EXTREMAL APPROXIMATELY CONVEX FUNCTIONS AND THE BEST CONSTANTS IN A THEOREM OF HYERS AND ULAM

S. J. DILWORTH, RALPH HOWARD, AND JAMES W. ROBERTS

Abstract. Let $n \geq 1$ and $B \geq 2$. A real-valued function f defined on the n-simplex Δ_n is approximately convex with respect to Δ_{B-1} if

$$f\left(\sum_{i=1}^{B} t_i x_i\right) \leq \sum_{i=1}^{B} t_i f(x_i) + 1$$

for all $x_1, \dots, x_B \in \Delta_n$ and all $(t_1, \dots, t_B) \in \Delta_{B-1}$. We determine the extremal function of this type which vanishes on the vertices of Δ_n. We also prove a stability theorem of Hyers-Ulam type which yields as a special case the best constants in the Hyers-Ulam stability theorem for ε-convex functions.

1. Introduction

Let U be a convex subset of a real vector space. Then a function $f\colon U \to \mathbb{R}$ is *ε-convex* iff

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \varepsilon$$

for all $t \in [0,1]$ and $x, y \in U$. In 1952 Hyers and Ulam [6] proved that any ε-convex function on a finite dimensional convex set can be approximated by a convex function. Since then several authors have considered the problem of improving the constants in this stability theorem. (See the book [5] for the complete history.) Here we find the best constants.

Theorem 1. *Suppose that $U \subseteq \mathbb{R}^n$ is convex and that $f\colon U \to \mathbb{R}$ is ε-convex. Then there exist convex functions $g, g_0\colon U \to \mathbb{R}$ such that*

$$g(x) \leq f(x) \leq g(x) + \kappa(n)\varepsilon \qquad \text{and} \qquad |f(x) - g_0(x)| \leq \frac{\kappa(n)\varepsilon}{2}$$

for all $x \in U$, where

$$\kappa(n) = \lfloor \log_2 n \rfloor + \frac{2(n+1-2^{\lfloor \log_2 n \rfloor})}{n+1}.$$

Moreover, $\kappa(n)$ is the best constant in these inequalities.

Date: August 23, 2001.

2000 *Mathematics Subject Classification.* Primary: 26B25, 41A44; Secondary: 39B72, 51M16, 52A40.

Key words and phrases. Convex functions, approximately convex functions, Hyers-Ulam Theorem, best constants.

The value $\kappa(2) = 5/3$ was first obtained by Green [4]. The value $\kappa(2^n - 1) = n$ was obtained by a different argument in [3]. Note that $\kappa(3) = 2$, $\kappa(4) = 12/5$, $\kappa(5) = 8/3$, $\kappa(6) = 20/7$, $\kappa(7) = 3$, etc. These values improve the constants obtained by Cholewa [1]. The best constants corresponding to $\kappa(n)$ for approximately midpoint-convex functions were obtained in [2].

Our methods give the best constants for a more general stability theorem. To explain this we fix some notation. The standard n-simplex Δ_n is defined by

$$\Delta_n = \Big\{(x(0), \dots, x(n)) : \sum_{j=0}^{n} x(j) = 1, x(j) \geq 0, 0 \leq j \leq n\Big\}.$$

The vertices of Δ_n are denoted by $e(j)$ $(0 \leq j \leq n)$. For $x \in \Delta_n$, the set $\{0 \leq j \leq n \colon x(j) \neq 0\}$ is denoted by $\operatorname{supp} x$. Fix $B \geq 2$ and $n \geq 1$, and let U be a convex subset of $\mathbb{R}^n$. We say that a function $f \colon U \to \mathbb{R}$ is *approximately convex with respect to* Δ_{B-1} iff

$$f\Big(\sum_{i=1}^{B} t_i x_i\Big) \leq \sum_{i=1}^{B} t_i f(x_i) + 1$$

for all $x_1, \dots, x_B \in U$ and all $(t_1, \dots, t_B) \in \Delta_{B-1}$. When $B = 2$ this is just the definition of 1-convex and by rescaling properties of ε-convex function reduce to those of 1-convex functions.

In Section 2 we consider real-valued functions with domain Δ_n that are approximately convex with respect to Δ_{B-1}. We show that there exists an extremal such function satisfying the following: (i) E is approximately convex with respect to Δ_{B-1}; (ii) E vanishes on the vertices of Δ_n; (iii) if $f \colon U \to \mathbb{R}$ is approximately convex with respect to Δ_{B-1} and satisfies $f(e(j)) \leq 0$ for $j = 0, \dots, n$, then $f(x) \leq E(x)$ for all $x \in \Delta_n$. Moreover, we obtain an explicit formula for E, and we show that E is concave and piecewise-linear on Δ_n and continuous on the interior of Δ_n. We also calculate the maximum value of E.

