \sum_{K\in\Omega_h} \left[\int_K \mathrm{grad}\, u \cdot \mathrm{grad}\, v \;-\int_{\partial K} \hat{q}\cdot n\; v \right]$$

$$Y = H^1(\Omega_h)$$
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Definition (of Q^{div}, the space where numerical flux $\hat{q}\cdot n$ lies)

Define the element-by-element trace operator tr_n by

$$\mathrm{tr}_n : H(\mathrm{div}, \Omega) \to \prod_{K\in\Omega_h} H^{-1/2}(\partial K), \qquad \mathrm{tr}_n\, r|_{\partial K} = r\cdot n|_{\partial K}.$$

Set Q^{div} = range of tr_n. It is complete under the norm

$$\|\hat{q}\cdot n\|_{Q^{\mathrm{div}}} = \inf_{r\in \mathrm{tr}_n{}^{-1}\{\hat{q}\cdot n\}} \|r\|_{H(\mathrm{div},\Omega)}.$$

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Broken and Unbroken forms

Broken form

$$b(\,(u, \hat{q}\cdot n), v) = \underbrace{(\operatorname{grad} u, \operatorname{grad} v)_h}_{b_0(u,v)} + \underbrace{\langle -\hat{q}\cdot n, v\rangle_h}_{\hat{b}(\hat{q}\cdot n, v)}$$

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Unbroken form

$$b_0(u, v) = (\operatorname{grad} u, \operatorname{grad} v)$$

Stability of the unbroken form on $H_0^1(\Omega) \times H_0^1(\Omega)$ is standard.
Stability of broken form?

Abstracting the structure

Suppose we have two further Hilbert spaces X_0 and $\hat{X}$ such that:

Abstract setting:

Dirichlet example:

- $X_0 = H^1_0(\Omega)$, $\hat{X} = Q^{\mathrm{div}}$, $Y = H^1(\Omega_h)$

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Suppose we have two further Hilbert spaces X_0 and $\hat{X}$ such that:

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- $Y_0 = H_0^1(\Omega)$.

From standard to broken forms: An abstract result

Assumption [H] — Hybrid form

$$\exists\, \hat{c} > 0 : \qquad \hat{c}\, \|\hat{x}\|_{\hat{X}} \le \sup_{y \in Y} \frac{|\hat{b}(\hat{x}, y)|}{\|y\|_Y} \quad \forall \hat{x} \in \hat{X}.$$

Theorem (Stability of unbroken form $\implies$ Stability of broken form)

Suppose Assumption **[H]** holds. Then

$$\left.\begin{array}{r} [\mathbf{U} + \mathbf{I}] \text{ holds for} \\ b_0 \text{ on } X_0 \times Y_0 \end{array}\right\} \implies \left\{\begin{array}{l} [\mathbf{U} + \mathbf{I}] \text{ holds for} \\ b = b_0 + \hat{b} \text{ on } X \times Y \end{array}\right.$$

[Carstensen+Demkowicz+G 2015]

Example: Maxwell cavity problem

Assuming all time variations are harmonic ($e^{-\imath\omega t}$), the electric (E) and magnetic (H) fields satisfy

$$\begin{aligned} \imath\omega\mu H - \operatorname{curl} E &= 0 && \text{on } \Omega \\ \imath\omega\varepsilon E + \operatorname{curl} H &= J && \text{on } \Omega \\ n \times E &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Find E: $$\begin{cases} \operatorname{curl} \mu^{-1} \operatorname{curl} E - \omega^2 \varepsilon E = \imath\omega J, & \text{on } \Omega \\ n \times E = 0, & \text{on } \partial\Omega. \end{cases}$$

If ω is not a cavity resonance, then this problem is wellposed.

