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Convex bodies of constant width and constant brightness

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Abstract

In 1926 Nakajima (= Matsumura) showed that any convex body in $\mathbf{R}^3$ with constant width, constant brightness, and boundary of class C^2 is a ball. We show that the regularity assumption on the boundary is unnecessary, so that balls are the only convex bodies of constant width and brightness.

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1. Introduction

A *convex body* in the n-dimensional Euclidean space $\mathbf{R}^n$ is a compact convex set with non-empty interior. A convex body K in three-dimensional Euclidean space has *constant width* w iff the orthogonal projection of K onto every line is an interval of length w. It has *constant brightness* b iff the orthogonal projection of K onto every plane is a region of area b.

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Theorem 1.1. *Any convex body in* $\mathbf{R}^3$ *of constant width and constant brightness is a Euclidean ball.*

Under the extra assumption that the boundary is of class C^2 this was proven by Nakajima (= Matsumura) [17] in 1926 (versions of Nakajima's proof can be found in the books of Bonnesen and Fenchel [1, Section 68] and Gardner [8, p. 117]). Since then the problem, often called the Nakajima problem, of determining if there is a non-smooth non-spherical convex body in $\mathbf{R}^3$ of constant width and constant brightness has become well-known among geometers studying convexity (cf. [3, p. 992; 4, p. 82; 6, Problem A10; 8, Problem 3.9, p. 119; 9, Question 2, p. 437; 10, p. 24; 11, p. 368],). Theorem 1.1 solves this problem.

For convex bodies with C^2 boundaries and positive curvature Nakajima's result was generalized by Chakerian [2] in 1967 to "relative geometry" where the width and brightness are measured with respect to some convex body K_0 symmetric about the origin called the *gauge body*. The following isolates the properties required of the gauge body. Recall the *Minkowski sum* of two subsets A and B of $\mathbf{R}^n$ is $A + B = \{a + b : a \in A, b \in B\}$.

Definition. A convex body K_0 is a *regular gauge* iff it is centrally symmetric about the origin and there are convex sets K_1, K_2 and Euclidean balls B_r and B_R such that $K_0 = K_1 + B_r$ and $B_R = K_0 + K_2$.

Any convex body symmetric about the origin with C^2 boundary and positive Gaussian curvature is a regular gauge (Corollary 2.4 below). For any linear subspace P of $\mathbf{R}^n$ let $K|P$ be the projection of K onto P (all projections in this paper are orthogonal). For a unit vector u let $w_K(u)$ be the width in the direction of u. For each positive integer k and any Borel subset A of $\mathbf{R}^n$ let be $V_k(A)$ be the k-dimensional volume of A (which in this paper is the k-dimensional Hausdorff measure of A). Two subsets A and B of $\mathbf{R}^n$ are *homothetic* iff there is a positive scalar λ and a vector v_0 such that $B = v_0 + \lambda A$.

Theorem 1.2. *Let* K_0 *be a regular gauge in* $\mathbf{R}^3$ *and let* K *be any convex body in* $\mathbf{R}^3$ *such that for some constants* α, β *the equalities* $w_K(u) = \alpha w_{K_0}(u)$ *and* $V_2(K|u^\perp) = \beta V_2(K_0|u^\perp)$ *hold for all* $u \in \mathbb{S}^2$. *Then* K *is homothetic to* K_0.

Letting K_0 be a Euclidean ball recovers Theorem 1.1. While we are assuming some regularity on the gauge body K_0, the main point is that no assumptions, other than convexity, are being put on K. It is likely that the result also holds with no restrictions on either K or K_0. One indication this may be the case is a beautiful and surprising result of Schneider [22] that almost every, in the sense of Baire category, centrally symmetric convex body K_0 is determined up to translation in the class of all convex bodies by just its width function. This contrasts strongly with the fact that for any regular gauge K_0 there is an infinite-dimensional family of convex bodies that have the same width function as K_0 (see Remark 2.7 below).

Two convex bodies K and K_0 in $\mathbf{R}^n$ have *proportional k-brightness* iff there is a constant γ such that $V_k(K|P) = \gamma V_k(K_0|P)$ for all k-dimensional subspaces P of $\mathbf{R}^n$. Theorem 1.2 implies a result, valid in all dimensions, about pairs of convex bodies that have both 1-brightness and 2-brightness proportional. If A and B are convex sets in $\mathbf{R}^n$ and L is a linear subspace of $\mathbf{R}^n$, then taking Minkowski sums commutes with projection onto L, that is $(A+B)|L = A|L + B|L$. As the projection of a Euclidean ball is a Euclidean ball, it follows that if K_0 is a regular gauge in $\mathbf{R}^n$, then $K_0|L$ is a regular gauge in L. Also, if P is a linear subspace of L, then $K|P = (K|L)|P$. Therefore, if K_0 is a regular gauge in $\mathbf{R}^n$ and K is a convex body such that K and K_0 have proportional 1-brightness and proportional 2-brightness, then for any three-dimensional subspace L of $\mathbf{R}^n$ the set $K_0|L$ is a regular gauge in L and $K_0|L$ and $K|L$ will have proportional 1-brightness and proportional 2-brightness as subsets of L. Thus by Theorem 1.2 $K|L$ is homothetic to $K_0|L$. However, if the projections $K_0|L$ and $K|L$ are homothetic for all three-dimensional subspaces L, then, [8, Theorem 3.1.3, p. 93], K is homothetic to K_0. Thus:

Corollary. *If K_0 is a regular gauge in $\mathbf{R}^n$, $n \geqslant 3$, and K is a convex body in $\mathbf{R}^n$ that has* 1*-brightness and* 2*-brightness proportional to those of K_0, then K is homothetic to K_0. In particular if K_0 is a Euclidean ball this implies any convex body K in $\mathbf{R}^n$ of constant* 1*-brightness and* 2*-brightness is also a Euclidean ball.*

The contents of this paper are as follows. In Section 2 some preliminaries about convex sets are given and a $C^{1,1}$ regularity result, Proposition 2.5, for the support functions of convex sets in $\mathbf{R}^n$ that appear is a summand in a convex set with $C^{1,1}$ support function is proven. (I am indebted to Daniel Hug for some of the results in this section). Section 3 gives explicit formulas, in terms of the support function, h, for the inverse of the Gauss map of the boundary of a convex set in $\mathbf{R}^n$ and conditions are given for two convex sets with $C^{1,1}$ boundary to have proportional brightness. It is important for our applications that some of these formulas (eg. Proposition 3.2) apply even when the function h is not the support function of a convex set. It is also shown that if the convex body, K, has its brightness function proportional to that of the gauge, K_0, then the support function h_K of K satisfies a Monge–Ampère-type equation. In Section 4 results about quasiregular maps are used to show that certain Monge–Ampère type equations on spheres have no odd solutions. In Section 5 the results of the previous sections are combined to prove Theorems 1.1 and 1.2.

