The Discontinuous Petrov Galerkin (DPG) Method with Optimal Test Functions: Lecture 2

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The University of Texas at Austin & Portland State University

Spring school, University of South Carolina, February 2018

Thanks: AFOSR

The philosophy

Numerical methods achieve stability in many different ways.

- Standard finite element method: coercivity & conformity
- Mixed methods: balanced pair of spaces
- SUPG methods: artificially added streamline diffusion
- DG methods: upwind stabilization & jump penalization
- HDG methods: difference between interior & interface unknowns

  ⋮ ⋮
- DPG methods: stability by automatic test space design

Outline of Lecture 2

1. Petrov-Galerkin schemes
2. Ideal & practical DPG methods
3. A priori error analysis
4. Least-squares interpretation
5. Interpretation as a mixed method
6. A posteriori error estimate

"Petrov-Galerkin" schemes (PG)

PG schemes are distinguished by different trial and test (Hilbert) spaces.

The problem: [P.D.E.+ boundary conditions.

$\downarrow$

Variational form: [Find x in a trial space X satisfying

$$b(x, y) = \ell(y)$$

for all y in a test space Y.

$\downarrow$

Discretization: [Find x_h in a discrete trial space $X_h \subset X$ satisfying

$$b(x_h, y_h) = \ell(y_h)$$

for all y_h in a discrete test space $Y_h \subset Y$.

For PG schemes, $X_h \neq Y_h$ in general.

Historical remarks

1. [B. G. Galerkin, 1915] "Series occurring in various questions concerning the elastic equilibrium of rods and plates", *Vestnik Inzhenerov*
   (Engineer's Bulletin).

2. [G. I. Petrov, 1940] "Application of the method of Galerkin to a problem involving the stationary flow of a viscous fluid. *Prikl. Matem. i Mekh.*
   (Journal of Applied Mathematics and Mechanics).

3. [S. G. Mikhlin, 1950] "Variational methods of solution of problems of mathematical physics", *Uspekhi Mat. Nauk*
   (Russian Math. Surveys).

Petrov's paper

А К А Д Е М И Я Н А У К С С С Р
USSR ACADEMY OF SCIENCES

ИНСТИТУТ МЕХАНИКИ
ЖУРНАЛ „ПРИКЛАДНАЯ МАТЕМАТИКА И МЕХАНИКА“

INSTITUTE OF MECHANICS
JOURNAL OF APPLIED MATHEMATICS AND MECHANICS

Т. IV, в. 3, 1940

ПРИМЕНЕНИЕ МЕТОДА ГАЛЕРКИНА К ЗАДАЧЕ ОБ УСТОЙЧИВОСТИ ТЕЧЕНИЯ ВЯЗКОЙ ЖИДКОСТИ

Г. И. ПЕТРОВ

(Москва)

При исследовании распространения колебаний в плоском прямолинейном потоке вязкой жидкости и устойчивости таких течений задача приводится к краевой задаче для уравнения:

$$L(\varphi) = \varphi^{IV}(y) - 2\alpha^2 \varphi''(y) + \qquad (A)$$

Petrov's paper

1. Имеем линейное уравнение

$$L(\varphi)-f=0 \tag{1.1}$$

⋮

φ_s — система функций, удовлетворяющая граничным условиям нашей задачи,

v_i — полная система ортогональных функций.

Подставим в уравнение (1.1) линейную комбинацию:

$$\varphi^{(n)}=\sum^{n} a_s \varphi_s; \tag{1.2}$$

⋮

Получим n уравнений для определения n коэффициентов a_s:

$$\int_0^1 L(\varphi^{(n)})\, v_i\, dy-\int_0^1 f v_i\, dy=0. \tag{1.4}$$

Petrov's paper

системы, стремящееся к точному при $n \to \infty$. Если функции v_i взять те же, что и φ_i, то уравнения (1.4) будут уравнениями метода Галеркина[7].