Deriving broken and unbroken formulation

Integrate by parts on Ω:

$$\int_\Omega \mu^{-1} \operatorname{curl} E \cdot \overline{\operatorname{curl} F} - \omega^2 \varepsilon E \cdot \overline{F} + \int_{\partial\Omega} n \times \mu^{-1} \operatorname{curl} E \cdot \overline{F} = 0$$

Deriving broken and unbroken formulation

Unbroken (standard) formulation Integrate by parts on Ω:

Find $E \in H_0(\mathrm{curl}, \Omega)$ satisfying

$$\underbrace{(\mu^{-1}\,\mathrm{curl}\,E, \mathrm{curl}\,F) - \omega^2(\varepsilon E, F)}_{b_0(E,F)} = (f, F)$$

for all $F \in H_0(\mathrm{curl}, \Omega)$.

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Broken formulation Integrate by parts element by element:

$$\sum_{K \in \Omega_h} \left[\int_K \mu^{-1}\,\mathrm{curl}\,E \cdot \overline{\mathrm{curl}\,F} - \omega^2 \varepsilon E \cdot \overline{F} + \int_{\partial K} n \times \underbrace{\mu^{-1}\mathrm{curl}\,E}_{\imath\omega\hat{H}} \cdot \overline{F} \right] = 0$$

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Primal DPG formulation for the Maxwell problem

$$b((E, n \times \hat{H}), F) = (\mu^{-1} \operatorname{curl} E, \operatorname{curl} F)_h - \omega^2(\varepsilon E, F) + \imath\omega\langle n \times \hat{H}, F\rangle_h$$

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$Q^{\operatorname{curl}} = \operatorname{range}(\operatorname{tr}_\times)$, normed by $\|n\times\hat F\|_{Q^{\operatorname{curl}}} = \inf_{G\in{\operatorname{tr}_\times}^{-1}\{n\times\hat F\}} \|G\|_{H(\operatorname{curl},\Omega)}$.

Fitting to the previous abstract structure

Abstract setting:

Maxwell example:

- $X_0 = H_0(\mathrm{curl}, \Omega)$, $\hat{X} = Q^{\mathrm{curl}}$,

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Recall the abstract result

Assumption [H] Hybrid form

$$\exists\, \hat{c} > 0 : \qquad \hat{c}\, \|\hat{x}\|_{\hat{X}} \le \sup_{y \in Y} \frac{|\hat{b}(\hat{x}, y)|}{\|y\|_Y} \quad \forall \hat{x} \in \hat{X}.$$

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Analysis of broken Maxwell and Laplace forms

The last theorem reduces analysis of wellposedness to verification of **[H]**.

[U+I] for broken Maxwell form will follow if **[H]** is proved:

$$\|n \times \hat{H}\|_{Q^{\mathrm{curl}}} \le \frac{1}{\hat{c}} \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat{H}, F\rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

[U+I] for broken Dirichlet form will follow if **[H]** is proved:

$$\|n \cdot \hat{q}\|_{Q^{\mathrm{div}}} \le \frac{1}{\hat{c}} \sup_{v \in H^1(\Omega_h)} \frac{|\langle \hat{q} \cdot n, v\rangle_h|}{\|v\|_{H^1(\Omega_h)}}$$

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$$\inf_{H \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|H\|_{H(\mathrm{curl},\Omega)} = \|n \times \hat{H}\|_{Q^{\mathrm{curl}}} \le \frac{1}{\hat{c}} \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat{H}, F\rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

[U+I] for broken Dirichlet form will follow if **[H]** is proved:

$$\inf_{r \in \mathrm{tr}_n{}^{-1}\{\hat{q}\cdot n\}} \|r\|_{H(\mathrm{div},\Omega)} = \|n \cdot \hat{q}\|_{Q^{\mathrm{div}}} \le \frac{1}{\hat{c}} \sup_{v \in H^1(\Omega_h)} \frac{|\langle \hat{q} \cdot n, v\rangle_h|}{\|v\|_{H^1(\Omega_h)}}$$

Interface (inf=sup) lemma

Lemma [Carstensen+Demkowicz+G 2015]

$$\inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},\Omega)} = \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat{H}, F \rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

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Interpreting the lemma for a one element mesh:

- Two types of traces of $F \in H(\mathrm{curl}, K)$ on one element boundary:

$$\mathrm{tr}_\times F = n\times F|_{\partial K}, \qquad \mathrm{tr}_\top F = (n\times F)\times n|_{\partial K}.$$

Interface (inf=sup) lemma

Lemma [Carstensen+Demkowicz+G 2015]

$$\inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat H\}} \|F\|_{H(\mathrm{curl},\Omega)} = \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat H, F\rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

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- $\mathrm{Range}(\mathrm{tr}_\times) = H^{-1/2}(\mathrm{div}, \partial K)$. $\qquad \mathrm{Range}(\mathrm{tr}_\top) = H^{-1/2}(\mathrm{curl}, \partial K)$.