2. Preliminaries on convexity

We assume that $\mathbf{R}^n$ has its standard inner product $\langle\, , \rangle$ and let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbf{R}^n$. For any convex body K contained $\mathbf{R}^n$ the support function $h = h_K$ of K is the function $h\colon \mathbb{S}^{n-1} \to \mathbf{R}$ given by $h(u) := \max_{y\in K}\langle y, u\rangle$. A convex body is uniquely determined by its support function. The Minkowski sum of K_1 and K_2 corresponds to the sum of the support functions: $h_{K_1+K_2} = h_{K_1} + h_{K_2}$. The *width function* of K is $w(u) = h(u) + h(-u)$. This is the length of the projection of K onto a line parallel to the vector u. In the terminology of Gardner, [8, p. 99], the *central symmetral* of a convex body K is the convex body $K_0 := \frac{1}{2}(K - K) = \{\frac{1}{2}(a-b) : a, b \in K\}$. The body K_0 is centrally symmetric about the origin and, denoting the support function of K_0 by h_0, it follows from $h_{\frac{1}{2}(K-K)} = \frac{1}{2}h_K + \frac{1}{2}h_{-K}$ that $h_0(u) = \frac{1}{2}(h(u) + h(-u))$. Therefore, K and K_0 have the same width in all directions. These definitions imply that a convex body has constant width w if and only if its central symmetral is a Euclidean ball of radius $w/2$.

We need the following, which is an elementary corollary of the Brunn–Minkowski theorem. For a proof see [8, Theorem 3.2.2, p. 100].

Proposition 2.1. *The volumes of a convex body K and its central symmetral $K_0 = \frac{1}{2}(K - K)$ satisfy $V(K_0) \geqslant V(K)$ with equality if and only if K is a translate of K_0.*

Recall that a function f defined on an open subset U of $\mathbf{R}^k$ is of class $C^{1,1}$ iff it is continuously differentiable and all the first partial derivatives satisfy a Lipschitz condition. A convex body K has $C^{1,1}$ boundary iff its boundary ∂K is locally the graph of a $C^{1,1}$ function.

There is a very nice geometric characterization of the convex bodies that have $C^{1,1}$ boundaries in terms of freely sliding bodies. Let K_1 and K_2 be convex bodies in $\mathbf{R}^n$. Then K_1 *slides freely inside of* K_2 iff for all $a \in \partial K_2$ there is a translate $y + K_1$ of K_1 such that $y + K_1 \subseteq K_2$ and $a \in y + K_1$. It is not hard to see, [21, Theorem 3.2.2, p. 143], that K_1 slides freely inside of K_2 if and only if K_1 is a Minkowski summand of K_2. That is, if and only if there is a convex set K such that $K + K_1 = K_2$. In what follows we will use the expressions "K_1 slides freely inside of K_2" and "K_1 is a Minkowski summand of K_2" interchangeably. A proof of the following can be found in [13, Proposition 1.4.3, p. 97].

Proposition 2.2. *A convex body K has $C^{1,1}$ boundary if and only if some Euclidean ball B_r slides freely inside of K.*

I learned the following elegant dual form of this theorem, with a somewhat different proof, from Hug.

Proposition 2.3 (*Hug [14]*). *The support function h of a convex body K is $C^{1,1}$ if and only if K slides freely inside of some Euclidean ball B_R.*

Proof. Assume that K slides freely inside of the ball B_R of radius R. Without loss of generality it may be assumed that the origin is in the interior of K. Let $K^\circ := \{y : \langle y, x\rangle \leqslant 1 \text{ for all } x \in K\}$ be the *polar body* of K. The radial function of K° (which is the positive real-valued function ρ on $\mathbb{S}^{n-1}$ such that $u \mapsto \rho(u)u$ parameterizes the boundary $\partial(K^\circ)$ of K°) is $\rho(u) = 1/h(u)$, [21, Remark 1.7.7, p. 44]. So it is enough to show that ρ is a $C^{1,1}$ function, and to show this it is enough to show that the boundary $\partial(K^\circ)$ is $C^{1,1}$. By Proposition 2.2 it is enough to show that some ball slides freely inside of K°. Let $\rho(u)u \in \partial(K^\circ)$. Because K slides freely inside a ball of radius R there is a ball $B_R(a)$ of radius R centered at some point a such that $K \subset B_R(a)$

and a point $x \in K \cap \partial B_R(a)$ such that u is the outward pointing normal to $B_R(a)$ at x. As the operation of taking polars is inclusion reversing, $B_R(a)^\circ$ is contained in K° and as u is the outward pointing unit normal to both K and $B_R(a)$ at x we also have $\rho(u)u \in \partial(B_R(a)^\circ)$. The support function of $B_R(a)$ is $h_{B_R(a)}(u) = R + \langle a, u\rangle$ and therefore the radial function of the polar $B_R(a)^\circ$ is $\rho_{B_R(a)^\circ}(u) = 1/(R + \langle a, u\rangle)$. Thus points on $\partial(B_R(a)^\circ)$ are of the form $y = (1/(R + \langle a, u\rangle))\, u$ for $u \in \mathbb{S}^{n-1}$. This implies $|y| = 1/(R + \langle a, u\rangle)$ and $\langle a, y\rangle = \langle a, u\rangle/(R + \langle a, u\rangle)$. If $\langle a, u\rangle$ is eliminated from these equations the result can be written as

$$R^2|y|^2 - \langle a, y\rangle^2 + 2\langle a, y\rangle = 1.$$

For each a this is an ellipsoid and an ellipsoid has positive rolling radius (which is the largest number r so that a ball of radius r slides freely inside of the body). By Blaschke's rolling theorem, [21, Corollary 3.2.10, p. 150], the rolling radius is the smallest radius of curvature of $\partial(B_R(v)^\circ)$ and this is a continuous function of the vector v. The set of v such that $B_R(v)$ contains K is a compact set and therefore, by the continuous dependence of the rolling radius of $\partial(B_R(v)^\circ)$ on v, there is a positive number r_0 such that a ball of radius r_0 slides freely inside of any $B_R(v)^\circ$ that contains K. In particular this is true of $B_R(a)^\circ$ and so K° contains an internally tangent ball of radius r_0 at $\rho(u)u$. But $\rho(u)u$ was an arbitrary point of $\partial(K^\circ)$ and hence a ball of radius r_0 slides freely inside of K° as required.