В некоторых случаях удобно пользоваться другой системой функций, так как функции v_i не должны обязательно удовлетворять граничным условиям.

Требование ортогональности принято нами только для удобства вывода

"…If functions v_i are taken to be the same as φ_i, then (1.4) is the equation of Galerkin's method.

In some cases, it can be useful to employ another set of v_i, since v_i need not satisfy the boundary conditions …"

However, there is no example, or analysis, in the rest of the paper with $v_i \neq \varphi_i$.

Mikhlin's review

ВАРИАЦИОННЫЕ МЕТОДЫ
РЕШЕНИЯ ЗАДАЧ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

С. Г. Михлин

СОДЕРЖАНИЕ

§ 1. ВВЕДЕНИЕ

Хорошо известно, что в ряде случаев задача интегрирования дифферен-

Mikhlin's review

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Mikhlin's review

§ 22. ОБОБЩЕНИЕ Г. И. ПЕТРОВА

В статье [13] Г. И. Петров предложил некоторое обобщение метода Галёркина. Это обобщение состоит в следующем: решая уравнение

$$Au - f = 0, \tag{1}$$

мы вводим две координатные последовательности $\{\varphi_n\}$ и $\{\psi_n\}$; приближённое решение строим в виде

$$u_n = \sum_{k=1}^{n} a_k \psi_k$$

и определяем коэффициенты a_k из условия, чтобы левая часть уравнения (1) после подстановки в неё u_n вместо u была ортогональна к $\varphi_1, \varphi_2, \ldots, \varphi_n$. Как и в предшествующих параграфах, мы примем, что $A = A_0 + K$, где A_0 — положительно-определённый самосопряжённый оператор, и $T = A_0^{-1} K$ — вполне-непрерывен в соответствующем пространстве H_0. Далее, мы будем считать, что $\psi_n \in D_A$ и $\varphi_n \in H_0$. Мы сформулируем, не приводя доказательства, результат Н. И. Польского [16], относящийся к проблеме сходимости обобщённого метода Галёркина.

Обозначим через L_n и M_n подпространства H_0, натянутые на элементы $\psi_1, \psi_2, \ldots, \psi_n$ и $\varphi_1, \varphi_2, \ldots, \varphi_n$ соответственно, и через P_n — оператор проектирования на M_n. Допустим, что для любого $u \in L_n$ выполняется неравенство

$$|u| \leqslant C\,|P_n u|, \tag{2}$$

где C — постоянная, которая не зависит от n. Тогда все теоремы §§ 19 и 21 остаются в силе. Условие (2) не только достаточно, но в некотором смысле и необходимо: если оно не выполнено, то, как показывают примеры, может случиться, что либо u_n невозможно построить при бесконечном множестве значений n, либо u_n не стремится к решению уравнения (1).

Mikhlin's review

- Mikhlin refers to Petrov's contribution as the

  "*generalized Galerkin method*".

- Mikhlin says that an analysis of such methods was performed in an early dissertation:
  - [N. I. Pol'sky, 1949] "On the convergence of approximation methods of Galerkin type", *Kiev State University*, Ph. D. dissertation (hand written).

- If anyone knows more about the analysis in this dissertation, please contact me!

Elements of modern theory

- *Variational formulation:* (BNB Theorem)

$$\left[\begin{array}{c}\text{Exact inf-sup condition}\\ c_1\|x\|_X \le \sup\limits_{y\in Y}\dfrac{|b(x,y)|}{\|y\|_Y}\end{array}\right] + \left[\begin{array}{c}\text{adjoint}\\ \text{uniqueness}\end{array}\right] \implies \text{wellposedness}$$

- *Babuška's theorem:* [Babuška 1970], [Xu+Zikatanov 2003]

$$\left[\begin{array}{c}\text{Discrete inf-sup condition}\\ c_2\|x_h\|_X \le \sup\limits_{y_h\in Y_h}\dfrac{|b(x_h,y_h)|}{\|y_h\|_Y}\end{array}\right] \implies \|x-x_h\|_X \le \frac{\|b\|}{c_2}\inf_{w_h\in X_h}\|x-w_h\|_X.$$

- *Difficulty:* Exact inf-sup condition $\not\Longrightarrow$ Discrete inf-sup condition

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- *Difficulty:* Exact inf-sup condition $\not\Longrightarrow$ Discrete inf-sup condition

- Is there a way to find a stable test space for *any* given trial space?