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Lemma [Carstensen+Demkowicz+G 2015]

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- $\mathrm{Range}(\mathrm{tr}_\times) = H^{-1/2}(\mathrm{div}, \partial K)$. $\mathrm{Range}(\mathrm{tr}_\top) = H^{-1/2}(\mathrm{curl}, \partial K)$.
- Lemma $\implies$ the inf $= \|n \times \hat{H}\|_{H^{-1/2}(\mathrm{div},\partial K)} =$ the sup $=$

$$= \sup_{F_\top \in H^{-1/2}(\mathrm{curl},\partial K)} \frac{|\langle n \times \hat{H}, F_\top\rangle_h|}{\|F_\top\|_{H^{-1/2}(\mathrm{curl},\partial K)}} = \|n \times \hat{H}\|_{[H^{-1/2}(\mathrm{curl},\partial K)]^*}.$$

Interface (inf=sup) lemma

Lemma [Carstensen+Demkowicz+G 2015]

$$\inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},\Omega)} = \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat{H}, F\rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

$\Longrightarrow$ The lemma, on one element K, says that the norms of

$H^{-1/2}(\mathrm{div}, \partial K)$ and $[H^{-1/2}(\mathrm{curl}, \partial K)]^*$ are equal.

Interface (inf=sup) lemma

Lemma [Carstensen+Demkowicz+G 2015]

$$\inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},\Omega)} = \sup_{F \in H(\mathrm{curl},\Omega_h)} \frac{|\langle n \times \hat{H}, F\rangle_h|}{\|F\|_{H(\mathrm{curl},\Omega_h)}}$$

Proof:

Given $n \times \hat{H}$ on element boundary ∂K, solve these:

Find $H \in H(\mathrm{curl}, K)$: $\begin{cases} n \times H = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, H + H = 0, & \text{in } K. \end{cases}$	Find $G \in H(\mathrm{curl}, K)$: $\begin{cases} n \times \mathrm{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, G + G = 0, & \text{in } K. \end{cases}$

One is related to the "inf" and the other is related to the "sup"...

Proof (continued)

Find $H \in H(\mathrm{curl}, K)$:

$$\begin{cases} n \times H = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, H + H = 0, & \text{in } K. \end{cases}$$

Find $G \in H(\mathrm{curl}, K)$:

$$\begin{cases} n \times \mathrm{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, G + G = 0, & \text{in } K. \end{cases}$$

$$\|H\|_{H(\mathrm{curl},K)} = \inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},K)} =: INF.$$

Proof (continued)

Find $H \in H(\mathrm{curl}, K)$:	Find $G \in H(\mathrm{curl}, K)$:
$\begin{cases} n \times H = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, H + H = 0, & \text{in } K. \end{cases}$	$\begin{cases} n \times \mathrm{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, G + G = 0, & \text{in } K. \end{cases}$

$$\|H\|_{H(\mathrm{curl},K)} = \inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},K)} =: INF.$$

$$\|G\|_{H(\mathrm{curl},K)} = \sup_{F \in H(\mathrm{curl},K)} \frac{|(\mathrm{curl}\, G, \mathrm{curl}\, F)_K + (G, F)_K|}{\|F\|_{H(\mathrm{curl},K)}}$$

$$= \sup_{F \in H(\mathrm{curl},K)} \frac{|\langle n \times \hat{H}, F \rangle|}{\|F\|_{H(\mathrm{curl},K)}} =: SUP.$$

Proof (continued)