Conversely assume that the support function, h, of K is $C^{1,1}$. Let $\tilde{h}$ be the extension of h to $\mathbf{R}^n$ that is homogeneous of degree 1. Explicitly

$$\tilde{h}(x) := \max_{y\in K}\langle y, x\rangle. \tag{2.1}$$

As h is $C^{1,1}$ the function $\tilde{h}$ is $C^{1,1}_{\mathrm{Loc}}$ on $\mathbf{R}^n \setminus \{0\}$. That is on each compact subset of $\mathbf{R}^n \setminus \{0\}$ the vector field $\partial\tilde{h}$ is Lipschitz. Therefore by Rademacher's theorem, [7, Theorem 3.1.6, p. 216], the second derivative $\partial^2\tilde{h}$ exists almost everywhere and is bounded on compact subsets of $\mathbf{R}^n \setminus \{0\}$. Moreover $\partial^2\tilde{h}(x)$ is symmetric at the points where it exists (cf. [7, p. 219]). As $\tilde{h}$ is homogeneous of degree 1, the second derivative $\partial^2\tilde{h}$ is homogeneous of degree -1. Likewise the function $|\cdot|$ is also homogeneous of degree 1 and $\partial^2|\cdot|$ is homogeneous of degree -1. Also $\partial^2|\cdot|$ is symmetric and positive semi-definite on $\mathbf{R}^n \setminus \{0\}$. Therefore, the boundedness of $\partial^2\tilde{h}$ on compact sets together with the homogeneity implies there is a constant $R > 0$ such that if $H_0 := R|\cdot| - \tilde{h}$, then $\partial^2 H_0(x)$ exists and is positive semi-definite for almost all $x \in \mathbf{R}^n \setminus \{0\}$. We now show that H_0 is convex. Let $\{\varphi_\ell\}_{\ell=1}^\infty$ be a C^∞ approximation to the identity such that φ_ℓ is non-negative and supported in the ball $B_{1/\ell}(0)$. Let H_ℓ be the convolution $H_\ell(x) := \int_{\mathbf{R}^n} H_0(y)\varphi_\ell(x-y)\,dy = \int_{\mathbf{R}^n} H_0(x-y)\varphi_\ell(y)\,dy$. Then H_ℓ is C^∞ and $H_\ell \to H_0$ informally on compact subsets of $\mathbf{R}^n$. Also on $\mathbf{R}^n \setminus B_{2/\ell}(0)$ Lebesgue's bounded convergence theorem implies the second derivative of H_ℓ is given by $\partial^2 H_\ell(x) = \int_{\mathbf{R}^n} \partial^2 H_0(x-y)\varphi_\ell(y)\,dy$, which shows that $\partial^2 H_\ell$ is positive semi-definite on $\mathbf{R}^n \setminus B_{2/\ell}(0)$. A smooth function with positive semi-definite Hessian defined on a convex open set is convex (cf. [21, Theorem 1.5.10, p. 29]). It follows that the restriction of H_ℓ to any convex open subset of $\mathbf{R}^n \setminus B_{2/\ell}(0)$ is convex. Let x_0 and x_1 be two points of $\mathbf{R}^n$ and $[x_0, x_1]$ the segment between them. If 0 is not on $[x_0, x_1]$, then there is a convex open set U containing $[x_0, x_1]$ such that for large ℓ the sets $B_{2/\ell}(0)$ and U are disjoint. Thus for large ℓ the restriction of H_ℓ to U is convex and therefore for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} H_0((1-t)x_0 + tx_1) &= \lim_{\ell\to\infty} H_\ell((1-t)x_0 + tx_1) \\ &\leqslant \lim_{\ell\to\infty} \big((1-t)H_\ell(x_0) + tH_\ell(x_1)\big) \\ &= (1-t)H_0(x_0) + tH_0(x_1). \end{aligned}$$

If $[x_0, x_1]$ does contain the origin, then $H_0((1-t)x_0 + tx_1) \leqslant (1-t)H_0(x_0) + tH_0(x_1)$ still holds as can be seen by approximating $[x_0, x_1]$ by a sequence of segments that do not contain the origin. Therefore H_0 is convex. But it is also homogeneous of degree 1 and thus, [21, Theorem 1.7.1, p. 38], the restriction $H_0\big|_{\mathbb{S}^{n-1}}$ is the support function of a unique compact convex set K_0. Then $H_0 + \tilde{h} = R|\cdot|$ implies that $K + K_0 = B_R(0)$. Therefore K is a summand in a ball. $\square$

Corollary 2.4. *Let K_0 be a convex body that is centrally symmetric about the origin, with ∂K_0 of class C^2 with positive Gauss curvature. Then K_0 is a regular gauge.*

Proof. It follows from a generalization of Blaschke's rolling theorem, [21, Corollary 3.2.10, p. 150], that if B_r is a Euclidean ball with r smaller than any of the radii of curvature of K_0, then B_r slides freely inside of K_0 and if R is larger than any of the radii of curvature of ∂K_0, then K_0 slides freely inside of B_R. $\square$

Proposition 2.5. *Let $K_1, \dots, K_k$ be convex bodies in $\mathbf{R}^n$ such that the Minkowski sum $K_1 + \cdots + K_k$ has $C^{1,1}$ support function. Then each summand K_j also has $C^{1,1}$ support function.*

Proof. If $K_1 + \cdots + K_k$ has $C^{1,1}$ support function, then, by Proposition 2.3, $K_1 + \cdots + K_k$ is a Minkowski summand in some ball B_R. But then each K_j is also a summand in B_R and therefore Proposition 2.3 yields that K_j has $C^{1,1}$ support function. $\square$