In Section 3 we prove a stability theorem of Hyers-Ulam type for approximately convex functions and show that the maximum value of the extremal function E gives the best constant in this theorem. The special case of $B = 2$ is Theorem 1.

More information about approximately convex functions and stability theorems can be found in the book [5]. Our earlier paper [2] gives a thorough treatment of extremal approximately midpoint-convex functions and related results.

Finally we remark on why the proofs for approximately convex functions are shorter and simpler than in the case of approximately midpoint-convex functions in [2]. An approximately convex function defined on an open set is easily seen to be locally bounded. However the existence of non-measurable solutions to the functional equation $f(x+y) = f(x) + f(y)$ shows that there are approximately midpoint-convex functions defined on all of $\mathbb{R}^n$ that are unbounded, both above and below, on every non-empty open subset of $\mathbb{R}^n$.

Thus the extremal approximately midpoint-convex function on the simplex Δ_n, corresponding to E of Theorem 2 in the current paper, is not pointwise largest in the set of all approximately midpoint-convex functions vanishing on the vertices of Δ_n, but only extremal in the set of Borel measurable approximately midpoint-convex functions vanishing on the vertices of Δ_n. These measure theoretic considerations are a major reason for the more complicated proofs in [2].

2. Extremal Approximately Convex Functions

Define a function $E\colon \Delta_n \to \mathbb{R}$ as follows (recall that $\operatorname{sgn} 0 = 0$ and $\operatorname{sgn} a = a/|a|$ if $a \neq 0$):

$$E(x) = \min\Big\{\sum_{j=0}^{n} m(j)x(j) : \sum_{j=0}^{n} \frac{\operatorname{sgn} x(j)}{B^{m(j)}} \leq 1,\ m(j) \geq 0, m(j) \in \mathbb{N}\Big\}. \quad (2.1)$$

If $x \in \Delta_n$ then $x(j) \geq 0$ and so $\operatorname{sgn} x(j)$ is either 0 or 1. Note that if $A = \operatorname{supp} x$, then

$$E(x) = \min\Big\{\sum_{j\in A} m(j)x(j) : \sum_{j\in A} \frac{1}{B^{m(j)}} \leq 1,\ m(j) \geq 0, m(j) \in \mathbb{N}\Big\}. \quad (2.2)$$

Proposition 1. *$E(e(j)) = 0$ for all j and E is approximately convex with respect to Δ_{B-1}.*

Proof. It is clear from (2.2) that $E(x) \geq 0$ for all x and that $E(e(j)) = 0$ for all j. Suppose that $x \in \Delta_n$ and that $x = \sum_{k=1}^{B} t_k x_k$ for some $x_1, \dots, x_B \in \Delta_n$. Let $A = \operatorname{supp} x$ and $A_k = \operatorname{supp} x_k$, and note that $A \subseteq \bigcup_{k=1}^{B} A_k$. For each $1 \leq k \leq B$, we have

$$E(x_k) = \sum_{j\in A_k} m_k(j)x_k(j)$$

for some $(m_k(j))_{j\in A_k}$ such that $\sum_{j\in A_k} 1/B^{m_k(j)} \leq 1$. For $j \in A$, let $C(j) = \{1 \leq k \leq B : j \in A_k\}$ and let

$$M(j) = \min\{m_k(j) : k \in C(j)\}.$$

Note that

$$\frac{1}{B^{M(j)+1}} = \frac{1}{B}\frac{1}{B^{M(j)}} \leq \frac{1}{B}\sum_{k\in C(j)} \frac{1}{B^{m_k(j)}}.$$

Thus,

$$\sum_{j\in A} \frac{1}{B^{M(j)+1}} \leq \sum_{j\in A} \frac{1}{B} \sum_{k\in C(j)} \frac{1}{B^{m_k(j)}} \leq \frac{1}{B}\sum_{k=1}^{B}\sum_{j\in A_k} \frac{1}{B^{m_k(j)}} \leq 1.$$

Hence

$$\begin{aligned}
E\Big(\sum_{k=1}^{B} t_k x_k\Big) = E(x) &\le \sum_{j\in A}(1+M(j))x(j) \\
&= \sum_{j\in A}(1+M(j))\sum_{k=1}^{B} t_k x_k(j) \\
&= 1+\sum_{k=1}^{B} t_k \sum_{j\in A} M(j)x_k(j) \\
&= 1+\sum_{k=1}^{B} t_k \sum_{j\in A_k} M(j)x_k(j)
\end{aligned}$$

(since $A_k \subseteq A$ if $t_k \neq 0$)

$$\begin{aligned}
&\le 1+\sum_{k=1}^{B} t_k \sum_{j\in A_k} m_k(j)x_k(j) \\
&= 1+\sum_{k=1}^{B} t_k E(x_k).
\end{aligned}$$

Thus, E is approximately convex with respect to Δ_{B-1}. □

Lemma 1. *If $m(j) \ge 1$ for each $0 \le j \le n$ and $\sum_{j=0}^{n} 1/B^{m(j)} \le 1$, then $\{0,1,\dots,n\}$ is the disjoint union of sets $P_1,\dots,P_B$ such that*

$$\sum_{j\in P_k} \frac{1}{B^{m(j)}} \le \frac{1}{B}$$

for $k = 1,\dots,B$.