The best test space & ideal DPG method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ satisfying

$$b(x_h, y) = \ell(y), \qquad \text{for all } y \in Y_h^{\text{opt}} \stackrel{\text{def}}{=} T(X_h),$$

where $T : X \mapsto Y$ is defined by $(Tw, y)_Y = b(w, y)$, for all $y \in Y$ and any $w \in X$.

Rationale:

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Rationale:

- *Q:* Which function y maximizes $\dfrac{|b(x,y)|}{\|y\|_Y}$ for any given x ?

$$\sup_{y \in Y} \frac{|b(x,y)|}{\|y\|_Y} = \sup_{y \in Y} \frac{|(Tx, y)_Y|}{\|y\|_Y}$$

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- *A:* $y = Tx$ is the maximizer. $\leftarrow$ Optimal test function.

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DPG Idea: If the discrete test space contains the optimal test functions,

exact inf-sup condition $\Longrightarrow$ discrete inf-sup condition.

Quasioptimality of the ideal DPG Method

Assumption [U] — Uniqueness

$$\{y \in Y : \; b(x,y) = 0 \text{ for all } x \in X\} = \{0\}.$$

Assumption [I] — Inf-Sup

$$\exists c_1 > 0 : \quad \forall x \in X, \qquad c_1 \|x\|_X \le \sup_{0 \ne y \in Y} \frac{|b(x,y)|}{\|y\|_Y} \le \|b\| \, \|x\|_X.$$

Theorem [Demkowicz+G 2011]

Assumptions **[U+I]** $\implies$

$$\|x - x_h\|_X \le \frac{\|b\|}{c_1} \inf_{w_h \in X_h} \|x - w_h\|_X.$$

Proof: Since the discrete inf-sup condition holds, apply Babuška's theorem. □

Example: An ODE

$$\text{1D transport: } \left[\begin{array}{rl} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{array} \right.$$

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1D transport: $\left[\begin{array}{ll} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{array}\right.$

L^2 weak form:

$$\left[\begin{array}{l} \text{Find } u \in L^2, \text{ and a number } \hat{u}_1 \in \mathbb{R}, \text{ satisfying} \\ \underbrace{-\int_0^1 u v' + \hat{u}_1 v(1)}_{b(\,(u,\hat{u}_1),\, v)} = \underbrace{\int_0^1 f v + u_0 v(0)}_{\ell(v)}, \qquad v \in H^1. \\ \text{Trial space: } X = L^2 \times \mathbb{R}, \qquad \|(u, \hat{u}_1)\|_X^2 = \|u\|^2 + |\hat{u}_1|^2. \\ \text{Test space: } Y = H^1, \qquad \|v\|_Y^2 = \|v'\|^2 + |v(0)|^2. \end{array}\right.$$

Example: An ODE

1D transport:

$$\begin{cases} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{cases}$$

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Trial space: $X = L^2 \times \mathbb{R}$, $\qquad \|(u, \hat{u}_1)\|_X^2 = \|u\|^2 + |\hat{u}_1|^2.$