Corollary 2.6. *Let K be a convex body such that its central symmetral has a $C^{1,1}$ support function. Then the support function of K is also $C^{1,1}$. In particular any convex body of constant width has $C^{1,1}$ support function.*

Proof. If K_0 is the central symmetral of K, then $K + (-K) = 2K_0$. As K_0 has $C^{1,1}$ support function, h_0, the support function, $2h_0$, of $2K_0$ is also $C^{1,1}$ and therefore the support function of K is $C^{1,1}$ by Proposition 2.5. $\square$

Remark 2.7. Corollary 2.6 is sharp in the sense that even when the support function, h_0, of the central symmetral, K_0, is C^∞, the most that can be said about the regularity of the support function, h, of K is that it is $C^{1,1}$. For example let h_0 be the support function of a regular gauge, K_0, and let p a $C^{1,1}$ function $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ with $p(-u) = -p(u)$. Then for sufficiently small $\varepsilon > 0$ the function $h := h_0 + \varepsilon p$ is the support function of a convex body with the same width function as K_0. But there are many choices of h_0 and p with h_0 of class C^∞ and h only of class $C^{1,1}$.

3. Support functions and the inverse of the Gauss map

We view vector fields ξ on subsets of U of $\mathbf{R}^n$ as functions $\xi\colon U \to \mathbf{R}^n$. A vector field on $\mathbb{S}^{n-1}$ is a function $\xi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ such that for all $u \in \mathbb{S}^{n-1}$ the vector $\xi(u) \in T_u\mathbb{S}^{n-1}$. As the tangent space, $T_u\mathbb{S}^{n-1}$, to $\mathbb{S}^{n-1}$ at u is just $u^\perp$, the orthogonal complement to u in $\mathbf{R}^n$, a vector field ξ on $\mathbb{S}^{n-1}$ can also be viewed as a map from $\mathbb{S}^{n-1}$ to $\mathbf{R}^n$ with $\xi(u) \perp u$ for all u. If $X \in T_u\mathbb{S}^{n-1}$ is a tangent vector to $\mathbb{S}^{n-1}$ at u, then a *curve fitting* X is a smooth curve $c\colon (a,b) \to \mathbb{S}^{n-1}$ defined on an interval about 0 with $c(0) = u$ and $c'(0) = X$. If ξ is a vector field on $\mathbb{S}^{n-1}$ that is differentiable at the point u, then for any $X \in T_u\mathbb{S}^{n-1}$ the *covariant derivative*, $(\nabla_X\xi)(u)$, *of* ξ *by* X is the projection of $\frac{d}{dt}\xi(c(t))\big|_{t=0}$ onto $T_u\mathbb{S}^{n-1}$ where c is any curve fitting X. This is independent of the choice of c fitting X and is given explicitly by

$$(\nabla_X\xi)(u) := \frac{d}{dt}\,\xi(c(t))\bigg|_{t=0} - \left\langle \frac{d}{dt}\xi(c(t))\bigg|_{t=0}, u \right\rangle u.$$

This definition implies that for any smooth curve $c\colon (a,b) \to \mathbb{S}^{n-1}$ and any vector field ξ on $\mathbb{S}^{n-1}$

$$\frac{d}{dt}\,\xi(c(t)) = (\nabla_X\xi)(c(t)) + \left\langle \frac{d}{dt}\xi(c(t)), c(t) \right\rangle c(t) \tag{3.1}$$

for any value t such that ξ is differentiable at $c(t)$.

For any C^1 function $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ the (spherical) *gradient* is the vector field, ∇p, on $\mathbb{S}^{n-1}$ such that $\langle \nabla p, X\rangle = dp(X)$ for all vectors X tangent to $\mathbb{S}^{n-1}$. At any point u where the vector field ∇p is differentiable the *second derivative* of p is the linear map $\nabla^2 p(u)\colon T_u\mathbb{S}^{n-1} \to T_u\mathbb{S}^{n-1}$ given by

$$\nabla^2 p(u)X := (\nabla_X \nabla p)(u).$$

Remark 3.1. There is a another way of viewing $\nabla^2 p$ that is useful. If p is defined on $\mathbb{S}^{n-1}$ then extend p to $\mathbf{R}^n$ to be homogeneous of degree one. That is let $\tilde{p}\colon \mathbf{R}^n \to \mathbf{R}$ be

$$\tilde{p}(x) = |x|\,p(|x|^{-1}x) \tag{3.2}$$

for $x \neq 0$ and $\tilde{p}(0) = 0$. Let $\partial \tilde{p}$ be the usual gradient of $\tilde{p}$, that is $\partial \tilde{p}$ is the vector with components $\partial_1 \tilde{p}, \partial_2 \tilde{p}, \ldots, \partial_n \tilde{p}$, and let $\partial^2 \tilde{p}$ be the field of linear maps on $\mathbf{R}^n \setminus \{0\}$ given by $\partial^2 \tilde{p}(x)Y := (\partial_Y \partial \tilde{p})(x)$ where ∂_Y is the usual directional derivative in the direction of the vector Y. The matrix of $\partial^2 \tilde{p}$ with respect to the coordinate basis is the usual Hessian matrix $\left[\partial_i \partial_j \tilde{p}\right]$. A straightforward calculation shows that $\partial^2 \tilde{p}$ and $\nabla^2 p$ are related by

$$\partial^2 \tilde{p}(x)Y = \frac{1}{|x|}\Big(\nabla^2 p(|x|^{-1}x) + p(|x|^{-1}x)I\Big)(Y - |x|^{-2}\langle Y, x\rangle x). \tag{3.3}$$