Proof. Without loss of generality we may assume that $1 \le m(0) \le m(1) \le \dots \le m(n)$. We shall prove that the result holds for all $n \ge 1$ by induction on $N = \sum_{j=0}^{n} m(j)$. Note that the result is vacuously true if $N = 1$ and is trivial if $n \le B$. So suppose that $N \ge 2$ and that $n > B$, so that $n-1 > B-1 \ge 1$. By inductive hypothesis, $\{0,1,\dots,n-1\}$ is the disjoint union of sets $F_1,\dots,F_B$ such that

$$\sum_{j\in F_k} \frac{1}{B^{m(j)}} \le \frac{1}{B}$$

for $k = 1,\dots,B$. Since $\sum_{j=0}^{n-1} 1/B^{m(j)} < 1$, and since $1 \le m(0) \le m(1) \le \dots \le m(n)$, there exists k_0 such that

$$\sum_{j\in F_{k_0}} \frac{1}{B^{m(j)}} \le \frac{1}{B} - \frac{1}{B^{m(n-1)}} \le \frac{1}{B} - \frac{1}{B^{m(n)}}. \tag{2.3}$$

Put $P_{k_0} = P_{k_0} \cup \{n\}$ and $P_k = F_k$ for $k \neq k_0$ to complete the induction. □

Theorem 2. *E is extremal, that is if $h\colon \Delta_n \to \mathbb{R}$ is approximately convex with respect to Δ_{B-1} and $h(e(j)) \le 0$ for $j = 0, 1, \ldots, n$, then*

$$h(x) \le E(x) \qquad \text{for all } x \in \Delta_n.$$

Proof. Let $s = |\operatorname{supp} x|$, so that $1 \le s \le n+1$. The proof is by induction on s. If $s = 1$ then $x = e(j)$ for some j, so that

$$E(x) = E(e(j)) = 0 \ge h(e(j)) = h(x).$$

As inductive hypothesis, we suppose that $h(x) \le E(x)$ whenever $|\operatorname{supp} x| < s$. Now suppose that $s \ge 2$ and that $|\operatorname{supp} x| = s$. Without loss of generality we may assume that $\operatorname{supp} x = \{0, \ldots, s-1\}$, so that $E(x) = \sum_{j=0}^{s-1} m(j)x(j)$, where $\sum_{j=0}^{s-1} 1/B^{m(j)} \le 1$. Note that each $m(j) \ge 1$ since $s \ge 2$.

If $\sum_{j=0}^{s-1} 1/B^{m(j)} \le 1/B$, let $P_1 = \{0, \ldots, s-2\}$, $P_2 = \{s-1\}$, and $P_k = \varnothing$ for $2 < k \le B$. Note that $|P_k| < s$ for $1 \le k \le B$ and that $\sum_{j \in P_k} 1/B^{m(j)} \le 1/B$.

On the other hand, if $\sum_{j=0}^{s-1} 1/B^{m(j)} > 1/B$, then applying Lemma 1 with $n = s-1$, we can write $\{0, 1, \ldots, s-1\}$ as the disjoint union of sets $P_1, \ldots, P_B$ such that $\sum_{j \in P_k} 1/B^{m(j)} \le 1/B$ for each $1 \le k \le B$. Note that this implies that $|P_k| < s$ for $1 \le k \le B$.

If $P_k \ne \varnothing$, let $x_k = (1/t_k) \sum_{j \in P_k} x(j)e(j)$, where $t_k = \sum_{j \in P_k} x(j)$. If $P_k = \varnothing$, let $x_k = e(0)$ and let $t_k = 0$. Thus $x = \sum_{k=1}^{B} t_k x_k$, where $t_k \ge 0$ and $\sum_{k=1}^{B} t_k = 1$. Note that

$$|\operatorname{supp} x_k| = \max\{1, |P_k|\} < s \qquad (1 \le k \le B).$$

If $P_k \ne \varnothing$, then $m(j) \ge 1$ for all $j \in P_k$, and $\sum_{j \in P_k} 1/B^{m(j)-1} \le 1$. Since $|\operatorname{supp} x_k| < s$, our inductive hypothesis implies that $h(x_k) \le E(x_k)$. Finally,

