Abstract

Determining Starlike Bodies By Their Curvature Integrals

Anamaria Rusu

Let $\mathfrak{A}_k^1(\mathbf{R}^n)$ be the collection of all C^1 starlike and symmetric about the origin bodies in $\mathbf{R}^n$, with the property that for all $k-$dimensional linear subspaces P of $\mathbf{R}^n$, $V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P)$ where $\mathbf{B}^n$ is the Euclidean ball. (That is $K \in \mathfrak{A}_k^1(\mathbf{R}^n)$ is a centrally symmetric starlike about the origin body with C^1 boundary and the property that the $(k-1)$-dimensional "perimeter" of $K \cap P$ is the same as that of $\mathbf{B}^n \cap P$ for all P, k-dimensional linear subspaces of $\mathbf{R}^n$.) In the first Chapter we show that in this class the Euclidean unit ball is isolated in the sense that all one parameter analytic deformations of the unit ball in $\mathfrak{A}_k^1(\mathbf{R}^n)$ are constant. This gives evidence to support the conjecture that if K_1 and K_2 are two starlike symmetric about the origin bodies whose sections by any k-dimensional plane through the origin have equal perimeters, then $K_1 = K_2$, a question posed by Richard Gardner in his book *Geometric Tomography* in the case $k = 2$ and $n = 3$.

In Chapter 2 we generalize and instead of considering perimeters of k-dimensional central sections, we introduce the integral invariants defined by order two $\mathbf{O}(n)$ invariant functions on $\mathbf{R}^n$, where $\mathbf{O}(n)$ is the orthogonal group of degree n over $\mathbf{R}$. An *order two* $\mathbf{O}(n)$ *invariant function* on $\mathbf{R}^n$ is a C^1 function $f\colon (0,\infty) \times \mathbf{R}^n \times \operatorname{sym}(\mathbf{R}^n) \to \mathbf{R}$ such that $f(r, gv, gAg^{-1}) = f(r, v, A)$ for all $g \in \mathbf{O}(n)$. If f is an order two $\mathbf{O}(n)$ invariant function on $\mathbf{R}^n$, then the *integral invariant defined by* f is the function $I_f\colon \mathcal{S}^2(\mathbf{R}^n) \to \mathbf{R}$ given by

$$I_f(K) := \int_{\mathbf{S}^{n-1}} f(\rho_K(u), \nabla \rho_K(u), \nabla^2 \rho_K(u))\, du$$

where ρ_K is the radial function of K and $\mathcal{S}^2(\mathbf{R}^n)$ is the space of all bodies in $\mathbf{R}^n$ that are starlike with respect to the origin and have C^2 boundaries. Let $\mathfrak{B}_k^2(\mathbf{R}^n)$ be the collection of all C^2 starlike and symmetric about the origin bodies in $\mathbf{R}^n$, with the property that for all k-dimensional linear subspaces P of $\mathbf{R}^n$, $I_f(K \cap P) = I_f(\mathbf{B}^n \cap P)$ where I_f is the integral invariant defined by f (an order two $\mathbf{O}(n)$ invariant function on $\mathbf{R}^n$) and $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$. We show that in this class the Euclidean unit ball is isolated in the sense that all one parameter analytic deformations of the unit ball in $\mathfrak{B}^2_k(\mathbf{R}^n)$ are constant. We also prove a Corollary of this result in which we replace the requirement $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$ with I_f being positively homogeneous of nonzero degree.

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Determining Starlike Bodies By Their Curvature Integrals

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Dedication

For my son Daniel and my husband Jack Burg

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Abstract

Let $\mathfrak{A}^1_k(\mathbf{R}^n)$ be the collection of all C^1 starlike and symmetric about the origin bodies in $\mathbf{R}^n$, with the property that for all $k-$dimensional linear subspaces P of $\mathbf{R}^n$, $V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P)$ where $\mathbf{B}^n$ is the Euclidean ball. (That is $K \in \mathfrak{A}^1_k(\mathbf{R}^n)$ is a centrally symmetric starlike about the origin body with C^1 boundary and the property that the $(k-1)$-dimensional "perimeter" of $K \cap P$ is the same as that of $\mathbf{B}^n \cap P$ for all P, k-dimensional linear subspaces of $\mathbf{R}^n$.) In the first Chapter we show that in this class the Euclidean unit ball is isolated in the sense that all one parameter analytic deformations of the unit ball in $\mathfrak{A}^1_k(\mathbf{R}^n)$ are constant. This gives evidence to support the conjecture that if K_1 and K_2 are two starlike symmetric about the origin bodies whose sections by any k-dimensional plane through the origin have equal perimeters, then $K_1 = K_2$, a question posed by Richard Gardner in his book *Geometric Tomography* in the case $k = 2$ and $n = 3$.

In Chapter 2 we generalize and instead of considering perimeters of k-dimensional central sections, we introduce the integral invariants defined by order two $\mathbf{O}(n)$ invariant functions on $\mathbf{R}^n$, where $\mathbf{O}(n)$ is the orthogonal group of degree n over $\mathbf{R}$. An *order two* $\mathbf{O}(n)$ *invariant function* on $\mathbf{R}^n$ is a C^1 function $f\colon (0,\infty) \times \mathbf{R}^n \times \mathrm{sym}(\mathbf{R}^n) \to \mathbf{R}$ such that $f(r, gv, gAg^{-1}) = f(r, v, A)$ for all $g \in \mathbf{O}(n)$. If f is an order two $\mathbf{O}(n)$ invariant function on $\mathbf{R}^n$, then the *integral invariant defined by* f is the function $I_f\colon \mathcal{S}^2(\mathbf{R}^n) \to \mathbf{R}$ given by

$$I_f(K) := \int_{\mathbf{S}^{n-1}} f(\rho_K(u), \nabla\rho_K(u), \nabla^2\rho_K(u))\, du$$

where ρ_K is the radial function of K and $\mathcal{S}^2(\mathbf{R}^n)$ is the space of all bodies in $\mathbf{R}^n$ that are starlike with respect to the origin and have C^2 boundaries. Let $\mathfrak{B}^2_k(\mathbf{R}^n)$ be the collection of all C^2 starlike and symmetric about the origin bodies in $\mathbf{R}^n$, with the property that for all k-dimensional linear subspaces P of $\mathbf{R}^n$, $I_f(K \cap P) = I_f(\mathbf{B}^n \cap P)$ where I_f is the integral invariant defined by f (an order two $\mathbf{O}(n)$ invariant function on $\mathbf{R}^n$) and $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$. We show that in this class the Euclidean unit ball is isolated in the sense that all one parameter analytic deformations of the unit ball in $\mathfrak{B}^2_k(\mathbf{R}^n)$ are constant. We also prove a Corollary of this result in which we replace the requirement $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$ with I_f being positively homogeneous of nonzero degree.

Contents

Introduction and statement of results

This thesis is motivated by the following open problem from Richard Gardner's book *Geometric Tomography*: "Let K_1 and K_2 be centered convex bodies in $\mathbf{R}^3$ whose sections by any plane through the origin have equal perimeters. Is $K_1 = K_2$? If the answer is positive, is the natural generalization to starlike bodies in $\mathbf{R}^n$ true?". This is not even known when one of the bodies is the Euclidean ball. While we can not solve this problem in full generality, we are able to show that it is true in an infinitesimal sense at the unit ball.

Definition 1. A set C in $\mathbf{R}^n$ is called ***convex*** if it contains the closed line segment joining any two of its points, or, equivalently, if $(1-t)x + ty \in C$ whenever x, $y \in C$ and $0 \leqslant t \leqslant 1$.

A ***convex body*** is a compact convex set whose interior is nonempty. Let $\mathcal{C}^n$ be the class of convex bodies in the Euclidean space $\mathbf{R}^n$ that are symmetric about the origin. For $1 \leqslant k \leqslant n-1$, we denote the Grassmann of all k-dimensional linear subspaces of $\mathbf{R}^n$ by $\mathbf{Gr}_k(\mathbf{R}^n)$. To put Gardner's question in a larger context we mention that there are several basic results in geometric tomography to the effect that members K of $\mathcal{C}^n$ are determined by measurements of either sections $K \cap P$ or projections $K|P$ (the usual orthogonal projections on a line or plane) for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$. In geometric tomography there is a remarkable correspondence between projections and central sections. We denote the origin, unit sphere, and closed unit ball in n-dimensional Euclidean space $\mathbf{R}^n$ by O, $\mathbf{S}^{n-1}$, and $\mathbf{B}^n$, respectively. If $u \in \mathbf{S}^{n-1}$, we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u. We write Vol_k for

$k-$dimensional Lebesque measure in $\mathbf{R}^n$, where $k \in \{0, \ldots, n\}$, and where we identify Vol_k with k-dimensional Hausdorff measure (Vol_0 is the counting measure). When no misunderstanding can arise (for example, when working with compact convex sets) we call the Vol_k-measure of a k- dimensional body in $\mathbf{R}^n$ its ***volume***. The notation dz will always mean $d\,\mathrm{Vol}_k(z)$ for the appropriate k with $k \in \{0, \ldots, n\}$. In particular, du signifies integration on $\mathbf{S}^{n-1}$ with respect to Vol_{n-1}, that in $\mathbf{S}^{n-1}$ is identified with spherical Lebesque measure. The notation dP will denote integration on the Grassmann $\mathbf{Gr}_k(\mathbf{R}^n)$ with respect to the canonical invariant probability measure, usually referred to as Haar measure in $\mathbf{Gr}_k(\mathbf{R}^n)$.