This implies that if $u \in \mathbb{S}^{n-1}$ and $Y \in T_u\mathbb{S}^{n-1} = u^{\perp}$, then

$$\partial^2 \tilde{p}(u)Y = (\nabla^2 p(u) + p(u)I)Y$$

and $\partial^2 \tilde{p}(u)u = 0$. Thus $T_u\mathbb{S}^{n-1}$ is invariant under $\partial^2 \tilde{p}$. The symmetry of the second partials implies that when p is C^2, so that $\tilde{p}$ is C^2 on $\mathbf{R}^n \setminus \{0\}$, then $\partial^2 \tilde{p}(x)$ is self-adjoint (that is $\langle \partial^2 \tilde{p}(x)X, Y\rangle = \langle X, \partial^2 \tilde{p}(x)Y\rangle$) for $x \in \mathbf{R}^n \setminus \{0\}$. But then $\nabla^2 p(u) = \partial^2 \tilde{p}(u)\big|_{T_u\mathbb{S}^{n-1}} - p(u)I$ implies that $\nabla^2 p(u)$ is self-adjoint on $T_u\mathbb{S}^{n-1}$. Formula (3.3) also implies that $\nabla^2 p$ exists at $u \in \mathbb{S}^{n-1}$ if and only if $\partial^2 \tilde{p}$ exists at all points tu with $t > 0$. This, combined with Fubini's Theorem, yields that $\nabla^2 p$ exists almost everywhere on $\mathbb{S}^{n-1}$ if and only if $\partial^2 \tilde{p}$ exists almost everywhere on $\mathbf{R}^n$.

The following proposition is known in the case that φ is the inverse Gauss map and p the support function of strictly convex C^2 convex body (cf. [15, Korollar p. 132]). As we will need the result when p is not the support function of a any convex body and only has $C^{1,1}$ smoothness we include a proof.

Proposition 3.2. *Let $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ be a Lipschitz map such that for all u where the derivative $\varphi'(u)$ exists it satisfies $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$. Then there is a unique $C^{1,1}$ function $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ such that*

$$\varphi(u) = p(u)u + \nabla p(u). \tag{3.4}$$

The derivative $\varphi'(u)$ exists at u if and only if the second derivative $\nabla^2 p(u)$ of p exists at u and at these points

$$\varphi'(u) = p(u)I + \nabla^2 p(u), \tag{3.5}$$

where I is the identity map on $T_u\mathbb{S}^{n-1}$. Conversely if p is $C^{1,1}$ and φ is given by 3.4 *then $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$ for all points u where φ is differentiable. Finally for $k \geqslant 1$ the function φ is C^k if and only if p is C^{k+1}.*

Proof. Any function $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ can be uniquely written as $\varphi(u) = p(u)u + \xi(u)$ where $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ and ξ is a vector field on $\mathbb{S}^{n-1}$. Because φ is Lipschitz, so are p and ξ. Therefore a theorem of Rademacher [7, Theorem 3.1.6, p. 216], implies that p and ξ are both differentiable almost everywhere on $\mathbb{S}^{n-1}$. Let E be the set of points where both p and ξ are differentiable. Then φ is also differentiable at u. Let $u \in E$, $X \in T_u\mathbb{S}^{n-1}$, and c a curve fitting X. Then, using (3.1),

$$\begin{aligned}\varphi'(u)X &= \frac{d}{dt}\bigg|_{t=0} \big(p(c(t))c(t) + \xi(c(t))\big)\\ &= dp_u(X)u + p(u)X + (\nabla_X\xi)(u) + \left\langle \frac{d}{dt}\bigg|_{t=0} \xi(c(t)), u\right\rangle u.\end{aligned}$$

But $dp_u(X) = \langle \nabla p(u), X\rangle$ and, using that $\langle \xi(c(t)), c(t)\rangle \equiv 0$,

$$\left\langle \frac{d}{dt}\bigg|_{t=0} \xi(c(t)), u\right\rangle = \frac{d}{dt}\bigg|_{t=0} \langle \xi(c(t)), c(t)\rangle - \langle \xi(c(t)), c'(t)\rangle\bigg|_{t=0} = -\langle \xi(u), X\rangle.$$

Therefore the formula for $\varphi'(u)X$ becomes

$$\varphi'(u)X = \langle \nabla p(u) - \xi(u), X\rangle u + p(u)X + (\nabla_X\xi)(u). \tag{3.6}$$

As $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ the component normal to $\mathbb{S}^{n-1}$ must vanish. Hence $\langle \nabla p(u) - \xi(u), X\rangle = 0$ for all $X \in T_u\mathbb{S}^{n-1}$. This implies

$$\xi(u) = \nabla p(u) \quad \text{at points } u \text{ where both } p \text{ and } \xi \text{ are differentiable.} \tag{3.7}$$

We now argue that p is continuously differentiable and that $\nabla p = \xi$ on all of $\mathbb{S}^{n-1}$. This will be based on the following elementary lemma, whose proof will be given after the proof of Proposition 3.2.

Lemma 3.3. *Let q be a real-valued Lipschitz function defined on an open subset U of $\mathbf{R}^N$. Assume that there are Lipschitz functions $q_1, \ldots, q_N$ on U and a set of full measure $S \subseteq U$ such that for all $x \in S$ the partial derivatives of q exist and satisfy $\partial_j q(x) = q_j(x)$ for all $x \in S$. Then q is of class $C^{1,1}$ and $\partial_j q = q_j$ on all of U.*

Near any point, u_0, of $\mathbb{S}^{n-1}$ there is a C^∞ parameterization $f\colon U \to V \subset \mathbb{S}^{n-1}$ of a neighborhood V of u_0, with U a bounded open set in $\mathbf{R}^{n-1}$, and f a C^∞ diffeomorphism. By making the domain U of f smaller we can assume that f and its derivatives are Lipschitz. To show that p is $C^{1,1}$ it is enough to show the function $q\colon U \to \mathbf{R}$ given by $q(x) := p(f(x))$ is $C^{1,1}$. Let S be the subset of points $x \in U$ where both p and ξ are differentiable at $f(x)$. As p and ξ are Lipschitz and f is a diffeomorphism this is a set of full measure in U and at all points of S we have, by (3.7), that $\nabla p(f(x)) = \xi(f(x))$. As ξ is Lipschitz there are real-valued Lipschitz functions $\xi^1, \ldots, \xi^{n-1}$ defined on U such that $\xi(f(x)) = \sum_{i=1}^{n-1} \xi^i(x)\partial_i f(x)$. Therefore at points x in S we have $\nabla p(f(x)) = \xi(f(x)) = \sum_{i=1}^{n-1} \xi^i(x)\partial_i f(x)$ and thus

$$\partial_j q(x) = dp_{f(x)}(\partial_j f) = \langle \nabla p(f(x)), \partial_j f\rangle = \sum_{i=1}^{n-1} \xi^i(f(x))\langle \partial_i f(x), \partial_j f(x)\rangle.$$

The functions $q_j(x) := \sum_{i=1}^{n-1} \xi^i(f(x))\langle \partial_i f(x), \partial_j f(x)\rangle$ are Lipschitz so Lemma 3.3 implies that q, and therefore also p, is a $C^{1,1}$ function and that ∇p is Lipschitz.