$$\begin{aligned}
h(x) = h\Big(\sum_{k=1}^{B} t_k x_k\Big) &\le 1 + \sum_{k=1}^{B} t_k h(x_k) \le 1 + \sum_{P_k \ne \varnothing} t_k E(x_k) \\
&\le 1 + \sum_{P_k \ne \varnothing} t_k \sum_{j \in P_k} (m(j)-1) x_k(j) \\
&= 1 + \sum_{P_k \ne \varnothing} \sum_{j \in P_k} (m(j)-1) x(j) \\
&= 1 + \sum_{j=0}^{s-1} m(j)x(j) - \sum_{j=0}^{s-1} x(j) \\
&= \sum_{j=0}^{s-1} m(j)x(j) = E(x).
\end{aligned}$$

This completes the induction. ∎

Following the convention that $x \log_B x = 0$ when $x = 0$, the *entropy* function $F\colon \Delta_n \to \mathbb{R}$ is defined as follows:

$$F(x) = -\sum x(j) \log_B x(j).$$

Proposition 2. *F is approximately convex with respect to Δ_{B-1} and satisfies*

$$F(x) \leq E(x) \leq F(x) + 1 \qquad (x \in \Delta_n).$$

Proof. Let $x \in \Delta_n$. A standard Lagrange multiplier calculation yields

$$F(x) = \min \Big\{ \sum_{j \in A} y(j)x(j) : \sum_{j \in A} \frac{1}{B^{y(j)}} \leq 1,\ y(j) \geq 0 \Big\}, \tag{2.4}$$

where $A = \operatorname{supp} x$. Using (2.4) in place of (2.2), minor changes in the proof of Proposition 1 show that F is approximately convex with respect to Δ_{B-1}. Suppose that

$$F(x) = \sum_{j \in A} y(j)x(j) \tag{2.5}$$

for some $y(j) \geq 0$ satisfying $\sum_{j \in A} 1/B^{y(j)} \leq 1$. Let $m(j) = \lceil y(j) \rceil$. Then $\sum_{j \in A} 1/B^{m(j)} \leq 1$, and so

$$E(x) \leq \sum_{j \in A} m(j)x(j) \leq \sum_{j \in A} (y(j)+1)x(j) = F(x) + 1.$$

On the other hand, since F is approximately convex with respect to Δ_{B-1}, it follows from Theorem 2 that $F(x) \leq E(x)$. $\square$

Recall that a *face* of a compact convex set A is an intersection of A with any of its supporting hyperplanes. An *open face* is the interior of a face in the minimal affine space containing it. When A is a simplex, the faces of A are just the sub-simplices of A of lower dimension.

Proposition 3. *(i) E is piecewise-linear and the restriction of E to each open face of Δ_n is continuous.*
(ii) E is lower semi-continuous;
(iii) E is concave.

Proof. To prove that E is piecewise linear it is enough to show that E is piecewise linear on the interior Δ_n° of Δ_n. For then by an induction on n we will have that E is piecewise linear on Δ_n° and the induction hypothesis implies that it is piecewise linear when restricted to any of the faces of Δ_n, which implies that E is piecewise linear on Δ_n. For fixed n and B let

$$\mathcal{F}(n,B) := \left\{ (m_0, \dots, m_n) : m_k \in \mathbb{N},\ \sum_{k=0}^{n} \frac{1}{B^{m_k}} \leq 1 \right\}$$

be the set of feasible $(n+1)$-tuples. For $(m_0, \dots, m_n) \in \mathcal{F}(n,B)$ let $\Lambda_{(m_0,\dots,m_n)}\Delta_n \to \mathbb{R}$ be the linear function

$$\Lambda_{(m_0,\dots,m_n)}(x_0, \dots, x_n) = m_0x_0 + m_1x_1 + \dots + m_nx_n$$

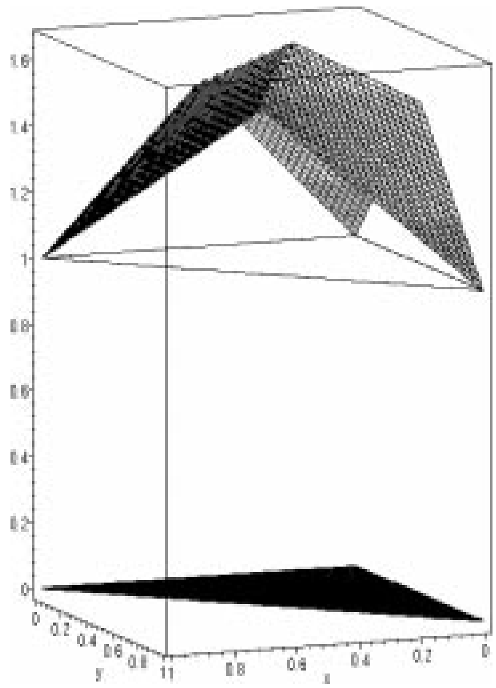

FIGURE 1. Graph of $y = E(x, y, 1-x-y)$ for $B = 2$ over the simplex $0 \le y \le 1 - x \le 1$ showing the discontinuity along the boundary. On the boundary E_S has the value 1 except at the three vertices where it has the value 0.