Test space: $Y = H^1$, $\qquad \|v\|_Y^2 = \|v'\|^2 + |v(0)|^2.$

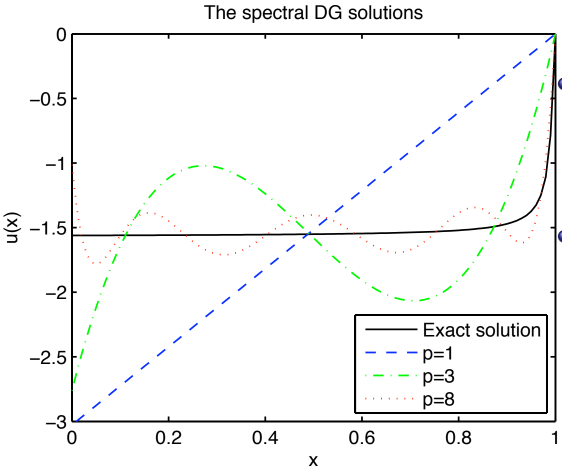
Ideal DPG:

Find $(u_p, \hat{u}_1) \in X_h \equiv P_p \times \mathbb{R}$, satisfying

$$b(\,(u_p, \hat{u}_1), v) = \ell(v), \qquad \text{for all } v \in Y_h^{\text{opt}} = T(X_h).$$

Example: An ODE

Exercise: For this example, $Y_h^{\text{opt}} = P_{p+1}$.



- Experiment: Solve ODE (the transport equation) using DG and ideal DPG on a *single element.*
- Exact solution has a sharp layer at $x = 1$.

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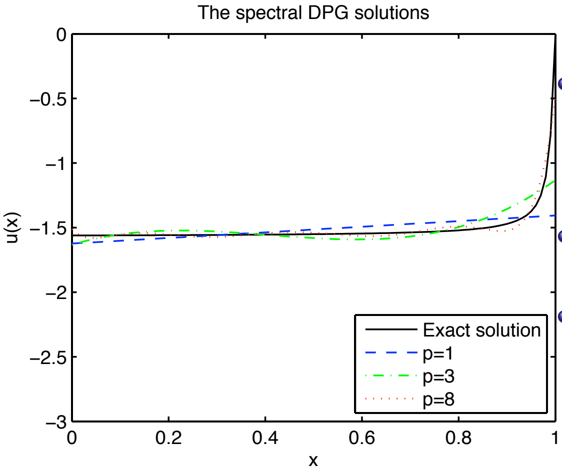

- Experiment: Solve ODE (the transport equation) using DG and ideal DPG on a *single element.*
- Exact solution has a sharp layer at $x = 1$.
- *DPG is more stable.* Solution oscillates an order of magnitude less.

The importance of "D" in "DPG"

- The ideal DPG method requires us to compute the *optimal test space* $Y_h^{\mathrm{opt}} = T(X_h)$ where $T : X \mapsto Y$ is defined by

$$(Tw, y)_Y = b(w, y), \qquad \text{for all } y \in Y, w \in X.$$

- Application of T decouples into element-by-element calculations when Y admits DG functions,

- When Y is infinite-dimensional, we must further approximate T to get a practical method.

The (practical) DPG method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

$$b(x_h, y) = \ell(y), \qquad \text{for all } y \in T(X_h),$$

where $T : X \mapsto Y$ is defined by $(Tw, y)_Y = b(w, y)$, for all $y \in Y$ and any $w \in X$.

Pick any $X_h \subseteq X$. The (practical) DPG method finds $x_h \in X_h$, using a finite-dimensional $Y_h \subseteq Y$, such that