By (3.7) $\nabla p(u) = \xi(u)$ on the dense set E and ∇p and ξ are continuous thus $\nabla p = \xi$ on all of $\mathbb{S}^{n-1}$. Therefore $\varphi(u)$ is given by (3.4) as required. When φ is of this form it is clear that φ is differentiable exactly at the points u where the second derivative $\nabla^2 p(u)$ exists. At such points use $\nabla p = \xi$ and $\nabla_X \xi(u) = (\nabla_X \nabla p)(u) = \nabla^2 p(u)X$ in (3.6) to see that (3.5) holds. This completes the proof that if $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ is a Lipschitz map with $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $u \in \mathbb{S}^{n-1}$ where φ is differentiable, then φ is given by (3.4) for a uniquely determined $C^{1,1}$ function p.

Conversely, if p is $C^{1,1}$ let $\xi = \nabla p$ in the calculations leading up to (3.6) to see that φ given by (3.4) satisfies $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $u \in \mathbb{S}^{n-1}$ where φ is differentiable.

Finally $\varphi(u) = p(u)u + \nabla p(u)$ makes it clear that if p is C^{k+1}, then φ is C^k. Conversely if φ is C^k, then $p(u) = \langle u, \varphi(u)\rangle$ implies p is C^k. Then $\nabla p(u) = \varphi(u) - p(u)u$ implies that ∇p is also C^k. But if ∇p is C^k, then p is C^{k+1}. $\square$

Proof of Lemma 3.3. We will show that the jth distributional derivative of q is q_j. By definition this means we need to show that for all C^∞ functions ψ with compact support contained in U that $\int_U q\partial_j\psi\,dx = -\int_U q_j\psi\,dx$. Let e_j be the jth coordinate vector. Then

$$\begin{aligned}
\int_U q(x)\partial_j\psi(x)\,dx &= \lim_{h\to 0}\int_U q(x)\frac{\psi(x+he_j)-\psi(x)}{h}\,dx \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_U q(x)\psi(x+he_j)\,dx - \frac{1}{h}\int_U q(x)\psi(x)\,dx\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_U q(x-he_j)\psi(x)\,dx - \frac{1}{h}\int_U q(x)\psi(x)\,dx\right) \\
&= \lim_{h\to 0}\int_U \frac{q(x-he_j)-q(x)}{h}\psi(x)\,dx.
\end{aligned}$$

But q is Lipschitz and therefore the quotients $(q(x-he_j)-q(x))/h$ are uniformly bounded. By assumption for all $x \in S$, $\lim_{h\to 0}(q(x-he_j)-q(x))/h = -\partial_j q(x) = -q_j(x)$ and S has full measure so this limit holds almost everywhere. Therefore Lebesgue's bounded convergence theorem implies $\lim_{h\to 0}\int_U((q(x-he_j)-q(x))/h)\,\psi(x))\,dx = -\int_U q_j q(x)\psi(x)\,dx$. Using this in the calculation above yields that $\int_U q\partial_j\psi\,dx = -\int_U q_j\psi\,dx$ holds, and thus the distributional partial derivatives $\partial_j q$ are q_j.

Then a standard result about distributional derivatives [12, Theorem 1.4.2, p. 10], implies that the classical partial derivatives $\partial_j q$ of q are equal to q_j in all of U. But a function with continuous partial derivatives is C^1. Finally $\partial_j q = q_j$ so the derivative is Lipschitz, that is q is of class $C^{1,1}$. $\square$

Proposition 3.4. *Let $p: \mathbb{S}^{n-1} \to \mathbf{R}$ be a $C^{1,1}$ function. Then for almost all $u \in \mathbb{S}^{n-1}$ the second derivative $\nabla^2 p(u)$ exists and is self-adjoint.*

Proof. If p is $C^{1,1}$ the vector field ∇p is Lipschitz and thus by Rademacher's Theorem $\nabla^2 p(u)$ exists for almost all u. We have seen, Remark 3.1, that if p is of class C^2, then $\nabla^2 p(u)$ is self-adjoint for all $u \in \mathbb{S}^{n-1}$. In the case that p is $C^{1,1}$, for each $\varepsilon > 0$ there is a C^2 function p_ε such that if $E_\varepsilon := \{u \in \mathbb{S}^{n-1} : p(u) = p_\varepsilon(u), \nabla p(u) = \nabla p_\varepsilon(u), \nabla^2 p(u) = \nabla^2 p_\varepsilon(u)\}$ then the measure of $\mathbb{S}^{n-1} \setminus E_\varepsilon$ is less than ε, [7, Theorem 3.1.15, p. 227]. As p_ε is C^2, $\nabla^2 p(u) = \nabla^2 p_\varepsilon(u)$ is self-adjoint for all $u \in E_\varepsilon$. Letting ε go to zero shows that $\nabla^2 p$ is self-adjoint almost everywhere on $\mathbb{S}^{n-1}$. $\square$

Before applying Proposition 3.2 to the support function of a convex set, it is useful to record some symmetry properties of the operators ∇ and ∇^2. Note that the tangent spaces $T_u\mathbb{S}^{n-1}$ and $T_{-u}\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$ at antipodal points u and $-u$ are both just the orthogonal complement $u^\perp$ to u. Therefore for a function p on $\mathbb{S}^{n-1}$ the vectors $\nabla p(u)$ and $\nabla p(-u)$ are in the same vector space, $u^\perp$, and the linear maps $\nabla^2 p(u)$ and $\nabla^2 p(-u)$ also both act on the space $u^\perp$. Recall that a function $p: \mathbb{S}^{n-1} \to \mathbf{R}$ is even (respectively, odd) iff $p(-u) = p(u)$ (respectively, $p(-u) = -p(u)$) for all $u \in \mathbb{S}^{n-1}$. These definitions extend in an obvious way to vector fields or fields of linear maps on $\mathbb{S}^{n-1}$. The proof of the following is elementary and left to the reader.