so that $E\colon \Delta_n \to \mathbb{R}$ is given by

$$E(x) = \min\{\Lambda_{(m_0,\dots,m_n)}(x) : (m_0, \dots, m_n) \in \mathcal{F}(n, B)\}.$$

Let

$$\begin{aligned}\mathcal{E}(n, B) := \{(m_0, \dots, m_n) \in \mathcal{F}(n, B) :&\\ \Lambda_{(m_0,\dots,m_n)}(x) = E(x) \text{ for some } x \in \Delta_n^\circ\}&\end{aligned}$$

be the set of extreme $(n+1)$-tuples. Then

$$E\big|_{\Delta_n^\circ}(x) = \min\{\Lambda_{(m_0,\dots,m_n)}(x) : (m_0, \dots, m_n) \in \mathcal{E}(n, B)\}$$

and therefore showing that $E\big|_{\Delta_n^\circ}$ is piecewise linear is equivalent to showing that $\mathcal{E}(n, B)$ is finite.

Lemma 2. *Let* $(m_0, \dots, m_n) \in \mathcal{E}(n, B)$ *and* $(m_0', \dots, m_n') \in \mathcal{F}(n, B)$ *with* $m_k' \le m_k$ *for* $0 \le k \le n$. *Then* $(m_0', \dots, m_n') = (m_0, \dots, m_n)$.

Proof. For if not then there is an index k with $m_k' < m_k$. As all the components of $x = (x_0, \dots, x_n)$ are positive on Δ_n° this implies that on $x \in \Delta_n^\circ$

$$\begin{aligned}E(x) &\le \Lambda_{(m_0',\dots,m_n')}(x) = \Lambda_{(m_0,\dots,m_n)}(x) + \Lambda_{(m_0',\dots,m_n')}(x) - \Lambda_{(m_0,\dots,m_n)}(x)\\ &\le \Lambda_{(m_0,\dots,m_n)}(x) + (m_k' - m_k)x_k < \Lambda_{(m_0,\dots,m_n)}(x).\end{aligned}$$

This contradicts that for $(m_0, \dots, m_n) \in \mathcal{E}(n, B)$ there is an $x \in \Delta_n^\circ$ with $\Lambda_{(m_0,\dots,m_n)}(x) = E(x)$. $\square$

Let $\operatorname{Perm}(n+1)$ be the group of permutations of $\{0, 1, \dots, n\}$. Then it is easily checked that $\mathcal{E}(n, B)$ is invariant under the action of $\operatorname{Perm}(n+1)$ given by $\sigma(m_0, m_1, \dots, m_n) = (m_{\sigma(0)}, m_{\sigma(1)}, \dots, m_{\sigma(n)})$. Therefore if $\mathcal{E}^*(n, B)$ is the set of monotone decreasing elements of $\mathcal{E}(n, B)$, that is

$$\mathcal{E}^*(n, B) := \{(m_0, \dots, m_n) \in \mathcal{E}(n, B) : m_0 \geq m_1 \geq \cdots \geq m_n\},$$

then

$$\mathcal{E}(n, B) = \{\sigma(m_0, \dots, m_n) : (m_0, \dots, m_n) \in \mathcal{E}^*(n, B), \sigma \in \operatorname{Perm}(n+1)\}$$

and to show that $\mathcal{E}(n, B)$ is finite it is enough to show that $\mathcal{E}^*(n, B)$ is finite.

Lemma 3. *Suppose that $n \geq 0$. Let $m_0 \geq m_1 \geq \cdots \geq m_n$ be a non-increasing sequence of $(n+1)$ positive integers, and let C be a positive real number such that*

$$\sum_{k=0}^{n} \frac{1}{B^{m_k}} \leq C,$$

and such that if $m_0', m_1', \dots, m_n'$ are any positive integers with $m_k' \leq m_k$ for $0 \leq k \leq n$, then

$$\sum_{k=0}^{n} \frac{1}{B^{m_k'}} \leq C$$

implies that $(m_0', \dots, m_n') = (m_0, \dots, m_n)$. (We will say that $(m_0, \dots, m_n)$ is extreme for *(n, C).) Let*

$$\eta = \eta(n, C) := \min\{j \geq 2 : CB^j \geq n + B\}.$$

Then $m_n < \eta(n, C)$. (The explicit value of η is $\eta(n, C) = \max\{2, \lceil \log_B((n+B)/C) \rceil\}$.)