Find $H \in H(\text{curl}, K)$:

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Find $G \in H(\text{curl}, K)$:

$$\begin{cases} n \times \text{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \text{curl}\,\text{curl}\, G + G = 0, & \text{in } K. \end{cases}$$

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Now, $H = \text{curl}\, G$

Proof (continued)

Find $H \in H(\mathrm{curl}, K)$:	Find $G \in H(\mathrm{curl}, K)$:
$\begin{cases} n \times H = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, H + H = 0, & \text{in } K. \end{cases}$	$\begin{cases} n \times \mathrm{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, G + G = 0, & \text{in } K. \end{cases}$

$$\|H\|_{H(\mathrm{curl},K)} = \inf_{F \in \mathrm{tr}_\times{}^{-1}\{n \times \hat{H}\}} \|F\|_{H(\mathrm{curl},K)} =: INF.$$

$$\|G\|_{H(\mathrm{curl},K)} = \sup_{F \in H(\mathrm{curl},K)} \frac{|(\mathrm{curl}\, G, \mathrm{curl}\, F)_K + (G, F)_K|}{\|F\|_{H(\mathrm{curl},K)}}$$

$$= \sup_{F \in H(\mathrm{curl},K)} \frac{|\langle n \times \hat{H}, F\rangle|}{\|F\|_{H(\mathrm{curl},K)}} =: SUP.$$

Now, $H = \mathrm{curl}\, G$ and $\|H\|_{H(\mathrm{curl},K)} = \|G\|_{H(\mathrm{curl},K)} \implies INF = SUP.$ $\square$

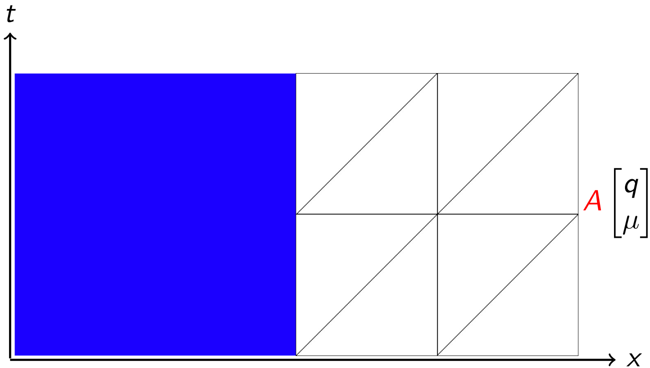
Summary of the technique

- Prove wellposedness (verify **[U+I]** of the *unbroken* often standard – formulation.
- Prove an "inf=sup" lemma to verify **[H]**.
- Conclude the wellposedness **[U+I]** of the *broken* formulation by our abstract theorem.

Spacetime DPG formulations are wellposed

The DPG methodology is well-suited to spacetime problems:

- Ready-made error estimator for spacetime adaptivity.
- Previous technique can be used to prove wellposedness.



$$A \begin{bmatrix} q \\ \mu \end{bmatrix} = \begin{bmatrix} \partial_t q - \operatorname{grad}_x \mu \\ \partial_t \mu - \operatorname{div}_x q \end{bmatrix}$$

Adaptive iterate 0

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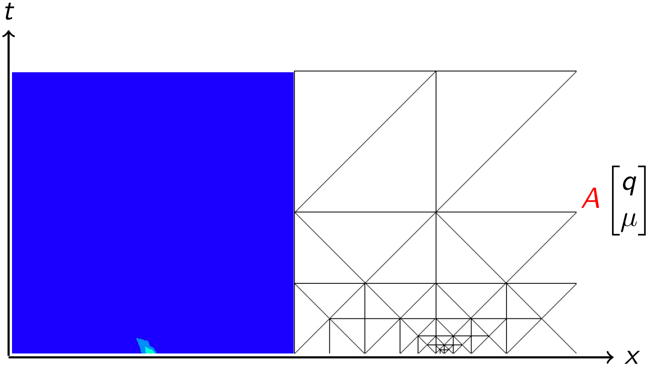

$$A \begin{bmatrix} \boldsymbol{q} \\ \mu \end{bmatrix} = \begin{bmatrix} \partial_t \boldsymbol{q} - \mathrm{grad}_x \mu \\ \partial_t \mu - \mathrm{div}_x \boldsymbol{q} \end{bmatrix}$$