Lemma 3.5. *Let $p: \mathbb{S}^{n-1} \to \mathbf{R}$ be a $C^{1,1}$ function. If p is even, then ∇p is odd, and $\nabla^2 p$ is even. If p is odd, then ∇p is even, and $\nabla^2 p$ is odd.* (*As p is $C^{1,1}$ the tensor $\nabla^2 p$ will only be defined almost everywhere. Saying that it is even* (*or odd*) *means that $\nabla^2 p(u)$ is defined if and only if $\nabla^2(-u)$ is defined and at these points $\nabla^2 p(-u) = \nabla^2 p(u)$* (*or $\nabla^2 p(-u) = -\nabla^2 p(u)$*).)

Recall that if K is a convex body with C^1 boundary ∂K, then the *Gauss map* is the function $v: \partial K \to \mathbb{S}^{n-1}$ where $v(x) = u$ iff u is the (unique as ∂K is C^1) outward pointing unit vector to K at x. If h is the support function of K, then it is not hard to see that $h(v(x)) = \langle x, v(x)\rangle$, Therefore, if v is injective, so that v^{-1} exists, then $h(u) = \langle v^{-1}(u), u\rangle$, [21, p. 106]. More generally when the support function h is C^1 the function $\varphi(u) = h(u)u + \nabla h(u)$ can still be viewed as the inverse of the Gauss map:

Proposition 3.6. *Let K be a convex body in $\mathbf{R}^n$ with C^1 support function h. Then the map $\varphi(u) = h(u)u + \nabla h(u)$ maps $\mathbb{S}^{n-1}$ onto ∂K with the property that $\varphi(u) = x$ if and only if u is an outward unit normal to K at x. Moreover $h(u) = \langle u, \varphi(u)\rangle$.*

Proof. We first assume that ∂K is C^∞ with positive curvature. Then the Gauss map $v: \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism. Let $\varphi := v^{-1}: \mathbb{S}^{n-1} \to \partial K$ be the inverse of v.

Then φ is a diffeomorphism and $T_u\mathbb{S}^{n-1}$ and $T_{\varphi(u)}\partial K$ are the same (as we are identifying subspaces that differ by a parallel translation). Hence $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$. By Proposition 3.2 this implies there is a unique smooth real-valued function p on $\mathbb{S}^{n-1}$ such that $\varphi(u) = p(u)u + \nabla p(u)$. Then $p(u) = \langle \varphi(u), u\rangle$. But, from the remarks above, the support function of K is also given by $h(u) = \langle \varphi(u), u\rangle$ and therefore $p = h$. So in this case $\varphi(u) = h(u)u + \nabla h(u)$ is the inverse of the Gauss map and so $\varphi(u) = x$ if and only if u is the outward normal to K at x is clear.

Now assume that h is C^1 and set $\varphi(u) = h(u)u + \nabla h(u)$. Then φ is a continuous map from $\mathbb{S}^{n-1}$ to $\mathbf{R}^n$. There are convex bodies $\{K_\ell\}_{\ell=1}^\infty$ whose boundaries are smooth with positive curvature and such that if the support function of K_ℓ is h_ℓ, then $h_\ell \to h$ in the C^1 topology, [21, pp. 158–160]. Therefore if $\varphi_\ell(u) := h_\ell(u)u + \nabla h_\ell(u)$, then $\varphi_\ell \to \varphi$ uniformly. The Hausdorff distance (see [21, p. 48] for the definition) between K and K_ℓ is given in terms of the support functions by $d_{\mathrm{Hau}}(K, K_\ell) = \|h - h_\ell\|_{L^\infty}$, [21, 1.8.11, p. 53], and so $K_\ell \to K$ in the Hausdorff metric. Because K and K_ℓ are convex this implies $\partial K_\ell \to \partial K$ in the Hausdorff metric. As $\varphi_\ell(u) \in \partial K_\ell$ this yields $\varphi(u) = \lim_{\ell\to\infty} \varphi_\ell(u) \in \partial K$. Therefore φ maps $\mathbb{S}^{n-1}$ into ∂K. Let $x \in \partial K$ and let u be an outward pointing unit normal to K at x. Then u is an outward pointing normal to K_ℓ at $\varphi_\ell(u)$. Therefore the half space $H_\ell^- := \{y \in \mathbf{R}^n : \langle y, u\rangle \leqslant h_\ell(u)\}$ contains K_ℓ and its boundary ∂H_ℓ^- is a supporting hyperplane to K_ℓ at $\varphi_\ell(u)$. Using that $h_\ell \to h$ uniformly, that $K_\ell \to K$ in the Hausdorff metric, and that $\varphi_\ell(u) \to \varphi(u)$ we see that K is contained in $H^+ := \{y \in \mathbf{R}^n : \langle y, u\rangle \leqslant h(u)\}$ and that $x \in \partial H^+$. Thus u is an outward pointing unit normal to K at $\varphi(u)$. But, [21, Corollary 1.7.3, p. 40], if the support function is differentiable, then the body is strictly convex. Therefore, K is strictly convex and thus a unit vector can be an outward unit normal to K in at most one point. So, as u is an outward unit normal to K at $\varphi(u)$ and at x, we have $\varphi(u) = x$.

Summarizing, if $x \in \partial K$ and u is an outward unit normal to K at x, then $\varphi(u) = x$. But for any point of ∂K there is at least one unit normal u to K at x, so $\varphi\colon \mathbb{S}^{n-1} \to \partial K$ is surjective. We need that if $\varphi(u) = x$, then u is an outward pointing unit normal to K at x. The vector u will be an outward pointing unit normal to K at some point $y \in \partial K$. But then $\varphi(u) = y$. Thus $x = y$ and u is an outward pointing unit vector to K at x. Finally we use that $\nabla h(u)$ is orthogonal to u to conclude that $\langle u, \varphi(u)\rangle = \langle u, p(u)u + \nabla p(u)\rangle = h(u)$. $\square$

Proposition 3.7. *Let K be a convex body with $C^{1,1}$ support function h. Then $hI + \nabla^2 h$ is positive semi-definite almost everywhere on $\mathbb{S}^{n-1}$. If in addition there is a Euclidean ball that slides freely inside of K, then there is a positive constant C_1 such that* $\det(hI + \nabla^2 h) \geqslant C_1$ *almost everywhere on $\mathbb{S}^{n-1}$.*

Proof. Let $\tilde{h}$ be the extension of h to $\mathbf{R}^n$ as a homogeneous function of degree one (thus $\tilde{h}$ is given by both formulas (2.1) and (3.2)). The function $\tilde{h}$ is convex, [21, Theorem 1.7.1, p. 38], and therefore its Hessian $\partial^2\tilde{h}$ is positive semi-definite at all points where it exists and is self-adjoint. But then formula (3.3) relating $\partial^2\tilde{h}$ and $\nabla^2 h$ together with Remark 3.1 and Proposition 3.4, shows that $hI + \nabla^2 h$ is positive semi-definite almost everywhere on $\mathbb{S}^{n-1}$.