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The (practical) DPG method

Pick any $X_h \subseteq X$. The ideal DPG method finds $x_h \in X_h$ such that

$$x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y^*}.$$

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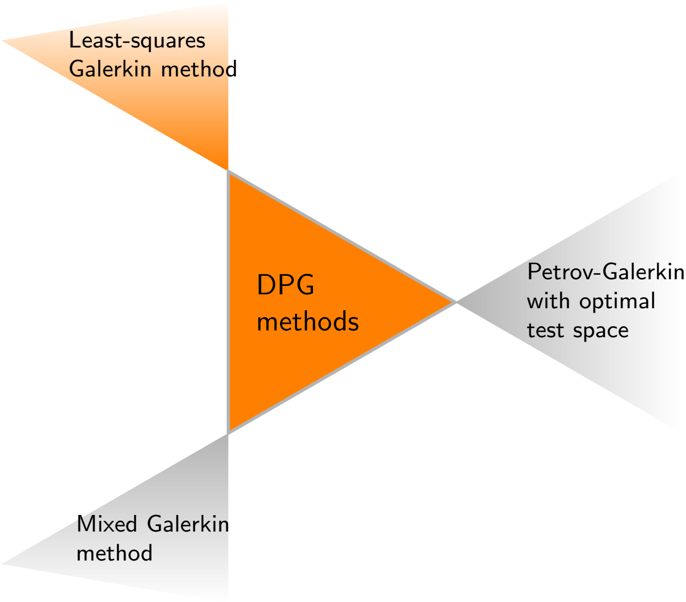
$$x_h^r = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*}.$$

Here $B : X \to Y^*$ is the operator generated by the form $b(x, y)$, i.e., $b(x, y) = (Bx)(y)$ for all $x \in X, y \in Y$.

Three avenues to DPG methods



Equivalent Least-Squares method

Find $x_h \in X_h$ satisfying

$$b(x_h, y) = \ell(y) \qquad \text{for all } y \in T_h(X_h) \tag{1}$$

where $T_h : X \to Y_h$ is defined by $(T_h z, y)_Y = b(z, y)$ for all $y \in Y_h$ for any $z \in X$.

Theorem

$$x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*} \quad \Longleftrightarrow \quad x_h \text{ solves (1).}$$

Equivalent Least-Squares method

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Theorem

$$x_h = \arg\min_{z_h \in X_h} \|\ell - Bz_h\|_{Y_h^*} \quad \Longleftrightarrow \quad x_h \text{ solves (1).}$$

Proof:

$$\begin{aligned} b(x - x_h, T_h z_h) = 0 &\iff (T_h(x - x_h), T_h z_h)_Y = 0 \\ &\iff x_h = \arg\min_{z_h \in X_h} \|T_h(x - z_h)\|_Y \qquad (T_h = R_{Y_h}^{-1} B) \\ &\iff x_h = \arg\min_{z_h \in X_h} \|B(x - z_h)\|_{Y_h^*}. \qquad \square \end{aligned}$$

Assumptions for error analysis

Let $b(x,y) = (Bx)(y)$, the sesquilinear form on $X \times Y$ generated by B.

Assumption [U] Uniqueness

$$\{y \in Y : \ b(x,y) = 0 \text{ for all } x \in X\} = \{0\}.$$

Assumption [I] Inf-Sup

$$\exists c_1 > 0 : \quad \forall x \in X, \qquad c_1 \|x\|_X \le \sup_{0 \ne y \in Y} \frac{|b(x,y)|}{\|y\|_Y} \le \|b\| \, \|x\|_X .$$

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Assumption [F] Fortin Operator

$\exists$ continuous linear $\varPi : Y \to Y_h$ such that

$$b(z_h, y - \varPi y) = 0 \qquad \text{for all } z_h \in X_h, \; y \in Y.$$

A priori error analysis

Theorem [G+Qiu 2013]

Assumptions **[U + I + F]** $\implies$

$$\|x - x_h\|_X \le \frac{\|b\| \|\Pi\|}{c_1} \inf_{z_h \in X_h} \|x - z_h\|_X.$$

A priori error analysis

Theorem [G+Qiu 2013]