DEFINITION 2. The ***k*-th *dimensional intrinsic volume*** of a compact convex body K in $\mathbf{R}^n$ is

$$V_k(K) = c_{n,k} \int_{\mathbf{Gr}_k(\mathbf{R}^n)} \mathrm{Vol}_k(K|P)\, dP \tag{0.0.1}$$

where the constants $c_{n,k}$ depend only on n and k.

THEOREM 3 (Gardner and Volčič). *If $K_1, K_2 \in \mathcal{C}^n$ and $V_k(K_1 \cap P) = V_k(K_2 \cap P)$ for some $k \in \{1, \ldots, n-1\}$ and all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$, then $K_1 = K_2$.*

Informally: convex, symmetric about the origin bodies are determined by areas of their central sections. This is a special case of a much more general theorem of Gardner and Volčič [**3**] (which can also be found in Gardner's book [**1**, Thm 7.2.3, p. 278]). The special case of $k = n-1$ was proven earlier by Larman and Tamvakis [**6**]. See [**1**, Note 7.4, p. 290] for more of the history. Dual to this there is:

THEOREM 4 (Alexandrov). *If $K_1, K_2 \in \mathcal{C}^n$ and for some i, k with $1 \leq i \leq k \leq n-1$ and all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ there holds $V_i(K_1|P) = V_i(K_2|P)$, then $K_1 = K_2$.*

This is the well known Alexandrov projection theorem (cf. [**1**, Thm 3.3.6, p. 115]). Dualizing Alexandrov's projection theorem we have the following:

CONJECTURE 1. *Let* $K_1, K_2 \in \mathcal{C}^n$ *and* $1 \le i \le k-1 \le n-2$ *and assume* $V_i(K_1 \cap P) = V_i(K_2 \cap P)$ *for all* $P \in \mathbf{Gr}_k(\mathbf{R}^n)$. *Then* $K_1 = K_2$.

The case of $n = 3$, $k = 2$ and $i = 1$ is the Gardner's perimeter problem that motivates this dissertation [**1**, Prob. 7.6, p. 289].

The most interesting special case of Conjecture 1 is when one of the bodies is the Euclidean ball $\mathbf{B}^n$ and $i = k-1$. For $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ the intrinsic volume $V_{k-1}(K \cap P)$ is a constant multiple of the surface area of the boundary $\partial(K \cap P)$.

CONJECTURE 2. *Let* $K \in \mathcal{C}^n$ *be so that* $V_{k-1}(K \cap P) = V_{k-1}(\mathbf{B}^n \cap P)$ *for all* $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ *where* $2 \le k \le n-1$. *Then* $K = \mathbf{B}^n$.

Unfortunately we have not been able to even settle this weaker conjecture in full generality. Our main result in Chapter 1 is that it holds in an infinitesimal sense near th Euclidean ball. To make this precise a few definitions are needed.

DEFINITION 5. A set $K \subset \mathbf{R}^n$ is ***starlike*** with respect to the origin O if the origin is an interior point of K and every line through O meets K in a line segment.

DEFINITION 6. If $K \subset \mathbf{R}^n$ is a convex body, or more general a body that is starlike with respect to the origin then its ***radial function*** is defined by $\rho_K\colon \mathbf{S}^{n-1} \to (0, \infty)$ $\rho_K(u) := \sup\{r : ru \in K\}$, where $\mathbf{S}^{n-1}$ is the unit sphere in $\mathbf{R}^n$.

We assume that the radial function is continuous and $u \mapsto \rho_K(u)u$ parametrizes the boundary ∂K, of K. Clearly a body K starlike about the origin is uniquely defined by its radial function and K is symmetric about the origin if and only if ρ is even (that is $\rho_K(-u) = \rho_K(u)$). Let $\mathcal{S}(\mathbf{R}^n)$ be the space of all bodies in $\mathbf{R}^n$ that are starlike with respect to the origin and $\mathcal{S}^\infty$ the elements of $\mathcal{S}(\mathbf{R}^n)$ that have smooth radial functions. Let $\mathcal{S}_{\mathrm{sym}}(\mathbf{R}^n)$ (respectively $\mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$) be the elements of $\mathcal{S}(\mathbf{R}^n)$ (respectively $\mathcal{S}^2(\mathbf{R}^n)$) that are symmetric about the origin.

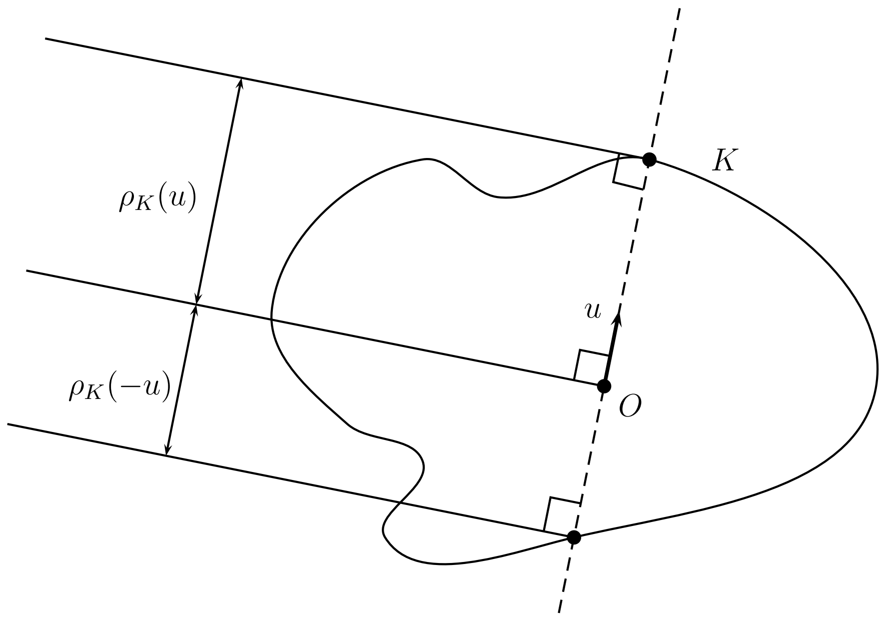

FIGURE 1. The radial function.

Let $C^k(\mathbf{S}^{n-1})$ be the Banach space of all C^k functions $f\colon \mathbf{S}^{n-1} \to \mathbf{R}$ with the norm

$$\|f\|_{C^k} := \sup_{u \in \mathbf{S}^{n-1}} \sum_{i=0}^{k} |\nabla^i f(u)|$$

where $\nabla^0 f(u) = f(u), \ldots, \nabla^k f(u) = \nabla(\nabla^{k-1} f(u))$. We will consider a one parameter family of functions $t \mapsto \rho_t$ in $C^k(\mathbf{S}^{n-1})$ that depends real analytically on t and with $\rho_0 = 1$ (the constant function 1). This means that for some $\delta > 0$ that there are $\rho_m \in C^k(\mathbf{S}^{n-1})$ such that there is a series expansion

$$\rho_t(u) = 1 + \sum_{m=1}^{\infty} \rho_m(u) t^m \tag{0.0.2}$$

which converges absolutely in $C^k(\mathbf{S}^{n-1})$ for $|t| < \delta$. (By absolute convergence in $C^k(\mathbf{S}^{n-1})$ we mean that $\sum_{m=1}^{\infty} \|\rho_m\|_{C^k} |t|^m$ converges.)

Considering a particular case when series (0.0.2) converges absolutely in $C^1(\mathbf{S}^{n-1})$ we can now state our main result from Chapter 1.

THEOREM 7. *Let K_t be a one parameter family of bodies starlike about the origin of $\mathbf{R}^n$ whose radial functions are given by the series* (0.0.2), *which is assumed to* *converge absolutely in $C^1(\mathbf{S}^{n-1})$ for $|t| < \delta$. Assume that each K_t is symmetric about the origin and that for each $P \in \mathbf{Gr}_k(\mathbf{R}^n)$*

$$V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P).$$

Then $\rho_t(u) = 1$ for all t and u and thus each K_t is the Euclidean ball $\mathbf{B}^n$.

Informally: "the unit ball of $\mathbf{R}^n$ is rigid in the class of C^1 bodies through analytic deformations preserving central symmetry and the perimeters of k-dimensional cantral sections". Of course this is far short of settling Conjecture 2, let alone Conjecture 1, but does give supporting evidence for them. It would be interesting to find other C^1 centrally symmetric convex bodies that are rigid in the class of C^1 bodies through analytic deformations preserving central symmetry and the perimeters of k-dimensional central sections. However our proof seems only to work for $\mathbf{B}^n$.

While central symmetry is vital in our proof, it is not clear if it necessary for the perimeters to have the same measure. The simplest open case is:

Problem 8. *Other than the ball $\mathbf{B}^3$, does there exist a convex body, K, in $\mathbf{R}^3$ such that*

$$\operatorname{Length}(P \cap \partial K) = 2\pi$$

for every plane $P \in \mathbf{Gr}_2(\mathbf{R}^3)$?

Another interpretation of our results is in terms of a nonlinear version of the Radon transform. Let $\mathbf{Gr}_k(\mathbf{R}^n)$ be the Grassmann of all k-dimensional linear subspaces of $\mathbf{R}^n$. The ***Radon transform*** is the linear map $R_{n,k} \colon C(\mathbf{S}^{n-1}) \to C(\mathbf{Gr}_k(\mathbf{R}^n))$ given by

$$R_{n,k}\rho(P) := \int_{\mathbf{S}^{n-1}\cap P} \rho(u)\, dV_{k-1}(u).$$

We are also going to use the following result of Helgason [**4**, Thm. 19, p. 289] (cf. [**5**, Thm. 4.7, p. 161]):

THEOREM 9. *If* $\rho\colon \mathbf{S}^{n-1} \to \mathbf{R}$ *is an even continuous function, and* $R_{n,k}\rho = 0$*, then* $\rho = 0$.