Proof. From the definition of η we have $\eta \geq 2$ and $CB^\eta \geq n + B$ which is equivalent to

$$\frac{n+1}{B^\eta} \leq C - \frac{1}{B^{\eta-1}} + \frac{1}{B^\eta}.$$

Assume, toward a contradiction, that $m_n \geq \eta$. Then

$$\frac{1}{B^{m_0}} + \cdots + \frac{1}{B^{m_{n-1}}} + \frac{1}{B^{m_n}} \leq \frac{n+1}{B^\eta} \leq C - \frac{1}{B^{\eta-1}} + \frac{1}{B^\eta}.$$

This can be rearranged to give

$$\frac{1}{B^{m_0}} + \cdots + \frac{1}{B^{m_{n-1}}} + \frac{1}{B^{\eta-1}} \leq C + \frac{1}{B^\eta} - \frac{1}{B^{m_n}} \leq C.$$

This contradicts that $(m_0, \dots, m_n)$ is (n, C) extreme and completes the proof. $\square$

We now prove $\mathcal{E}^*(n,B)$ is finite. First some notation. For positive integers $l_1,\ldots,l_j$ let $C(l_1,\ldots,l_j) := 1-\sum_{i=1}^{j} 1/B^{l_j}$. If $(m_0,\ldots,m_n)\in\mathcal{E}^*(n,B)$ then by Lemma 2 (and with the terminology of Lemma 3) for each j with $1\le j\le n$ the tuple $(m_0,\ldots,m_{n-j})$ is $(n-j, C(m_{n-j+1},\ldots,m_n))$ extreme, and $(m_0,\ldots,m_n)$ itself is $(n,1)$ extreme. Therefore, by Lemma 3, $m_n < \eta(n,1)$, whence there are only a finite number of possible choices for m_n. For each of these choices of m_n we can use Lemma 3 again to get $m_{n-1} < \eta(n-1, C(m_n))$, and so there are only finitely many choices for the ordered pair (m_{n-1},m_n). And for each of these pairs (m_{n-1},m_n) we have that so there are only finitely many possibilities for m_{n-2}. Continuing in this manner it follows that $\mathcal{E}^*(n,B)$ is finite. This completes the proof that $E_S^{\Delta_n}$ is piecewise linear and thus point (i) of Propsition 3

To prove point (ii) let A be a nonempty subset of $\{0,1,\ldots,n\}$. In proving point (i) we have seen that there is a finite collection $\mathcal{L}(A)$ of linear mappings $\Lambda\colon \Delta_n\to\mathbb{R}$, each one of the form $\Lambda(x)=\sum_{j\in A} m(j)x(j)$ for some nonnegative integers $m(j)$, $j=0,1,\ldots,n$, with $\sum_{j\in A} 1/B^{m(j)}\le 1$, such that

$$E(x)=\min\{\Lambda(x):\Lambda\in\mathcal{L}(A)\} \tag{2.6}$$

for all $x\in\Delta_n$ such that $\operatorname{supp} x = A$. Clearly, we may also assume that $\mathcal{L}(B)\subseteq\mathcal{L}(A)$ whenever $A\subseteq B$. Suppose that $(x_i)_{i=1}^{\infty}\subseteq\Delta_n$ and that $x_i\to x$ as $i\to\infty$. Note that $\operatorname{supp} x\subseteq \operatorname{supp} x_i$ for all sufficiently large i, so that $\mathcal{L}(\operatorname{supp} x_i)\subseteq\mathcal{L}(\operatorname{supp} x)$ for all sufficiently large i. Thus,

$$\begin{aligned}
E(x) &= \min\{T(x): T\in\mathcal{L}(\operatorname{supp} x)\}\\
&= \lim_{i\to\infty}\min\{T(x_i): T\in\mathcal{L}(\operatorname{supp} x)\}\\
&\le \liminf_{i\to\infty}\min\{T(x_i): T\in\mathcal{L}(\operatorname{supp} x_i)\}\\
&= \liminf_{i\to\infty} E(x_i).
\end{aligned}$$

Thus, E is lower semi-continuous.