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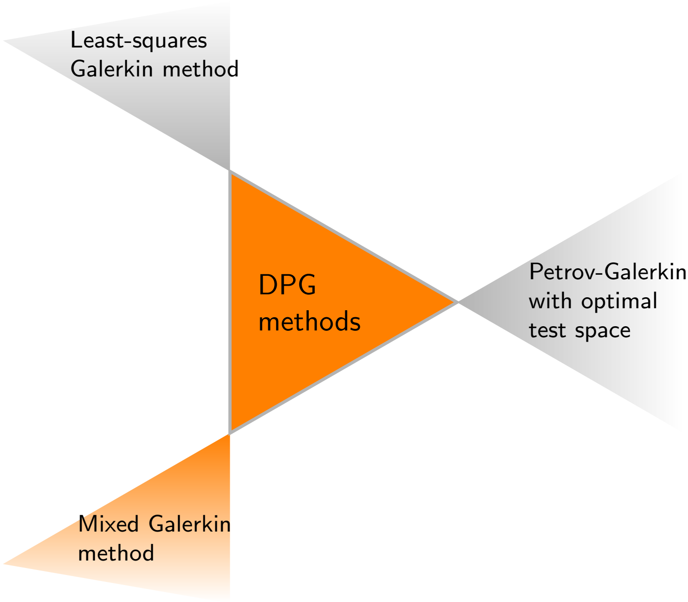
$$\|x - x_h\|_X \le \frac{\|b\|\|\Pi\|}{c_1} \inf_{z_h \in X_h} \|x - z_h\|_X.$$

Proof: For any $x_h \in X_h \subseteq X$,

$$\begin{aligned}
c_1 \|x_h\|_X &\le \sup_{0 \ne y \in Y} \frac{|b(x_h, y)|}{\|y\|_Y} && \text{by } \textbf{[I]} \\
&= \sup_{0 \ne y \in Y} \frac{|b(x_h, \Pi y)|}{\|y\|_Y} && \text{by } \textbf{[F]} \\
&\le \|\Pi\| \sup_{0 \ne y \in Y} \frac{|b(x_h, \Pi y)|}{\|\Pi y\|_Y} \le \|\Pi\| \sup_{0 \ne y_h \in Y_h} \frac{|b(x_h, y_h)|}{\|y_h\|_Y}.
\end{aligned}$$

Now apply the Babuška's theorem. □

The avenue of mixed methods



Equivalent mixed method

The function $e_h \in Y_h$ solving [Dahmen+Huang+Schwab+Welper 2012]

$$(e_h, y)_Y = \ell(y) - b(x_h, y) \qquad \text{for all } y \in Y_h$$

is called the **approximate error representation** function.

Theorem

An $x_h \in X_h$ together with some $e_h \in Y_h$ solves

$$\begin{aligned} (e_h, y)_Y + b(x_h, y) &= \ell(y) && \text{for all } y \in Y_h, \\ b(z, e_h) &= 0 && \text{for all } z \in X_h, \end{aligned}$$

if and only if

$$x_h = \arg\min_{z_h \in X_h} \|\ell - B z_h\|_{Y_h^*}.$$

Equivalent mixed method

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if and only if

$$x_h = \arg\min_{z_h \in X_h} \|\ell - Bz_h\|_{Y_h^*}.$$

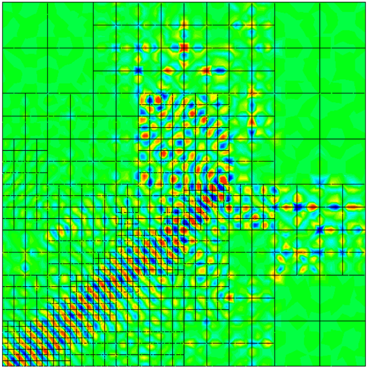
Proof:

$$\begin{aligned} b(z_h, e_h) &= (T_h z_h, e_h)_Y = (T_h z_h, R_{Y_h}^{-1}(\ell - Bx_h))_Y \\ &= (T_h z_h, T_h(x - x_h))_Y = b(x - x_h, T_h z_h) = 0. \end{aligned}$$

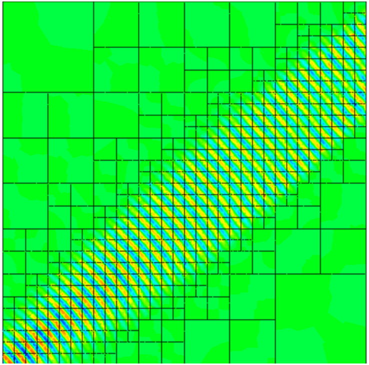
□

Next

Adaptive DPG applied to the Helmholtz equation



Standard finite elements



DPG method

Experiment: Use an adaptive algorithm with standard FEM and DPG methods, for simulating a Gaussian beam solution of Helmholtz equation.