Adaptive iterate 5

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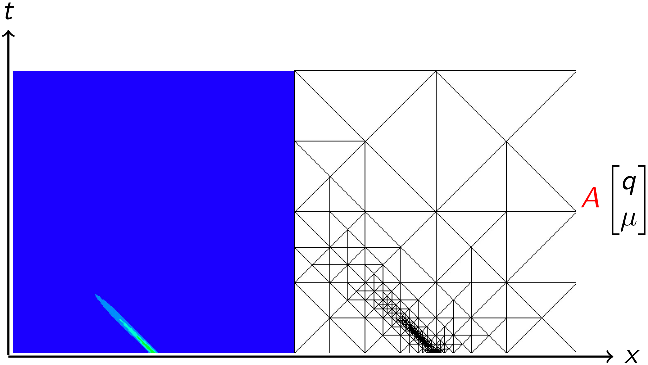

$$A \begin{bmatrix} q \\ \mu \end{bmatrix} = \begin{bmatrix} \partial_t q - \mathrm{grad}_x \mu \\ \partial_t \mu - \mathrm{div}_x q \end{bmatrix}$$

Adaptive iterate 10

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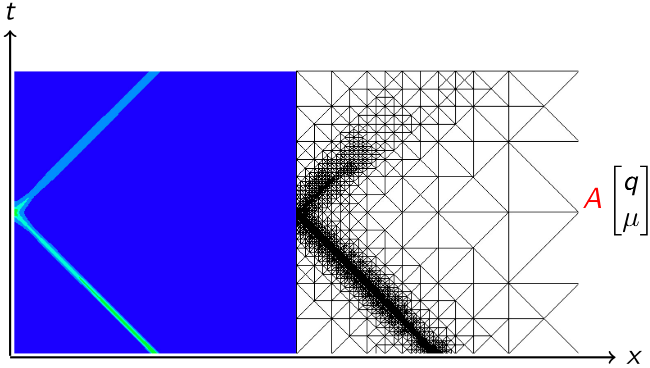

$$A \begin{bmatrix} \boldsymbol{q} \\ \mu \end{bmatrix} = \begin{bmatrix} \partial_t \boldsymbol{q} - \mathrm{grad}_x \mu \\ \partial_t \mu - \mathrm{div}_x \boldsymbol{q} \end{bmatrix}$$

Adaptive iterate 15

Spacetime DPG formulations are wellposed

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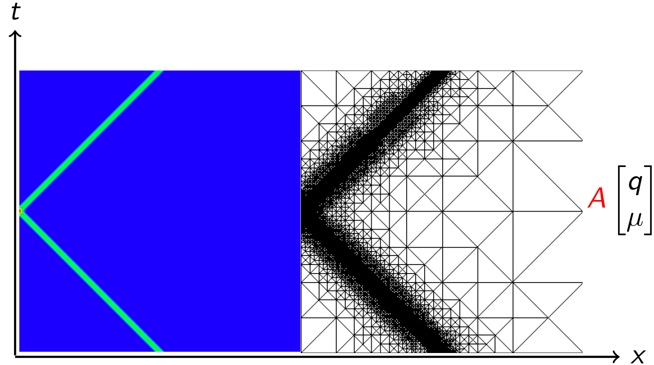

$$A \begin{bmatrix} \boldsymbol{q} \\ \mu \end{bmatrix} = \begin{bmatrix} \partial_t \boldsymbol{q} - \mathrm{grad}_x \mu \\ \partial_t \mu - \mathrm{div}_x \boldsymbol{q} \end{bmatrix}$$