Finally we prove point (iii). It follows from (2.6) that the restriction of E to the interior of any face is the minimum of a finite collection of linear functions, and hence is continuous and concave. The lower semi-continuity of E forces E to be concave on all of Δ_n. □

Remark. The algorithm implicit in the proof that $\mathcal{E}^*(n,B)$ is finite is rather effective for small values of n. In the case of most interest, when $B=2$ so that $S=\Delta_1$, it can be used to show

$$\begin{aligned}
&\mathcal{E}^*(2,2)=\{(2,2,1)\}, \qquad \mathcal{E}^*(3,2)=\{(3,3,2,1),(2,2,2,2)\}\\
&\mathcal{E}^*(4,2)=\{(4,4,3,2,1),(3,3,2,2,2)\},\\
&\mathcal{E}^*(5,2)=\{5,5,4,3,2,1),(3,3,3,3,2,2)\}.
\end{aligned}$$

When $n=2$ this leads to the explicit formula

$$E(x,y,1-x-y)=\min\{1+x+y,2-x,2-y\}$$

for $0 < x < 1 - y < 1$. (Cf. Figure 1). The sets $\mathcal{E}^*(n, 2)$ can be used to give messier, but equally explicit formulas, for higher values of n. $\square$

Proposition 4. *The maximum of E is given by*

$$\kappa(n, B) = \lfloor \log_B n \rfloor + \frac{\lceil B(n + 1 - B^{\lfloor \log_B n \rfloor})/(B - 1) \rceil}{n + 1} \tag{2.7}$$

For small values of B and n, $\kappa_S(n)$ is given in Table 1.

$B\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1.0	1.6667	2.0000	2.4000	2.6667	2.8571	3.0000	3.1111	3.4000	3.5455
3	1.0	1.0	1.5000	1.6000	1.8333	1.8571	2.0000	2.0000	2.2000	2.2727
4	1.0	1.0	1.0	1.4000	1.5000	1.5714	1.7500	1.7778	1.8000	1.9091
5	1.0	1.0	1.0	1.0	1.3333	1.4286	1.5000	1.5556	1.7000	1.7273
6	1.0	1.0	1.0	1.0	1.0	1.2857	1.3750	1.4444	1.5000	1.5455
7	1.0	1.0	1.0	1.0	1.0	1.0	1.2500	1.3333	1.4000	1.4545
8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.2222	1.3000	1.3636
9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.2000	1.2727
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.1818
11	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE 1. Values of $\kappa(n, B)$ for $2 \leq B \leq 11$ and $1 \leq n \leq 10$.

Proof. E is a symmetric function of $x(0), \ldots, x(n)$ and E is also concave. Thus E achieves its maximum at the barycenter $\overline{x} = (1/(n+1)) \sum_{j=0}^n e(j)$. So there exist nonnegative integers $m(j)$ ($j = 0, 1, \ldots, n$) such that $E(\overline{x}) = (1/(n+1)) \sum_{j=0}^n m(j)$ and $\sum_{j=0}^n 1/B^{m(j)} \leq 1$. We may also assume that $(m(j))_{j=0}^n$ have been chosen to minimize $\sum_{j=0}^n 1/B^{m(j)}$ among all possible choices of $(m(j))_{j=0}^n$. Suppose that there exist i and k such that $m(k) \geq m(i) + 2$. Note that

$$\frac{1}{B^{m(i)+1}} + \frac{1}{B^{m(k)-1}} \leq \frac{2}{B^{m(i)+1}} \leq \frac{B}{B^{m(i)+1}} < \frac{1}{B^{m(i)}} + \frac{1}{B^{m(k)}}. \tag{2.8}$$

Thus replacing $m(i)$ by $m(i) + 1$ and replacing $m(k)$ by $m(k) - 1$ leaves $(1/(n+1)) \sum_{j=0}^n m(j)$ unchanged while it reduces $\sum_{j=0}^n 1/B^{m(j)}$, which contradicts the choice of $(m(j))_{j=0}^n$. Thus $|m(i) - m(k)| \leq 1$ for all i, k. It follows that there exist integers $\ell \geq 0$ and $1 \leq s \leq n + 1$ such that

$$\kappa(n, B) = \frac{\ell(n + 1 - s) + (\ell + 1)s}{n + 1} = \ell + \frac{s}{n + 1} \tag{2.9}$$

and

$$\frac{n + 1 - s}{B^\ell} + \frac{s}{B^{\ell+1}} \leq 1. \tag{2.10}$$

Moreover, it is clear from (2.9) that ℓ is the least nonnegative integer satsifying (2.10) for some $1 \leq s \leq n + 1$, i.e.

$$\ell = \lfloor \log_B n \rfloor.$$

For this value of ℓ it is clear from (2.9) that s is the smallest integer in the range $1 \le s \le n+1$ satisfying (2.10), i.e.

$$s = \left\lceil \frac{B(n+1) - B^{\ell+1}}{B-1} \right\rceil = \left\lceil \frac{B}{B-1}(n+1-B^{\ell}) \right\rceil .$$

Substituting these values for ℓ and s into (2.9) gives (2.7). $\square$

3. Best Constants in Stabilty Theorems of Hyers-Ulam Type

Hyers and Ulam [6] introduced the following definition. Fix $\varepsilon > 0$. A function $f\colon U \to \mathbb{R}$, where U is a convex subset of $\mathbb{R}^n$, is ε-convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon$$

for all $x, y \in U$ and all $t \in [0,1]$.