Informally: "the Radon transforms determine even continuous functions on the sphere". Minkowski proved the particular case $n = 3$ of Theorem 9.

In Chapter 2 instead of considering perimeters of k-dimensional central sections, we generalize and introduce the integral invariants defined by order two $\mathbf{O}(n)$ invariant functions on $\mathbf{R}^n$, where $\mathbf{O}(n)$ is the orthogonal group of degree n over $\mathbf{R}$. An ***order two*** $\mathbf{O}(n)$ ***invariant function*** on $\mathbf{R}^n$ is a C^1 function $f\colon (0,\infty) \times \mathbf{R}^n \times \mathrm{sym}(\mathbf{R}^n) \to \mathbf{R}$ such that $f(r, gv, gAg^{-1}) = f(r, v, A)$ for all $g \in \mathbf{O}(n)$. If f is an order two $\mathbf{O}(n)$ invariant function on $\mathbf{R}^n$, then the ***integral invariant defined by*** f is the function $I_f\colon \mathcal{S}^2(\mathbf{R}^n) \to \mathbf{R}$ given by

$$I_f(K) := \int_{\mathbf{S}^{n-1}} f(\rho_K(u), \nabla\rho_K(u), \nabla^2\rho_K(u))\, du$$

where ρ_K is the radial function of K. We are interested in I_f transforms when f has some natural invariance properties and determine even positive C^2 functions on the sphere. The main result in Chapter 2 is the following theorem that generalizes Theorem 7.

THEOREM 10. *Let* K_t *be a one parameter family of* C^2 *boundaries starlike bodies in* $\mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$ *whose radial functions are given by the series 0.0.2 that converges absolutely in* $C^2(\mathbf{S}^{n-1})$*. Assume that for all* $P \in \mathbf{Gr}_l(\mathbf{R}^n)$

$$I_f(K_t \cap P) = I_f(\mathbf{B}^n \cap P)$$

where I_f *is the integral invariant defined by* f *(an order two* $\mathbf{O}(n)$ *invariant function on* $\mathbf{R}^n$*) and* $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$*. Then each* K_t *is the Euclidean unit ball of* $\mathbf{R}^n$*.*

In other words the unit ball of $\mathbf{R}^n$ is rigid in the class of C^2 starlike bodies through analytic deformations preserving central symmetry and integral invariants of k-dimensional central sections. We also prove a Corollary of Theorem 10 in which we replace the requirement $\frac{d}{dt}I_f(t\mathbf{B}^n)|_{t=1} \neq 0$ with I_f being positively homogeneous of nonzero degree. So our results need invariance and homogeneity but are only perturbation results that still do not prove Conjecture 2.

Chapter 1

Determining starlike bodies by the perimeters of their sections with linear subspaces of $\mathbf{R}^n$

1.1. Preliminaries

If $f\colon \mathbf{R}^n \to \mathbf{R}$, then the ***Euclidean gradient*** of f is the vector function ∂f defined by $\partial f(x_1, x_2, \ldots, x_n) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ and the ***Euclidean Hessian*** of f is the square matrix of the second-order partial derivatives of f

$$\partial^2 f(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

We also want the gradient and Hessian for functions defined on $\mathbf{S}^{n-1}$. Let X and Y be vector fields on $\mathbf{S}^{n-1}$. Then denote by $\partial_X Y$ the directional derivative of Y in the direction X. Explicitly, to compute $\partial_X Y$ at the point $p \in \mathbf{S}^{n-1}$, choose a smooth curve $c\colon (-\varepsilon, \varepsilon) \to \mathbf{S}^{n-1}$ with $c(0) = p$ and $c'(0) = X(p)$. Then

$$\partial_X Y\big|_p = \frac{d}{dt} Y(c(t))\bigg|_{t=0}.$$

In this formula it is not required that Y be tangent to $\mathbf{S}^{n-1}$. A case of interest is the position vector u on $\mathbf{S}^{n-1}$, which is also the inclusion map of $\mathbf{S}^{n-1}$ into $\mathbf{R}^n$. Therefore

$$\partial_X u = \frac{d}{dt} u(c(t))\bigg|_{t=0} = \frac{d}{dt} c(t)\bigg|_{t=0} = c'(0) = X.$$

It is elementary that for smooth vector fields X, Y, Z on $\mathbf{S}^{n-1}$ that the product rule

$$\partial_X \langle Y, Z\rangle = \langle \partial_X Y, Z\rangle + \langle Y, \partial_X Z\rangle$$

holds. The ***covariant derivative***, $\nabla_X Y$, of Y by X is the orthogonal projection of $\partial_X Y$ onto the tangent space of $\mathbf{S}^{n-1}$. That is for $u \in \mathbf{S}^{n-1}$

$$\nabla_X Y(u) = \partial_X Y - \langle \partial_X Y, u\rangle u,$$

which, by the product rule, that $\langle Y, u\rangle = 0$ and $\partial_X u = X$ implies

$$\begin{aligned}
\partial_X Y - \nabla_X Y &= \langle \partial_X Y, u\rangle u \\
&= \big(\partial_X \langle Y, u\rangle - \langle Y, \partial_X u\rangle\big) u \\
&= -\langle X, Y\rangle u.
\end{aligned}$$

If $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is a C^1 function, then ∇h is the vector field tangent to $\mathbf{S}^{n-1}$ such that for all vector X tangent to $\mathbf{S}^{n-1}$

$$dh(X) = \langle \nabla h, X\rangle.$$

This is the ***spherical gradient*** of h, which we will just refer to as the ***gradient***. Another method to define ∇h is to extend it to $\mathbf{R}^n \setminus \{0\}$ to be homogeneous of degree nonzero, that is as

$$\tilde{h}(x) = h(\|x\|^{-1} x)$$

then if $\partial \tilde{h}$ is the usual Euclidean gradient its restriction to $\mathbf{S}^{n-1}$ agrees with ∇h. The ***spherical Hessian***, or just ***Hessian***, of a C^2 function $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is the field of linear maps on tangent spaces to $\mathbf{S}^{n-1}$ given by

$$\nabla^2 h(u) X := \nabla_X \nabla h.$$

This is self-adjoint in the sense that

$$\langle \nabla^2 h X, Y\rangle = \langle X, \nabla^2 h Y\rangle.$$

It can also be given a definition in terms of the Euclidean Hessian of the extended function $\tilde{h}$ defined above. If $\partial^2\tilde{h}$ is the Euclidean Hessian on $\mathbf{R}^n \setminus \{0\}$, which is a symmetric matrix and thus also be viewed as a self-adjoint linear map on $\mathbf{R}^n$. For $u \in \mathbf{S}^{n-1}$, $\partial^2\tilde{h}(u)u = 0$ and as $\partial^2\tilde{h}(u)$ is self-adjoint and u is an eigvenvector of $\partial^2\tilde{h}(u)$, the orthogonal compliment $u^\perp = T_u\mathbf{S}^{n-1}$ is also invariant under $\partial^2\tilde{h}(u)$. The restriction of $\partial^2\tilde{h}(u)$ to $T_u\mathbf{S}^{n-1}$ agrees with $\nabla^2 h$. The ***Laplacian*** of h is

$$\Delta h := \operatorname{tr}(\nabla^2 h).$$

The following is a standard corollary of the divergence theorem on compact oriented manifolds.

PROPOSITION 11. *Let $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$ be a C^2 function. Then*

$$\int_{\mathbf{S}^{n-1}} \operatorname{tr}(\nabla^2 h)\, du = \int_{\mathbf{S}^{n-1}} \Delta h\, du = 0$$

where du is the volume measure on $\mathbf{S}^{n-1}$. $\square$

Let $\mathbf{O}(n)$ be the orthogonal group of degree n over $\mathbf{R}$.

PROPOSITION 12. *Let $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$, $h \in C^2(\mathbf{S}^{n-1})$, $g \in \mathbf{O}(n)$, and $\tilde{h} = h \circ g$. Then*

$$\nabla\tilde{h}(u) = g\nabla h(g^{-1}u)$$

and

$$\nabla^2\tilde{h}(u) = g\nabla^2 h(g^{-1}u)g^{-1}$$

PROOF. Let X be a vector field on $\mathbf{S}^{n-1}$. Choose a smooth curve $c\colon (-\varepsilon, \varepsilon) \to \mathbf{S}^{n-1}$ with $c(0) = p$ and $c'(0) = X(p)$. Then

$$\begin{aligned}\langle \nabla\tilde{h}(u), X\rangle &= \frac{d}{dt}\tilde{h}(c(t))|_{t=0}\\ &= \frac{d}{dt}(h \circ g)(c(t))|_{t=0}\\ &= \frac{d}{dt}h(g^{-1}c(t))|_{t=0}\end{aligned}$$

$$= \langle \nabla h(g^{-1}u), g^{-1}X \rangle$$

$$= \langle g\nabla h(g^{-1}u), X \rangle$$

Therefore, the gradient is invariant under orthogonal transformations

$$\nabla \tilde{h}(u) = g\nabla h(g^{-1}u).$$

Also

$$\begin{aligned}
\nabla^2 \tilde{h}(u) &= \nabla(\nabla \tilde{h}(u)) \\
&= \nabla(g\nabla h(g^{-1}u)) = \\
&= g\nabla^2 h(g^{-1}u)\nabla(g^{-1}u) \\
&= g\nabla^2 h(g^{-1}u)g^{-1}.
\end{aligned}$$