Adaptive iterate 20

The lemma's idea can extended far

Find $H \in H(\mathrm{curl}, K)$: $$\begin{cases} n \times H = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, H + H = 0, & \text{in } K. \end{cases}$$	Find $G \in H(\mathrm{curl}, K)$: $$\begin{cases} n \times \mathrm{curl}\, G = n \times \hat{H}, & \text{on } \partial K, \\ \mathrm{curl}\,\mathrm{curl}\, G + G = 0, & \text{in } K. \end{cases}$$
Find $H \in W(K)$: $$\begin{cases} DH = \hat{q}, & \text{on } \partial K, \\ A^* A\, H + H = 0, & \text{in } K. \end{cases}$$	Find $G \in W^*(K)$: $$\begin{cases} DA^* G = \hat{q}, & \text{on } \partial K, \\ AA^*\, G + G = 0, & \text{in } K. \end{cases}$$

$[Au]_i = \partial^\alpha (a_{ij\alpha} u_j)$.

$W(K) = \{u \in L^2 : Au \in L^2\}$.

$\langle Dw, w^* \rangle_{W^*} = (Aw, w^*) - (w, A^* w^*)$.

Operator A generalizes curl.

$W(K)$ generalizes $H(\mathrm{curl}, K)$.

D generalizes $n \times \cdot|_{\partial K}$.

⬇

INF = SUP

(for much more general operators)

[Demkowicz+G+Nagaraj+Sepulveda 2017]

Next

1. The importance of Y .... ✓
2. "Broken" forms for Laplace & Maxwell equations .... ✓
3. Verification of **[U+I]** .... ✓
4. Verification of **[F]**

Recall the third assumption

Assumption [U] Uniqueness

$$\{y \in Y : \; b(x,y) = 0 \text{ for all } x \in X\} = \{0\}.$$

Assumption [I] Inf-Sup

$$\exists c_1 > 0 : \quad \forall x \in X, \qquad c_1 \|x\|_X \le \sup_{0 \ne y \in Y} \frac{|b(x,y)|}{\|y\|_Y} \le \|b\| \, \|x\|_X.$$

Assumption [F] Fortin Operator

$\exists$ continuous linear $\Pi : Y \to Y_h$ such that

$$b(z_h, y - \Pi y) = 0 \qquad \text{for all } z_h \in X_h, \; y \in Y.$$

Example: Discrete spaces for 3D Laplace case

$$b(\,(u, \hat{q}\cdot n), v) = (\mathrm{grad}\, u, \mathrm{grad}\, v)_h - \langle \hat{q}\cdot n, v\rangle_h$$
$$Y = H^1(\Omega_h)$$
$$X = H^1_0(\Omega) \times Q^{\mathrm{div}}$$

Given an X_h, we want a discrete space Y_h satisfying Assumption **[F]**:

$$\begin{aligned} 0 &= b(\,(w_h, \hat{r}_h\cdot n), v - \Pi v) \\ &= -(\Delta w_h, v - \Pi v)_h + \langle n\cdot \mathrm{grad}\, w_h - \hat{r}_h \cdot n, v - \Pi v\rangle_h. \end{aligned}$$

If degree$(w_h|_K) \le p+1$ and degree$(\hat{r}_h\cdot n) \le p$, then moment conditions

$$(P_{p-1}(K), v - \Pi v)_K = 0 \quad \text{(needed for Laplace example)}$$
$$\langle n\cdot R_{p+1}(K), v - \Pi v\rangle_{\partial K} = 0 \quad \text{(needed for Laplace example)}$$

are sufficient.

Fortin operators with moment conditions

For Maxwell, and other applications, we need continuous linear operators

$$\Pi_{p+3}^{\text{grad}} : H^1(K) \to P_{p+3}(K),$$
$$\Pi_{p+3}^{\text{curl}} : H(\text{curl}, K) \to N_{p+3}(K),$$
$$\Pi_{p+3}^{\text{div}} : H(\text{div}, K) \to R_{p+3}(K),$$

satisfying these **moment conditions** on a tetrahedral element:

$$(P_{p-1}(K), \Pi_{p+3}^{\text{grad}} v - v) = 0$$ ← (needed for Laplace example)

$$\langle n \cdot R_{p+1}(K), \Pi_{p+3}^{\text{grad}} v - v \rangle = 0$$ ← (needed for Laplace example)

$$(P_p(K)^3, \Pi_{p+3}^{\text{curl}} E - E) = 0$$
$$\langle n \times P_{p+1}(K)^3, \Pi_{p+3}^{\text{curl}} E - E \rangle = 0$$
$$(P_{p+1}(K)^3, \Pi_{p+3}^{\text{div}} \tau - \tau) = 0$$
$$\langle n\, P_{p+2}(K), \Pi_{p+3}^{\text{div}} \tau - \tau \rangle = 0$$