Estimating error

Residual:

$$\rho = \ell - Bx_h.$$

Error representation function:

$$e_h = R_{Y_h}^{-1}(\ell - Bx_h).$$

Error estimator:

$$\eta = \|e_h\|_Y = \|\rho\|_{Y_h^*} = \|\ell - Bx_h\|_{Y_h^*}.$$

Petrov-Galerkin solve → e_h by local postprocessing

Least-squares → e_h is Riesz inverse of residual

Mixed method → e_h is one of the variables

Estimating error

Residual:
$$\rho = \ell - Bx_h.$$

Error representation function:
$$e_h = R_{Y_h}^{-1}(\ell - Bx_h).$$

Error estimator:
$$\eta = \|e_h\|_Y = \|\rho\|_{Y_h^*} = \|\ell - Bx_h\|_{Y_h^*}.$$

When Y_h consists of DG functions:

- e_h can be computed element-by-element

$$(e_h, y)_Y = \ell(y) - b(x_h, y), \qquad \text{for all } y \in Y_h.$$

- Its element-wise norm serves as a local error indicator.

A posteriori error control

Theorem [Carstensen+Demkowicz+G 2014]

Assumptions **[U + I + F]** $\implies$

$$c_1 \|x - x_h\|_X \le \eta^2 + (\|\Pi\|\, \eta + \operatorname{osc}(\ell))^2, \qquad \text{(reliability)}$$
$$\eta \le \|b\|\, \|x - x_h\|_X \qquad \text{(efficiency).}$$

The data approximation $\operatorname{osc}(\ell) = \|\ell \circ (I - \Pi)\|_{Y^*}$ is efficient in the sense

$$\operatorname{osc}(\ell) \le \|B\|\, \|I - \Pi\| \min_{z_h \in X_h} \|x - z_h\|_X.$$

A posteriori error control

Theorem [Carstensen+Demkowicz+G 2014]

Assumptions **[U + I + F]** $\Longrightarrow$

$$c_1 \|x - x_h\|_X \le \eta^2 + (\|\Pi\|\, \eta + \operatorname{osc}(\ell))^2, \qquad \text{(reliability)}$$
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The data approximation $\operatorname{osc}(\ell) = \|\ell \circ (I - \Pi)\|_{Y^*}$ is efficient in the sense

$$\operatorname{osc}(\ell) \le \|B\|\, \|I - \Pi\| \min_{z_h \in X_h} \|x - z_h\|_X.$$

Proof: The efficiency estimate is immediate:

$$\eta = \|B(x - x_h)\|_{Y_h^*} \le \|b\|\, \|x - x_h\|_X.$$

Proof of reliability

To prove $c_1^2\|x - x_h\|_X^2 \le \eta^2 + (\|\Pi\|\,\eta + \operatorname{osc}(\ell))^2$, we use these:

$$\begin{aligned} e \in Y: &\quad (e, y)_Y = \ell(y) - b(x_h, y) && \text{for all } y \in Y,\\ e_h \in Y_h: &\quad (e_h, y)_Y = \ell(y) - b(x_h, y) && \text{for all } y \in Y_h. \end{aligned}$$

Reliability of e is immediate:

$$c_1\|x - x_h\|_X \le \sup_{y\in Y} \frac{|b(x - x_h, y)|}{\|y\|_Y} = \|e\|_Y.$$

But we need reliability of e_h.