$\square$

Proposition 13. *Let $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$ be a $C^1(\mathbf{S}^{n-1})$ function and $u \in \mathbf{S}^{n-1}$. If $f\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ is given by $f(u) := h(u)u$, then the Jacobian of f is*

$$J(f)(u) = h^{n-2}\sqrt{h^2 + |\nabla h|^2}.$$

Proof. We first compute the derivative of f. Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis of the tangent space $T_u\mathbf{S}^{n-1}$ (or, what is the same thing, $e_1, \ldots, e_{n-1}$ is an orthonormal basis of $P \cap u^\perp$). On this set of basis vectors the derivative is given by

$$f'(u)e_j = he_j + dh(e_j)u.$$

Using that $u \wedge u = 0$,

$$\begin{aligned}
f'(u)e_1 \wedge \cdots \wedge f'(u)e_{n-1} &= (he_1 + dh(e_1)u) \wedge \cdots \wedge (he_{n-1} + dh(e_{n-1})u) \\
&= h^{n-1}e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \\
&\quad + h^{n-2}\sum_{i=1}^{n-1} dh(e_i)e_1 \wedge \ldots \wedge e_{i-1} \wedge u \wedge e_{i+1} \wedge \cdots \wedge e_{n-1}.
\end{aligned}$$

As $\{u, e_1, e_2, \ldots, e_{n-1}\}$ is an orthonormal set in $\mathbf{R}^n$, the set

$$\{e_1 \wedge \cdots \wedge e_{n-1}, u \wedge e_2 \wedge \cdots \wedge e_{n-1}, e_1 \wedge u \wedge e_3 \wedge \cdots \wedge e_{n-1}, \ldots, e_1 \wedge e_2 \wedge \ldots e_{n-2} \wedge u\}$$

is orthonormal in $\bigwedge^{n-1} \mathbf{R}^n$. Whence the Jacobian is

$$\begin{aligned} J(f)(u) &= |f'(u)e_1 \wedge f'(u)e_2 \wedge \cdots \wedge f'(u)e_{n-1}| \\ &= \sqrt{\left(h^{n-1}\right)^2 + \sum_{i=1}^{n-1} \left(h^{n-2} dh(e_i)\right)^2} \\ &= h^{n-2} \sqrt{h^2 + \sum_{i=1}^{n-1} dh(e_i)^2} \\ &= h^{n-2} \sqrt{h^2 + |\nabla h|^2}. \end{aligned}$$

$\square$

Definition 14. Let $f \colon \mathbf{S}^{n-1} \to \mathbf{R}$ be a C^∞ function and x^0 be any point in $\mathbf{S}^{n-1}$. The series

$$\sum_{(\alpha_1,\ldots,\alpha_n)} \frac{D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} f(x^0)}{\alpha_1! \alpha_2! \ldots \alpha_n!} (x_1 - x_1^0)^{\alpha_1} (x_2 - x_2^0)^{\alpha_2} \ldots (x_n - x_n^0)^{\alpha_n} \tag{1.1.1}$$

is called the ***Taylor series*** of f about x^0. In (1.1.1), $D_j = \frac{\partial}{\partial x_j}$, and α_j is a non-negative integer, $j = 1, \ldots, n$. Thus

$$D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} f = \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

The summation in (1.1.1) is taken over all n-tuples of integers greater or equal to 0, $(\alpha_1, \ldots, \alpha_n)$. The series (1.1.1) can be written in a shorter form if we introduce the notation

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n) \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \\ D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} \end{aligned}$$

$$\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

Then the Taylor series of f about x^0 can be written in the form

$$\sum_{|\alpha|\geqslant 0} \frac{D^\alpha f(x^0)}{\alpha!}(x - x^0)^\alpha. \tag{1.1.2}$$

We will also need the following result:

Lemma 15. *Let $x_1, x_2, \dots$ be a sequence of variables and assume*

$$\alpha = \sum_{m\geqslant 1} \alpha_m(x_1, \dots, x_m)t^m, \qquad \beta = \sum_{m\geqslant 1} \beta_m(x_1, \dots, x_m)t^m$$

where $\alpha_m(x_1, \dots, x_m)$ and $\beta_m(x_1, \dots, x_m)$ are polynomials in $x_1, \dots, x_m$ with the property that $\alpha_m(0, \dots, 0) = \beta_m(0, \dots, 0) = 0$. Then the product is of the form

$$\alpha\beta = \sum_{m\geqslant 2} P_m(x_1, \dots, x_{m-1})t^m$$

where $P_m(x_1, \dots, x_{m-1})$ is a polynomial in $x_1, \dots, x_{m-1}$ and $P_m(0, \dots, 0) = 0$.

The main point is that in the product the coefficient of t^m depends on one fewer of the variables x_j and that it vanishes if $x_1, \dots, x_{m-1}$ all vanish.

Proof. The coefficient of t^m in $\alpha\beta$ is

$$P_m = \sum_{j=1}^{m-1} \alpha_j(x_1, \dots, x_j)\beta_{m-j}(x_1, \dots, x_{m-j})$$

which clearly has the stated properties. □

1.2. Determining starlike bodies by the perimeters of their central sections

Theorem 16. *Let K_t be a one parameter family of bodies starlike about the origin of $\mathbf{R}^n$ whose radial functions are given by the series* (0.0.2)*, which is assumed to* *converge absolutely in* $C^1(\mathbf{S}^{n-1})$ *for small* t*. Assume that each* K_t *is symmetric about the origin and that for each* $P \in \mathbf{Gr}_k(\mathbf{R}^n)$

$$V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P).$$

Then $\rho_t(u) = 1$ *for all* t *and* u *and thus each* K_t *is the Euclidean ball* $\mathbf{B}^n$*.*

Proof. Let K_t be a one parameter family of C^1 starlike bodies in $\mathbf{R}^n$ symmetric about the origin and with radial functions ρ_t given by the series (0.0.2). The symmetry of the K_t implies that ρ_t is an even function on $\mathbf{S}^{n-1}$ for all t. The series expansion (0.0.2) then implies that each coefficient ρ_m is an even function. Let $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ and denote by $\mathbf{S}^{k-1}$ the intersection $P \cap \mathbf{S}^{n-1}$ and let h_t^P be the restriction $\rho_t\big|_{P\cap\mathbf{S}^{n-1}}$. It then follows from (0.0.2) that

$$\begin{aligned} h_t^P &= 1 + \sum_{m\geqslant 1} \rho_m\big|_{\mathbf{S}^{k-1}} t^m \\ &= 1 + \sum_{m\geqslant 1} h_m^P t^m \end{aligned}$$

where $h_m^P := \rho_m\big|_{\mathbf{S}^{k-1}}$. Now assume for some $\delta > 0$ that $h = h_t^P$ depends on a parameter $t \in [-\delta, \delta]$ and has a convergent (in $C^1(\mathbf{S}^{k-1})$) expansion

$$h = 1 + \sum_{m\geqslant 1} h_m t^m.$$

Then

$$\nabla h = \sum_{m\geqslant 1} \nabla h_m t^m$$

and this series converges uniformly.

Meanwhile, for $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ the quantity $V_{k-1}(\partial K_t \cap P)$ is just the $(k-1)$-dimensional "perimeter"(that is the surface area, or $(k-1)$-dimensional Hausdorff measure) of $K_t \cap P$. The function $f\colon \mathbf{S}^{k-1} \to \mathbf{R}^k$ given by

$$f(u) := h(u)u$$

parameterizes $\partial K_t \cap P$. The area of $\partial K_t \cap P$ can be computed by the usual calculus method of integrating the Jacobian of f over its domain. Using Proposition 13 we have that:

$$
\begin{aligned}
(1.2.1) \qquad V_{k-1}(\partial K_t \cap P) &= \int_{\mathbf{S}^{k-1}} J(f)(u)\, du \\
&= \int_{\mathbf{S}^{k-1}} h^{k-2}\sqrt{h^2 + |\nabla h|^2}\, du
\end{aligned}
$$

where du is the surface area (*i.e.* $(k-1)$-dimensional Hausdorff) measure on $\mathbf{S}^{k-1}$. We now expand the integral of (1.2.1) as a power series in t. The exact form of the coefficients will not be needed, only certain elementary properties of them will be required. In what follows we will use Lemma 15 and an obvious variant for series with vector coefficients, repeatedly without quoting it explicitly. Also we use the notation $Q = Q(x_1, \ldots, x_{m-1})$ to indicate that Q is a *polynomial* in $x_1, \ldots, x_{m-1}$ (when some or all of the x_j's are vectors this means that Q is a polynomial in the components of the vectors). First

$$
\begin{aligned}
h^2 &= \Big(1 + \sum_{m \geqslant 1} h_m t^m\Big)^2 \\
&= 1 + 2\sum_{m \geqslant 1} h_m t^m + \Big(\sum_{m \geqslant 1} h_m t^m\Big)^2 \\
&= 1 + 2\sum_{m \geqslant 1} h_m t^m + \sum_{m \geqslant 2} A_m t^m
\end{aligned}
$$

where $A_m = A_m(h_1, \ldots, h_{m-1})$ and $A_m(0, \ldots, 0) = 0$. Likewise

$$
(1.2.2) \qquad h^{k-2} = 1 + (k-2)\sum_{m \geqslant 1} h_m t^m + \sum_{m \geqslant 2} B_m t^m
$$

where $B_m = B_m(h_1, \ldots, h_{m-1})$ and $B_m(0, \ldots, 0) = 0$ (when $k = 2$ each B_m is the zero polynomial). Also

$$
|\nabla h|^2 = \Big|\sum_{m \geqslant 1} \nabla h_m t^m\Big|^2
$$

$$= \sum_{m\geqslant 2} C_m t^m$$

where $C_m = C_m(\nabla h_1, \dots, \nabla h_{m-1})$ and $C_m(0, \dots, 0) = 0$. Adding gives

$$h^2 + |\nabla h|^2 = 1 + 2\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2}(A_m + C_m)t^m.$$