Fortin operators with moment conditions

Theorem [Carstensen+Demkowicz+G 2015]

On any tetrahedron K, there are continuous linear operators

$$\Pi^{\text{grad}}_{p+3} : H^1(K) \to P_{p+3}(K),$$
$$\Pi^{\text{curl}}_{p+3} : H(\text{curl}, K) \to N_{p+3}(K),$$
$$\Pi^{\text{div}}_{p+3} : H(\text{div}, K) \to R_{p+3}(K),$$

such that the diagram

$$\begin{array}{ccccccc}
H^1(K)/\mathbb{R} & \xrightarrow{\text{grad}} & H(\text{curl}, K) & \xrightarrow{\text{curl}} & H(\text{div}, K) & \xrightarrow{\text{div}} & L^2(K) \\
\downarrow \Pi^{\text{grad}}_{p+3} & & \downarrow \Pi^{\text{curl}}_{p+3} & & \downarrow \Pi^{\text{div}}_{p+3} & & \downarrow \Pi_{p+2} \\
P_{p+3}(K)/\mathbb{R} & \xrightarrow{\text{grad}} & N_{p+3}(K) & \xrightarrow{\text{curl}} & R_{p+3}(K) & \xrightarrow{\text{div}} & P_{p+2}(K)
\end{array}$$

commutes and the **moment conditions** of the previous slide hold.

The DPG method for the Dirichlet problem

$$b(\,(u, \hat{q}\cdot n), v) = (\mathrm{grad}\, u, \mathrm{grad}\, v)_h - \langle \hat{q}\cdot n, v\rangle_h$$

$$Y = H^1(\Omega_h)$$

$$X = H^1_0(\Omega) \times Q^{\mathrm{div}}$$

$$Y_h = \{\ y \in Y : y|_K \in P_{p+3}(K)\ \}$$

$$X_h = \{\ (w_h, \hat{r}_h \cdot n) \in X :\ w_h|_K \in P_{p+1}(K),\ \hat{r}_h|_K \in R_{p+1}(K)\ \}$$

We have indicated how to verify **[U + I + F]** in this setting.
Hence *a priori* and *a posteriori* error estimates follow.

The DPG method for the Maxwell problem

$$b(\,(E, n \times \hat{H}), F) = (\mu^{-1}\,\mathrm{curl}\,E, \mathrm{curl}\,F)_h - \omega^2(\varepsilon E, F) + \imath\omega\langle n \times \hat{H}, F\rangle_h$$

$$Y = H(\mathrm{curl}, \Omega_h)$$

$$X = H(\mathrm{curl}, \Omega) \times Q^{\mathrm{curl}}$$

$$Y_h = \{F \in Y : F|_K \in N_{p+3}(K)\}$$

$$X_h = \{(E, n \times \hat{H}) \in X : E|_K \in P_p(K)^3, \hat{H}|_K \in P_{p+1}(K)^3\}$$

We have indicated how to verify **[U + I + F]** in this setting.
Hence *a priori* and *a posteriori* error estimates follow.

The DPG method for spacetime problems

- Discussed techniques are useful to prove **[U + I]** also for many spacetime operators (wave, Schrödinger, etc.)
- However, verification of **[F]** is an open problem for spacetime operators.

Conclusion of Lecture 3

1. The importance of Y .... ✓
2. "Broken" forms for Laplace & Maxwell equations .... ✓
3. Verification of **[U+I]** .... ✓
4. Verification of **[F]** .... ✓