Proof of reliability

To prove $c_1^2 \|x - x_h\|_X^2 \le \eta^2 + (\|\Pi\|\,\eta + \text{osc}(\ell))^2$, we use these:

$$
\begin{aligned}
e \in Y: \quad & (e, y)_Y = \ell(y) - b(x_h, y) && \text{for all } y \in Y,\\
e_h \in Y_h: \quad & (e_h, y)_Y = \ell(y) - b(x_h, y) && \text{for all } y \in Y_h.
\end{aligned}
$$

Reliability of e is immediate:

$$
c_1 \|x - x_h\|_X \le \sup_{y \in Y} \frac{|b(x - x_h, y)|}{\|y\|_Y} = \|e\|_Y.
$$

To obtain reliability of e_h, observe that $\|e\|_Y^2 = \|e_h\|_Y^2 + \|\underbrace{e - e_h}_{\delta}\|_Y^2$ and

$$
\begin{aligned}
\|\delta\|_Y^2 &= (\delta, \delta - \Pi\delta)_Y = (e - e_h, \delta - \Pi\delta)_Y \\
&= \ell(\delta - \Pi\delta) - b(x_h, \delta - \Pi\delta) - (e_h,\, \delta - \Pi\delta)_Y
\end{aligned}
$$

□

Proof of reliability

To prove $c_1^2\|x - x_h\|_X^2 \le \eta^2 + (\|\Pi\|\,\eta + \mathrm{osc}(\ell))^2$, we use these:

$$
\begin{aligned}
e \in Y: &\quad (e, y)_Y = \ell(y) - b(x_h, y) &\qquad \text{for all } y \in Y,\\
e_h \in Y_h: &\quad (e_h, y)_Y = \ell(y) - b(x_h, y) &\qquad \text{for all } y \in Y_h.
\end{aligned}
$$

Reliability of e is immediate:

$$
c_1\|x - x_h\|_X \le \sup_{y \in Y} \frac{|b(x - x_h, y)|}{\|y\|_Y} = \|e\|_Y.
$$

To obtain reliability of e_h, observe that $\|e\|_Y^2 = \|e_h\|_Y^2 + \|\underbrace{e - e_h}_{\delta}\|_Y^2$ and

$$
\begin{aligned}
\|\delta\|_Y^2 &= (\delta, \delta - \Pi\delta)_Y = (e - e_h, \delta - \Pi\delta)_Y \\
&= \ell(\delta - \Pi\delta) - b(x_h, \delta - \Pi\delta) - (e_h, \delta - \Pi\delta)_Y \\
&\le \mathrm{osc}(\ell)\|\delta\|_Y + \|e_h\|_Y\|\Pi\|\|\delta\|_Y.
\end{aligned}
$$

□

Remarks on the theorem

- The proof never used the fact the x_h is the DPG solution.
- Hence the theorem holds in fact for any $\tilde{x}_h \in X_h$, such as an inexactly computed solution. The residual $\tilde{\eta} = \|\ell - B\tilde{x}_h\|_{Y_h^*}$ can be used for error estimation.
- The reliability estimate

$$c_1 \|x - x_h\|_X \le \eta^2 + \Big(\eta \|\Pi\| + \operatorname{osc}(\ell\Big)^2$$

was improved to

$$c_1 \|x - x_h\|_X \le \eta^2 + \Big(\eta \sqrt{\|\Pi\|^2 - 1} + \operatorname{osc}(\ell)\Big)^2$$

by [Keith+Vaziriastaneh+Demkowicz 2017] when Π is a projection.

Conclusion of Lecture 2

1. Petrov-Galerkin schemes . . . . ✓
2. Ideal & practical DPG methods . . . . ✓
3. A priori error analysis . . . . ✓
4. Least-squares interpretation . . . . ✓
5. Interpretation as a mixed method . . . . ✓
6. A posteriori error estimate . . . . ✓