By the binomial theorem

$$\begin{aligned}
\sqrt{h^2 + |\nabla h|^2} &= \Bigg(1 + 2\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2}(A_m + C_m)t^m\Bigg)^{1/2} \\
&= 1 + \frac{1}{2}\Bigg(2\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2}(A_m + C_m)t^m\Bigg) \\
&\qquad + \sum_{\ell\geqslant 2}\binom{1/2}{\ell}\Bigg(2\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2}(A_m + C_m)t^m\Bigg)^{\ell} \\
&= 1 + \sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2} D_m t^m
\end{aligned}$$

where $D_m = D_m(h_1, \dots, h_{m-1}, \nabla h_1, \dots, \nabla h_{m-1})$ and $D_m(0, \dots, 0, 0, \dots, 0) = 0$. This and (1.2.2) gives

$$\begin{aligned}
h^{k-2}\sqrt{h^2 + |\nabla h|^2} &= \Bigg(1 + (k-2)\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2} B_m t^m\Bigg) \\
&\qquad \times \Bigg(1 + \sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2} D_m t^m\Bigg) \\
&= 1 + (k-1)\sum_{m\geqslant 1} h_m t^m + \sum_{m\geqslant 2} E_m t^m \qquad (1.2.3)
\end{aligned}$$

where $E_m = E_m(h_1, \dots, h_{m-1}, \nabla h_1, \dots, \nabla h_{m-1})$ and $E_m(0, \dots, 0, 0, \dots, 0) = 0$.

The hypothesis of Theorem 16 is that for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$, $V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P)$. But $V_{k-1}(\partial \mathbf{B}^n \cap P) = V_{k-1}(\mathbf{S}^{k-1})$. In light of (1.2.1) and (1.2.3) $V_{k-1}(\partial K_t \cap P) = V_{k-1}(\partial \mathbf{B}^n \cap P)$ is the same as

$$\begin{aligned}
V_{k-1}(\mathbf{S}^{k-1}) &= \int_{\mathbf{S}^{k-1}} h^{k-2}\sqrt{h^2 + |\nabla h|^2}\, du \\
&= V_{k-1}(\mathbf{S}^{k-1}) + (k-1)t\int_{\mathbf{S}^{k-1}} h_1(u)\, du \qquad (1.2.4)
\end{aligned}$$

$$+\sum_{m\geqslant 2} t^m \int_{\mathbf{S}^{k-1}} ((k-1)h_m + E_m)\,du$$

Equating the coefficients of t^m on the two sides of (1.2.4) for $m \geqslant 1$ gives

$$\int_{\mathbf{S}^{k-1}} h_1\,du = 0 \tag{1.2.5}$$

$$\int_{\mathbf{S}^{k-1}} ((k-1)h_m + E_m)\,du = 0 \qquad \text{for } m \geqslant 2 \tag{1.2.6}$$

for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$. The first of these is equivalent to

$$\int_{\mathbf{S}^{n-1}\cap P} \rho_1\,du = 0$$

for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$. As ρ_1 is an even function this implies $\rho_1 \equiv 0$ by Theorem 9. Now assume that $\rho_j \equiv 0$ for $j = 1, 2, \ldots, m-1$. Then for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ and $j = 1, 2, \ldots, m-1$ we have $h_j \equiv 0$ and thus also $\nabla h_j = 0$ which in turn implies $E_m = E_m(0, \ldots, 0, 0, \ldots, 0) \equiv 0$. Using this in (1.2.6) yields that for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$

$$\int_{\mathbf{S}^{n-1}\cap P} \rho_m\,du = \int_{\mathbf{S}^{k-1}} h_m\,du = 0.$$

As ρ_m is an even function, we again use Theorem 9 to conclude $\rho_m \equiv 0$. Inductively $\rho_m = 0$ for all $m \geqslant 1$. Thus $\rho_t(u) = 1$ which completes the proof of Theorem 16. □

If we let $n = 3$ and $k = 2$ in Theorem 16 then we obtain the following Corollary:

Corollary 17. *Let K_t be a one parameter family of bodies starlike about the origin of $\mathbf{R}^3$ whose radial functions are given by the series* (0.0.2). *Assume that each K_t is symmetric about the origin and that for each plane $P \in \mathbf{Gr}_2(\mathbf{R}^3)$*

$$\text{perimeter}(K_t \cap P) = 2\pi.$$

Then $\rho_t(u) = 1$ for all t and u and thus each K_t is just the Euclidean unit ball of $\mathbf{R}^3$.

Chapter 2

Determining starlike bodies by their curvature integrals

2.1. Invariantaly defined second order integral invariants.

We wish to define integral invariants of bodies starlike about the origin that are invariant under orthogonal maps. We restrict our selves to bodies that have C^2 boundaries and integrands that only depend on the first two derivatives of the radial function of the body. If V is a finite dimensional real inner product space on let $\operatorname{sym}(V)$ be the space of all selfadjoint linear maps on V.

Definition 18. An ***order two*** $\mathbf{O}(n)$ ***invariant function*** on $\mathbf{R}^n$ is a C^1 function $f\colon (0,\infty)\times \mathbf{R}^n \times \operatorname{sym}(\mathbf{R}^n) \to \mathbf{R}$ such that

$$f(r, gv, gAg^{-1}) = f(r, v, A)$$

for all $g \in \mathbf{O}(n)$. We denote the space of all order two $\mathbf{O}(n)$ invariant functions on $\mathbf{R}^n$ by $\mathcal{F}_2(\mathbf{R}^n)$. □

Remark 19. Here "order two" refers to second derivatives. In our applications we will be evaluating f at $(\rho, \nabla\rho, \nabla^2\rho)$ where ρ is the radial function of a starlike body. □

Remark 20. If V is any n-dimensional real vector space and $f \in \mathcal{F}_2(\mathbf{R}^n)$, and $(r, v, A) \in (0,\infty)\times V\times \operatorname{sym}(V)$ is well defined. To see this let $e_1', \dots, e_n'$ and $e_1'', \dots, e_n''$ be two orthonormal basis of V. Then v and A have coordinate vector and matrix representations representations v', v'', A' and A'' with respect to these basis. Let $g \in \mathbf{O}(n)$ be the change of basis matrix between $e'_1, \dots, e'_n$ and $e''_1, \dots, e''_n$. Then $v'' = gv'$ and $A'' = gA'g^{-1}$. Therefore

$$\begin{aligned} f(r, v'', A'') &= f(r, gv', gA'g^{-1}) \\ &= f(r, v', A'). \end{aligned}$$

Thus $f(r, v, A)$ can be defined as $f(r, v', A')$ with respect to any orthonormal basis $e'_1, \dots, e'_n$ of V. $\square$

DEFINITION 21. Let $f \in \mathcal{F}_2(\mathbf{R}^n)$. Then the ***integral invariant defined by*** f is the function $I_f \colon \mathcal{S}^2(\mathbf{R}^n) \to \mathbf{R}$ given by

$$I_f(K) := \int_{\mathbf{S}^{n-1}} f(\rho_K(u), \nabla \rho_K(u), \nabla^2 \rho_K(u))\, du$$

where ρ_K is the radial function of K. (This is well defined by Remark 20) $\square$

All the usual types of curvature (Gauss-Kronecker curvature, mean curvature etc.) are covered by this definition. See Section 2.2.

PROPOSITION 22. *If $f \in \mathcal{F}_2(\mathbf{R}^n)$, $K \in \mathcal{S}^2(\mathbf{R}^n)$ then $I_f(K)$ is invariant under $\mathbf{O}(n)$ in the sense that*

$$I_f(gK) = I_f(K)$$

for all $g \in \mathbf{O}(n)$.

PROOF. Let $f \in \mathcal{F}_2(\mathbf{R}^n)$, $K \in \mathcal{S}^2(\mathbf{R}^n)$ and $g \in \mathbf{O}(n)$. Using Definition 6 we have

$$\begin{aligned} \rho_{gK}(u) &= \sup\{r : ru \in gK\} \\ &= \sup\{r : rg^{-1}u \in K\} \\ &= \rho_K(g^{-1}u) \end{aligned}$$

Using Definition 21 and Proposition 12 we have

$$
\begin{aligned}
I_f(gK) &:= \int_{\mathbf{S}^{n-1}} f(\rho_{gK}(u), \nabla \rho_{gK}(u), \nabla^2 \rho_{gK}(u))\, du \\
&= \int_{\mathbf{S}^{n-1}} f(\rho_K(g^{-1}u), g\nabla \rho_K(g^{-1}u), g\nabla^2 \rho_K(g^{-1}u)g^{-1})\, du
\end{aligned}
$$

We make the change of variable $g^{-1}u = v$. Then $u = gv$ and thus $du = dv$ since g is an orthogonal transformation. Using Definition 18 and continuing the above calculations we have

$$
\begin{aligned}
I_f(gK) &= \int_{\mathbf{S}^{n-1}} f(\rho_K(v), g\nabla(\rho_K)(v), g\nabla^2(\rho_K)(v)g^{-1})\, dv \\
&= \int_{\mathbf{S}^{n-1}} f(\rho_K(v), \nabla \rho_K(v), \nabla^2 \rho_K(v))\, dv \\
&= I_f(K)
\end{aligned}
$$

□

Let $f \in \mathcal{F}_2(\mathbf{R}^n)$. Then f has a formal first order Taylor series expansion about $(1,0,0) \in \mathbf{R} \times \mathbf{R}^n \times \mathrm{sym}(\mathbf{R}^n)$ of the form

$$
\begin{aligned}
f(r,v,A) =& f(1,0,0) + f_{1,0,0}(r-1) + f_{0,1,0}v + f_{0,0,1}A \\
&+ C(r-1,v,A)\sqrt{(r-1)^2 + \|v\|^2 + \|A\|^2}
\end{aligned}
\tag{2.1.1}
$$

where

$$
\begin{aligned}
f_{1,0,0}(r,0,0) &= \frac{d}{dt} f(1+tr,0,0)|_{t=0}, \\
f_{0,1,0}(1,v,0) &= \frac{d}{dt} f(1,tv,0)|_{t=0}, \\
f_{0,0,1}(1,0,A) &= \frac{d}{dt} f(1,0,tA)|_{t=0} \quad \text{and}
\end{aligned}
$$

$$
\lim_{(r,v,A)\to(1,0,0)} C(r-1,v,A) = 0
$$

as $C(r-1,v,A)$ is the remainder of the Taylor series.