Note that f is ε-convex if and only if $(1/\varepsilon)f$ is approximately convex with respect to Δ_1. So let us generalize this notion by defining f to be *ε-convex with respect to Δ_{B-1}* if $(1/\varepsilon)f$ is approximately convex with respect to Δ_{B-1}.

The proof of the following theorem is adapted from Cholewa's proof [1] of the Hyers-Ulam stability theorem for ε-convex functions.

Theorem 3. *Suppose that $U \subseteq \mathbb{R}^n$ is convex and that $f\colon U \to \mathbb{R}$ is ε-convex with respect to Δ_{B-1}. Then there exist convex functions $g, g_0\colon U \to \mathbb{R}$ such that*

$$g(x) \le f(x) \le g(x) + \kappa(n,B)\varepsilon \qquad \text{and} \qquad |f(x) - g_0(x)| \le \frac{\kappa(n,B)\varepsilon}{2}$$

for all $x \in U$. Moreover, $\kappa(n,B)$ is the best constant in these inequalities.

Proof. By replacing f by f/ε, we may assume that $\varepsilon = 1$. Set $W = \{(x,y) \in U \times \mathbb{R} : y \ge f(x)\} \subseteq \mathbb{R}^{n+1}$ and define g by

$$g(x) = \inf\{y : (x,y) \in \mathrm{Co}(W)\}. \tag{3.1}$$

Clearly $-\infty \le g(x) \le f(x)$. Suppose that $(x,y) \in \mathrm{Co}(W)$. By Caratheodory's Theorem (see e.g. [7, Thm. 17.1]) there exist $n+2$ points $(x_0,y_0),\ldots,(x_{n+1},y_{n+1}) \in W$ such that $(x,y) \in \Delta := \mathrm{Co}(\{(x_0,y_0),\ldots,(x_{n+1},y_{n+1})\})$. Let $\overline{y} = \min\{\eta : (x,\eta) \in \Delta\}$. Then $(x,\overline{y})$ lies on the boundary of Δ and so it is a convex combination of $n+1$ of the points $(x_0,y_0),\ldots,(x_{n+1},y_{n+1})$. Without loss of generality, $(x,\overline{y}) = \sum_{j=0}^{n} t_j(x_j,y_j)$ for some $(t_0,\ldots,t_n) \in \Delta_n$. Note that

$$h\Big(\sum_{j=0}^{n} x(j)e(j)\Big) := f\Big(\sum_{j=0}^{n} x(j)x_j\Big) - \sum_{j=0}^{n} x(j)f(x_j) \qquad (x \in \Delta_n)$$

is approximately convex with respect to Δ_{B-1} and satisfies $h(e(j)) = 0$ for $j = 0, 1, \ldots, n$. By Proposition 4, $\max_{x\in\Delta_n} h(x) \le \kappa(n, B)$. Thus

$$\begin{aligned}
y \ge \overline{y} &= \sum_{j=0}^{n} t_j y_j = \sum_{j=0}^{n} t_j f(x_j) \\
&= f\Big(\sum_{j=0}^{n} t_j x_j\Big) - h\Big(\sum_{j=0}^{n} t_j e(j)\Big) \\
&\ge f\Big(\sum_{j=0}^{n} t_j x_j\Big) - \kappa(n, B) \\
&= f(x) - \kappa(n, B).
\end{aligned}$$

Taking the infimum over all y yields $g(x) \ge f(x) - \kappa(n, B)$, i.e. $f(x) \le g(x) + \kappa(n, B)$. Finally, set $g_0(x) = g(x) + \kappa(n, B)/2$.

The fact that $\kappa(n, B)$ is the best constant follows by taking f to be E, where E is the extremal approximately convex function (with respect to Δ_{B-1}) with domain Δ_n. $\square$

Setting $B = 2$ in Theorem 3, gives the best constants in the Hyers-Ulam stability theorem and completes the proof of Theorem 1.

References

[1] Piotr W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.

[2] S. J. Dilworth, Ralph Howard and James W. Roberts, *Extremal approximately convex functions and estimating the size of convex hulls*, Adv. in Math. **148** (1999), 1–43.

[3] S. J. Dilworth, Ralph Howard and James W. Roberts, *On the size of approximately convex sets in normed spaces*, Studia Math. **140** (2000), 213–241.

[4] John W. Green, *Approximately subharmonic functions*, Duke Math. J. **19** (1952), 499–504.

[5] Donald H. Hyers, George Isac and Themistocles M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Boston, 1998.

[6] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821–828.

[7] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: `dilworth@math.sc.edu`

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: `howard@math.sc.edu`

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: `roberts@math.sc.edu`