Proposition 23. *Let $f \in \mathcal{F}_2(\mathbf{R}^n)$, I_f the integral invariant defined by f, and $\mathbf{B}^n$ the Euclidean ball in $\mathbf{R}^n$. Then $f_{1,0,0}(1,0,0) \neq 0$ is equivalent to $\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} \neq 0$.*

Proof. Let $f \in \mathcal{F}_2(\mathbf{R}^n)$. Then $f\colon (0,\infty) \times \mathbf{R}^n \times \mathrm{sym}(\mathbf{R}^n) \to \mathbf{R}$ and according to Definition 21 the integral invariant defined by f for $t\mathbf{B}^n$ is

$$I_f\colon \mathcal{S}^2(\mathbf{R}^n) \to \mathbf{R}$$

and

$$\begin{aligned} I_f(t\mathbf{B}^n) &:= \int_{\mathbf{S}^{n-1}} f(\rho_{t\mathbf{B}^n}(u), \nabla \rho_{t\mathbf{B}^n}(u), \nabla^2 \rho_{t\mathbf{B}^n}(u))\, du \\ &= \int_{\mathbf{S}^{n-1}} f(t,0,0)\, du \end{aligned}$$

since $\rho_{t\mathbf{B}^n}\colon \mathbf{S}^{n-1} \to (0,\infty)$ $\rho_{t\mathbf{B}^n}(u) := \sup\{r : ru \in t\mathbf{B}^n\} = t$. Consider:

$$\begin{aligned} \frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} &= \frac{d}{dt} \int_{\mathbf{S}^{n-1}} f(t,0,0)\, du|_{t=1} \\ &= \int_{\mathbf{S}^{n-1}} \frac{d}{dt} f(t,0,0)|_{t=1}\, du \\ &= \int_{\mathbf{S}^{n-1}} \frac{d}{dt} f(1+rt,0,0)|_{t=0}\, du \\ &= \int_{\mathbf{S}^{n-1}} f_{1,0,0}(1,0,0)\, du \end{aligned}$$

and the result is now clear. □

Corollary 24. *Let $f \in \mathcal{F}_2(\mathbf{R}^n)$, I_f the integral invariant defined by f, and $\mathbf{B}^n$ the Euclidean ball in $\mathbf{R}^n$. If I_f is positively homogeneous of degree $a \neq 0$ (that is $I_f(\lambda K) = \lambda^a I_f(K)$ for $\lambda > 0$) and $I_f(\mathbf{B}^n) \neq 0$ then $f_{1,0,0}(1,0,0) \neq 0$.*

Proof. Let $f \in \mathcal{F}_2(\mathbf{R}^n)$ and I_f the integral invariant defined by f for $t\mathbf{B}^n$. Since I_f is homogeneous of degree $a \neq 0$ we have:

$$I_f(t\mathbf{B}^n) = t^a I_f(\mathbf{B}^n)$$

for $t > 0$. Then

$$\begin{aligned}
\frac{d}{dt} I_f(t\mathbf{B}^n)|_{t=1} &= \frac{d}{dt}(t^a I_f(\mathbf{B}^n))|_{t=1} \\
&= at^{a-1} I_f(\mathbf{B}^n)|_{t=1} \\
&= aI_f(\mathbf{B}^n) \\
&\neq 0
\end{aligned}$$

since $a \neq 0$ and $I_f(\mathbf{B}^n) \neq 0$ by hypothesis. Applying Proposition 23 we obtain

$$f_{1,0,0}(1,0,0) \neq 0$$

□

Proposition 25. *Let $L\colon \mathbf{R}^n \to \mathbf{R}$ linear functional invariant under $\mathbf{O}(n)$ in the sense that $L(gv) = L(v)$ for all $g \in \mathbf{O}(n)$. Then*

$$L(v) = 0$$

.

Proof. Let $L\colon \mathbf{R}^n \to \mathbf{R}$ linear functional. Since L is invariant under $O(n)$ we have $L(gv) = L(v)$ for all $g \in \mathbf{O}(n)$. Let $g = -I$. Then $L(-v) = L(v)$. Using that L is linear we have $-L(v) = L(v)$. Thus $L(v) = 0$. □

Proposition 26. *Let $L\colon \mathrm{sym}(\mathbf{R}^n) \to \mathbf{R}$ linear functional invariant under $\mathbf{O}(n)$ in the sense that $L(gAg^{-1}) = L(A)$ for all $g \in \mathbf{O}(n)$. Then*

$$L(A) = \lambda\,\mathrm{tr}(A)$$

for some $\lambda \in \mathbf{R}$.

Proof. Let $L\colon \mathrm{sym}(\mathbf{R}^n) \to \mathbf{R}$ linear functional and $g \in \mathbf{O}(n)$. By hypothesis we have $L(gAg^{-1}) = L(A)$. Using that L is linear we can write

$$L(gAg^{-1} - A) = 0.$$

This means $(gAg^{-1} - A) \in \mathrm{Ker}(L)$. Then

$$\mathrm{Ker}(L) \supseteq \mathrm{Span}\{gAg^{-1} - A | A \in \mathrm{sym}(\mathbf{R}^n),\ g \in \mathbf{O}(n)\}$$

Using the fact that the trace is invariant under similarity transformation,

$$\mathrm{tr}(gAg^{-1} - A) = 0.$$

We have:

$$\begin{aligned}
&\mathrm{Span}\{gAg^{-1} - A | A \in \mathrm{sym}(\mathbf{R}^n),\ g \in \mathbf{O}(n)\} \\
&\supseteq \mathrm{Span}\{E_{ij} | i,j = 1,\dots,n,\ i \neq j,\ 1 \,\mathrm{on}\, (i,j)^{\mathrm{th}} \,\mathrm{position},\ 0 \,\mathrm{in\ rest}\} \\
&\qquad \cup \mathrm{Span}\{E_i | i = 2,\dots,n,\ (-1) \,\mathrm{on}\, (1,1) \,\mathrm{position},\ 1 \,\mathrm{on}\, (i,i)^{\mathrm{th}} \,\mathrm{position},\ 0 \,\mathrm{in\ rest}\} \\
&= \mathrm{Span}\{A |\, \mathrm{tr}(A) = 0\} \\
&= \mathrm{Ker}(\mathrm{tr})
\end{aligned}$$

Since $\mathrm{Ker}(\mathrm{tr}) \subseteq \mathrm{Ker}(L)$ then $L(A) = \lambda\,\mathrm{tr}(A)$ for some $\lambda \in \mathbf{R}$ and all $A \in \mathrm{sym}(\mathbf{R}^n)$.

□

2.2. The Weingarten map of starlike body in terms of the radial function.

Let K be a body in $\mathbf{R}^n$ that is starlike about the origin. Then the ***radial function*** of K is the function $\rho\colon \mathbf{S}^{n-1} \to (0, \infty)$ such that $\varphi\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ given by

$$\varphi(u) = \rho(u)u$$

parametrizes ∂K. We assume that ρ is C^2. The of φ derivative is

$$\varphi'(u)Y = \langle \nabla\rho, Y\rangle u + \rho Y$$

and the second derivative is

$$\varphi''(u)XY := \partial_X(\varphi'(u)Y) - \varphi'(u)\nabla_X Y$$

$$
\begin{aligned}
&= \partial_X(\langle \nabla\rho, Y\rangle u + \rho Y) - (\langle \nabla\rho, \nabla_X Y\rangle u + \rho\nabla_X Y) \\
&= \langle \nabla^2 X, Y\rangle u + \langle \nabla\rho, \nabla_X Y\rangle u + \langle \nabla\rho, Y\rangle X + \langle \nabla\rho, X\rangle Y + \rho\partial_X Y \\
&\quad - (\langle \nabla\rho, \nabla_X Y\rangle u + \rho\nabla_X Y) \\
&= \langle \nabla^2\rho X, Y\rangle u + \langle \nabla\rho, Y\rangle X + \langle \nabla\rho, X\rangle Y + \rho\,(\partial_X Y - \nabla_X Y) \\
&= \langle \nabla^2\rho X, Y\rangle u + \langle \nabla\rho, Y\rangle X + \langle \nabla\rho, X\rangle Y - \rho\langle X, Y\rangle u
\end{aligned}
$$

For $X \in T_u\mathbf{S}^{n-1}$ the derivative of φ in the direction X is

$$\varphi'(u)X = \langle \nabla\rho(u), X\rangle u + \rho(u)X.$$

Choose an orthonormal basis $e_1, \ldots, e_{n-2}$ of $\nabla\rho(u)^\perp$ in $T_u\mathbf{S}^{n-1}$. Then

$$e_1, \ldots, e_{n-2}, e_{n-1} := \|\nabla\rho\|^{-1}\nabla\rho(u)$$

is an orthonormal basis of $T_u\mathbf{S}^{n-1}$,

$$\varphi'(u)e_j = \rho(u)e_j \qquad \text{for} \qquad 1 \le j \le n-2$$

and

$$
\begin{aligned}
\varphi'(u)e_{n-1} &= \varphi'(u)\|\nabla\rho\|^{-1}\nabla\rho(u) \\
&= \|\nabla\rho(u)\|u + \rho(u)\|\nabla\rho\|^{-1}\nabla\rho(u) \\
&= \|\nabla\rho\|u + \rho e_{n-1}.
\end{aligned}
$$

Therefore with respect to basis $e_1, \ldots, e_{n-1}$ the first fundamental form of φ is given by the diagonal matrix with entries $g_{ij} = \langle \varphi'(u)e_i, \varphi'(u)e_j\rangle$

$$
[g_{i\,j}] = \begin{bmatrix}
\rho^2 & 0 & \cdots & 0 & 0 \\
0 & \rho^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \rho^2 & 0 \\
0 & 0 & \cdots & 0 & \rho^2 + \|\nabla\rho\|^2
\end{bmatrix}.
$$

Whence the volume element is

$$d\,\mathrm{Vol}_{n-1} = \sqrt{\det[g_{i\,j}]}\,du = \rho^{n-2}\sqrt{\rho^2 + \|\nabla\rho\|^2}\,du.$$

The vectors $E_1, \ldots, E_{n-1}$ given by

$$E_1 := \frac{1}{\rho}e_1, \ldots, E_{n-2} := \frac{1}{\rho}e_{n-2}, E_{n-1} := \frac{1}{\rho^2 + \|\nabla\rho\|^2}(\|\nabla\rho\|u + \rho e_{n-1})$$

are an orthonormal basis of $T_{\varphi(u)}\partial K$. The vector field

$$\rho(u)u - \nabla\rho(u)$$

is easily checked to be orthogonal to all of $\varphi'(u)e_j$ for $1 \leq j \leq n-1$. Therefore

$$\nu(u) = \frac{1}{\sqrt{\rho^2 + \|\nabla\rho\|^2}}(\rho(u)u - \nabla\rho(u))$$

is the unit normal field along φ. The components of the second fundamental form are, by definition,

$$L_{i\,j} := \langle\varphi''(u)e_i e_j, \nu\rangle.$$

For $1 \leq i, j \leq n-2$

$$L_{i\,j} = \big\langle\langle\nabla^2\rho e_i, e_j\rangle u - \rho\delta_{i\,j}u, \nu\big\rangle = \big(\langle\nabla^2\rho e_i, e_j\rangle - \rho\delta_{i\,j}\big)\,\langle u, \nu\rangle,$$

for $1 \leq i \leq n-2$

$$L_{i\,n-1} = L_{n-1\,i} = \big\langle\langle\nabla^2\rho e_i, e_{n-1}\rangle u + \|\nabla\rho\|e_i, \nu\big\rangle = \langle\nabla^2\rho e_i, e_{n-1}\rangle\langle u, \nu\rangle,$$

and

$$\begin{aligned} L_{n-1\,n-1} &= \big\langle\langle\nabla^2\rho e_{n-1}, e_{n-1}\rangle u + 2\langle\nabla\rho, e_{n-1}\rangle e_{n-1} - \rho u, \nu\big\rangle \\ &= \big(\langle\nabla^2\rho, e_{n-1}, e_{n-1}\rangle - \rho\big)\,\langle u, \nu\rangle + 2\|\nabla\rho\|\langle e_{n-1}, \nu\rangle \end{aligned}$$

The components of the Weingarten map are (see [**8**, vol.4, pp. 50–51])

$$L_i^j = \sum_k g^{i\,k}L_{k\,j}$$

where $[g^{ij}]$ is the matrix inverse to $[g_{ij}]$. This is

$$[g^{ij}] = \begin{bmatrix} \frac{1}{\rho^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\rho^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\rho^2 + \|\nabla\rho\|^2} \end{bmatrix}.$$

And so

$$L_i^j = \frac{L_{ij}}{\rho^2 + \delta_{i\,n-1}\|\nabla\rho\|^2}$$

A chase through the definitions shows that the matrix $L = [L_i^j]$ is positively homogeneous of degree -1 in ρ. That is if $\lambda > 0$ is constant and ρ is replaced by $\lambda\rho$, then L is replaced by $\lambda^{-1}L$. Let $\sigma_k(L)$ be the k-th elementary function of the eigenvalues of L. These can be defined by

$$\det(I + tL) = 1 + \sigma_1(L)t + \sigma_2(L)t^2 + \cdots + \sigma_{n-2}(L)t^{n-2} + \sigma_{n-1}(L)t^{n-1}.$$

Then $\sigma_k(L)$ is positively homogeneous of degree $-k$. Recall that the intrinsic volumes V_k of a convex body with C^2 boundary can be computed in terms of the Weingarten map of the boundary ∂K:

Proposition 27. *If K is a convex body with C^2 boundary, then the intrinsic volumes $V_k(K)$ are given in terms of the Weingarten map by*

$$V_k(K) = c_{n,k} \int_{\partial K} \sigma_{n-k-1}(L)\, dx$$

where the constant $c_{n,k}$ only depends on n and k and dx is $(n-1)$-dimensional volume measure on ∂K.

To put this in our frame of work let $Q_k(\rho, \nabla\rho, \nabla^2\rho)$ be the function

$$Q_k(\rho, \nabla\rho, \nabla^2\rho) := \sigma_{n-k-1}(L)\rho^{n-2}\sqrt{\rho^2 + \|\nabla\rho\|^2}.$$

Using the formula for the volume element given above we have

PROPOSITION 28. *The function Q_k is a second order invariant that is homogeneous of degree k. There are constants $c_{n,k}$ such that the intrinsic volumes are given by*

$$V_k(K) = c_{n,k} \int_{\mathbf{S}^{n-1}} Q_k(\rho, \nabla\rho, \nabla^2\rho)\, du = I_{Q_k}(K).$$

where ρ is the radial function of K, starlike body about the origin in $\mathbf{R}^n$.

2.3. MAIN RESULT

THEOREM 29. *Let K_t be a one parameter family of C^2 boundaries starlike bodies in $\mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$ whose radial functions are given by the series 0.0.2 that converges absolutely in $C^2(\mathbf{S}^{n-1})$. Assume that for all $P \in \mathbf{Gr}_l(\mathbf{R}^n)$*

$$I_f(K_t \cap P) = I_f(\mathbf{B}^n \cap P) \tag{2.3.1}$$

where I_f is the integral invariant defined by $f \in \mathcal{F}_2(\mathbf{R}^l)$ and $f_{1,0,0}(1,0,0) \neq 0$. Then each K_t is the Euclidean unit ball of $\mathbf{R}^n$.

REMARK 30. The requirement $f_{1,0,0}(1,0,0) \neq 0$ is very important to avoid the trivial case when $I_f(K_t \cap P)$ is constant. Consider $K_t \in \mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$ with the corresponding radial functions ρ_t, $P \in \mathbf{Gr}_l(\mathbf{R}^n)$ and $f(\rho_t \nabla\rho_t, \nabla^2\rho_t) = \mathrm{tr}(\nabla^2\rho_t)$. Notice that $f \in \mathcal{F}_2(\mathbf{R}^l)$. Then

$$\begin{aligned} I_f(K_t \cap P) &= \int_{\mathbf{S}^{n-1}\cap P} f(\rho_t, \nabla\rho_t, \nabla^2\rho_t)\, du \\ &= \int_{\mathbf{S}^{n-1}\cap P} \mathrm{tr}(\nabla^2\rho_t)\, du \\ &= \int_{\mathbf{S}^{n-1}\cap P} \triangle\rho_t\, du = 0 \end{aligned}$$

by Proposition 11 and thus the Euclidean ball is not isolated in the class of $\mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$ bodies.

PROOF. Let $K_t \in \mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$. Note that $\rho_t(-u) = \rho_t(u)$ for all t as each K_t is centrally symmetric. Thus $\rho_m(-u) = \rho_m(u)$ for all $m \geqslant 1$ and therefore each ρ_m is an even function on $\mathbf{S}^{n-1}$. Let $P \in \mathbf{Gr}_l(\mathbf{R}^n)$. Then $\mathbf{S}^{n-1} \cap P = \mathbf{S}^{l-1}$. If we denote $h_t^P := \rho_t\big|_{\mathbf{S}^{n-1}\cap P}$ then h_t^P is the radial function of $K_t \cap P$ and it follows from (0.0.2) that

$$\begin{aligned} h_t^P &= 1 + \sum_{m\geqslant 1} \rho_m\big|_{\mathbf{S}^{n-1}\cap P} t^m \\ &= 1 + \sum_{m\geqslant 1} h_m^P t^m, \end{aligned}$$

where $h_m^P := \rho_m\big|_{\mathbf{S}^{n-1}\cap P}$. Now assume for some $\delta > 0$ that $h = h_t^P$ depends on a parameter $t \in [-\delta, \delta]$ and has a convergent (in $C^2(\mathbf{S}^{l-1})$) expansion

$$h = 1 + \sum_{m\geqslant 1} h_m t^m.$$

Then

$$\nabla h = \sum_{m\geqslant 1} \nabla h_m t^m$$

and

$$\nabla^2 h = \sum_{m\geqslant 1} \nabla^2 h_m t^m$$

and these series converge uniformly. Define

$$\begin{aligned} h^{[k]} &= 1 + \sum_{m\geqslant k} h_m t^m \\ &= 1 + h_k t^k + t^{k+1} R^{[k]}(t) \end{aligned}$$

where

$$R^{[k]}(t) = \sum_{m\geqslant k+1} h_m t^{m-k-1}.$$

Then

$$\begin{aligned} \nabla h^{[k]} &= \sum_{m\geqslant k} \nabla h_m t^m \\ &= \nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t) \end{aligned}$$

where

$$\nabla R^{[k]}(t) = \sum_{m \geqslant k+1} \nabla h_m t^{m-k-1},$$

and

$$\begin{aligned}\nabla^2 h^{[k]} &= \sum_{m \geqslant k} \nabla^2 h_m t^m \\ &= \nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t).\end{aligned}$$

where

$$\nabla^2 R^{[k]}(t) = \sum_{m \geqslant k+1} \nabla^2 h_m t^{m-k-1}.$$

Writing $h^{[k]} = h^{[k]}(u)$, $\nabla h^{[k]} = \nabla h^{[k]}(u)$, $\nabla^2 h^{[k]} = \nabla^2 h^{[k]}(u)$, and using the Taylor expansion (2.1.1) for f we have:

$$f\big(h^{[k]}, \nabla h^{[k]}, \nabla^2 h^{[k]}\big) = f\big(1 + h_k t^k + t^{k+1} R^{[k]}(t), \nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t),$$

$$\nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t)\big) = f(1,0,0) + f_{1,0,0}(h_k t^k + t^{k+1} R^{[k]}(t))$$

$$+ f_{0,1,0}(\nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t)) + f_{0,0,1}(\nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t))$$

$$+ C(h_k t^k + t^{k+1} R^{[k]}(t), \nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t), \nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t))$$

$$\sqrt{(h_k t^k + t^{k+1} R^{[k]}(t))^2 + \|\nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t)\|^2 + \|\nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t)\|^2}$$

$$= f(1,0,0) + t^k\big(f_{1,0,0} h_k + f_{0,1,0} \nabla h_k + f_{0,0,1} \nabla^2 h_k\big) + t^k \Bigg(f_{1,0,0} t R^{[k]}$$

$$+ f_{0,1,0} t \nabla R^{[k]} + f_{0,0,1} t \nabla^2 R^{[k]} + C(h_k + t R^{[k]}(t), \nabla h_k + t \nabla R^{[k]}(t), \nabla^2 h_k + t \nabla^2 R^{[k]}(t))$$

$$\sqrt{(h_k t^k + t^{k+1} R^{[k]}(t))^2 + \|\nabla h_k t^k + t^{k+1} \nabla R^{[k]}(t)\|^2 + \|\nabla^2 h_k t^k + t^{k+1} \nabla^2 R^{[k]}(t)\|^2} \Bigg)$$

$$= f(1,0,0) + t^k (f_{1,0,0} h_k + f_{0,1,0} \nabla h_k + f_{0,0,1} \nabla^2 h_k) + o(t^k).$$

The integral invariant defined by f is

$$\begin{aligned} I_f(K_t \cap P) &:= \int_{\mathbf{S}^{l-1}} f(h^{[k]}, \nabla h^{[k]}, \nabla^2 h^{[k]})\, du \\ &= \int_{\mathbf{S}^{l-1}} \Big(f(1,0,0) + t^k (f_{1,0,0} h_k + f_{0,1,0} \nabla h_k + f_{0,0,1} \nabla^2 h_k) + o(t^k) \Big)\, du. \end{aligned}$$

Note that

$$I_f(\mathbf{B}^n \cap P) = \int_{\mathbf{S}^{l-1}} f(\rho(u), \nabla\rho(u), \nabla^2\rho(u))\, du,$$

where $\rho(u)$ is the radial function of $\mathbf{B}^n \cap P = \mathbf{B}^l$. Since $\rho(u) = 1$ for $\mathbf{B}^l$ the above integral is $\int_{\mathbf{S}^{l-1}} f(1,0,0)\, du$. Using condition (2.3.1) of the hypothesis we have

$$\int_{\mathbf{S}^{l-1}} f(1,0,0)\, du + t^k \int_{\mathbf{S}^{l-1}} \left(f_{1,0,0}h_k + f_{0,1,0}\nabla h_k + f_{0,0,1}\nabla^2 h_k\right) du$$
$$+ \int_{\mathbf{S}^{l-1}} o(t^k)\, du = \int_{\mathbf{S}^{l-1}} f(1,0,0)\, du.$$

Subtracting $\int_{\mathbf{S}^{l-1}} f(1,0,0)\, du$ on both sides of the equation and dividing by t^k we obtain

$$\int_{\mathbf{S}^{l-1}} \left(f_{1,0,0}h_k + f_{0,1,0}\nabla h_k + f_{0,0,1}\nabla^2 h_k\right) du + \frac{1}{t^k}\int_{\mathbf{S}^{l-1}} o(t^k)\, du = 0.$$

Let $t \to 0$. Then

$$\int_{\mathbf{S}^{l-1}} \left(f_{1,0,0}h_k + f_{0,1,0}\nabla h_k + f_{0,0,1}\nabla^2 h_k\right) du = 0. \tag{2.3.2}$$

Using Proposition 25 for $L = f_{0,1,0}$ we have that $f_{0,1,0} = 0$ at $(1,0,0)$. Applying Proposition 26 for $L = f_{0,0,1}$ we have that for some $c \in \mathbf{R}$

$$f_{0,0,1}(1,0,0,)\nabla^2 h_k = c\,\mathrm{tr}(\nabla^2 h_k) = c\triangle h_k,$$

where $\triangle$ is the Laplace-Beltrami operator. Then by Proposition 11

$$\int_{\mathbf{S}^{l-1}} f_{0,0,1}\nabla^2 h_k\, du = c\int_{\mathbf{S}^{l-1}} \triangle h_k\, du = 0.$$

Equation (2.3.2) is thus equivalent to

$$\int_{\mathbf{S}^{l-1}} f_{1,0,0}h_k\, du = 0.$$

As $h_k = \rho_k|_{\mathbf{S}^{n-1}\cap P}$ we have

$$\int_{\mathbf{S}^{n-1}\cap P} f_{1,0,0}\rho_k\, du = \int_{\mathbf{S}^{l-1}} f_{1,0,0}h_k\, du = 0$$

for all $P \in \mathbf{Gr}_l(\mathbf{R}^n)$. As $f_{1,0,0} \neq 0$ by hypothesis, ρ_k is an even function that has all its integrals over great S^{l-1}'s equal to zero. We can use theorem 9 to conclude that

$\rho_k = 0$. Then $h_k = 0$. Since $h_k = 0$ we have $h^{[k]} = h^{[k+1]}$ for $k \geqslant 1$. For $k = 1$ we have $h_1 = 0$ and then $h = h^{[1]} = h^{[2]}$. Inductively $h = h^{[k]}$ for all $k \geqslant 1$. Thus $h = 1$. Then the radial function of each K_t is $\rho(u) = 1$ meaning that every K_t is the Euclidean unit ball of $\mathbf{R}^n$. □

Corollary 31. *Let K_t be a one parameter family of C^2 boundaries starlike bodies in $\mathcal{S}^2_{\mathrm{sym}}(\mathbf{R}^n)$ whose radial functions are given by the series 0.0.2 that converges absolutely in $C^2(\mathbf{S}^{n-1})$. Assume that for all $P \in \mathbf{Gr}_l(\mathbf{R}^n)$*

$$I_f(K_t \cap P) = I_f(\mathbf{B}^n \cap P)$$

where I_f is positively homogeneous of nonzero degree integral invariant defined by $f \in \mathcal{F}_2(\mathbf{R}^l)$. Then each K_t is the Euclidean unit ball of $\mathbf{R}^n$.

Proof. Since I_f is homogeneous of degree nonzero by hypothesis, according to Corollary 24 $f_{1,0,0} \neq 0$ so we can apply now Theorem 29. □

Remark 32. Corollary 31 applies to intrinsic volumes V_k, since according to Proposition 28, they are homogeneous of degree k integral invariants.

When we let $k = 2$ and f, the order two $\mathbf{O}(l)$ invariant function on $\mathbf{R}^l$, to be a power of the curvature of the body's central section in Theorem 29 then we obtain the Corollary:

Corollary 33. *Let K_t be a one parameter family of C^2 bodies starlike about the origin of $\mathbf{R}^n$, $O \in \mathrm{int}(K_t)$, whose radial functions are given by the series* 0.0.2 *that converges absolutely in $C^2(\mathbf{S}^{n-1})$. Assume that each K_t is symmetric about the origin and for all $P \in \mathbf{Gr}_2(\mathbf{R}^n)$*

$$I_\alpha(K_t \cap P) = \int_{\partial K_t \cap P} \kappa^\alpha \, ds = 2\pi \tag{2.3.3}$$

where κ is the curvature of $\partial K_t \cap P$, $\alpha \neq 1$, and s is the arclength of $\partial K_t \cap P$. Then $\rho_t(u) = 1$ for all t and u and thus each K_t is the Euclidean unit ball of $\mathbf{R}^n$.

Remark 34. If K is any starlike body in $\mathbf{R}^3$ and $P \in \mathbf{Gr}_2(\mathbf{R}^3)$ then $\partial K \cap P$ is a closed curve. For $\alpha = 1$ we have $I_1(K \cap P) = \int_{\partial K \cap P} \kappa \, ds$. Using the definition of the curvature

$$\kappa = \frac{d\tau}{ds}$$

where τ is the angle of the oriented tangent to $\partial K \cap P$ with the x-axis. So

$$\begin{aligned} I_1(K \cap P) &= \int_{\partial K \cap P} d\tau \\ &= 2\pi \end{aligned}$$

since $I_1(K \cap P)$ gives the total variation of τ when the tangent describes $\partial K \cap P$ that is a closed curve. So clearly the Euclidean unit ball is not determined by $I_1(K \cap u^{\perp}) = 2\pi$ for all $u \in \mathbf{S}^2$.

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