Assume that the Euclidean ball B_{2r} of radius $2r$ slides freely inside of K. Then there is a convex set K_1 such that $K_1 + B_{2r} = K$. However K_1 may not be a convex body (its interior might be empty). But $K_1 + B_{2r} = (K_1 + B_r) + B_r$ and $K_1 + B_r$ is a convex body. So by replacing K_1 by $K_1 + B_r$ we can assume $K_1 + B_r = K$ with K_1 a convex body. Let h_1 be the support function of K_1. Then, as the support function of B_r is the constant r, $h_1 + r = h$. This implies that h_1 is also $C^{1,1}$ and therefore $(h_1 I + \nabla^2 h_1)$ is positive semi-definite almost everywhere. But for any positive semi-definite matrices A and B the inequality $\det(A + B) \geqslant \det(A)$ holds. Therefore

$$\det(hI + \nabla^2 h) = \det(rI + (h_1 I + \nabla^2 h_1)) \geqslant \det(rI) = r^{n-1} =: C_1.$$

almost everywhere. $\square$

Lemma 3.8. *Let K be a convex body in $\mathbf{R}^n$ with $C^{1,1}$ support function h. Then for any unit vector $a \in \mathbf{R}^n$,*

$$2V_{n-1}(K|a^\perp) = \int_{\mathbb{S}^{n-1}} \det(hI + \nabla^2 h)|\langle a, u\rangle|\, dV_{n-1}(u).$$

Proof. Let h be the support function of K and let $\varphi\colon \mathbb{S}^{n-1} \to \partial K$ be $\varphi(u) = h(u)u + \nabla h(u)$. By Proposition 3.6 φ maps $\mathbb{S}^{n-1}$ onto ∂K and, as h is $C^{1,1}$, the map φ is Lipschitz. As φ is Lipschitz it is differentiable almost everywhere and by Proposition 3.2 at the points u where it is differentiable $\varphi'(u) = h(u)I + \nabla^2 h(u)$. Let $f\colon \mathbb{S}^{n-1} \to K|a^\perp$ be the function $f(u) = \varphi(u)|a^\perp$. This maps $\mathbb{S}^{n-1}$ onto $K|a^\perp$. An elementary computation shows that the Jacobian, $J(f)(u) := \det(f'(u))$, of f is given by $J(f)(u) = \det\left(h(u)I + \nabla^2 h(u)\right)\langle a, u\rangle$. The area theorem, [7, Theorem 3.2.3, p. 243], (note that the definition of Jacobian used in the area theorem is the absolute value of the one being used here) implies

$$\begin{aligned}\int_{K|a^\perp} \#(f^{-1}[y])\, dV_{n-1}(y) &= \int_{\mathbb{S}^{n-1}} |J(f)(u)|\, dV_{n-1}(u) \\ &= \int_{\mathbb{S}^{n-1}} \det\Big(h(u)I + \nabla^2 h(u)\Big)\, |\langle a, u\rangle|\, dV_{n-1}(u),\end{aligned}$$

where $\#(f^{-1}[y])$ is the number of points in the preimage $f^{-1}[y] := \{x : f(x) = y\}$. To complete the proof it is enough to show $\#(f^{-1}[y]) = 2$ for almost all $y \in K|a^\perp$.

As $K|a^\perp$ is convex its boundary $\partial(K|a^\perp)$ has measure zero. Therefore we only need consider y in the interior, $\operatorname{int}\left(K|a^\perp\right)$, of $K|a^\perp$. If $y \in \operatorname{int}\left(K|a^\perp\right)$ then there are exactly two points $x_1, x_2 \in \partial K$ with $x_j|a^\perp = y$. Thus $f^{-1}[y]$ is the disjoint union of $\varphi^{-1}[x_1]$ and $\varphi^{-1}[x_2]$. But, [21, Theorem 2.2.4, p. 74], the set, P, of points x in ∂K such that there is more than one outward unit normal to K at x is a set of measure zero. So if $x_1, x_2 \notin P$, each of the sets $\varphi^{-1}[x_1]$ and $\varphi^{-1}[x_2]$ will have just one element and therefore $\#(f^{-1}[y]) = 2$. The map $y \mapsto y|a^\perp$ is Lipschitz and hence it maps sets of measure zero to sets of measure zero. Thus $P|a^{\perp}$ is a set of measure zero. Hence for $y \in \operatorname{int}\left(K|a^{\perp}\right) \setminus P|a^{\perp}$, and therefore for almost all $y \in K|a^{\perp}$, $\#(f^{-1}[y]) = 2$ which finishes the proof. □

Proposition 3.9. *Let K_1 and K_2 be convex bodies in $\mathbf{R}^n$ with $C^{1,1}$ support functions h_1 and h_2, respectively. Then there is a constant β such that $V_{n-1}(K_1|a^{\perp}) = \beta V_{n-1}(K_2|a^{\perp})$ for all $a \in \mathbb{S}^{n-1}$ if and only if*

$$\det(h_1 I + \nabla^2 h_1) = \beta \det(h_2 I + \nabla^2 h_2) + q, \quad \textit{with } q \textit{ an odd function}.$$

Proof. By Lemma 3.8 $V_{n-1}(K_1|a^{\perp}) = \beta V_{n-1}(K_2|a^{\perp})$ for all $a \in \mathbb{S}^{n-1}$ if and only if $\int_{\mathbb{S}^{n-1}} q(u)|\langle a, u\rangle|\,du = 0$ for all $a \in \mathbb{S}^{n-1}$ where $q = \det(h_1 I + \nabla^2 h_1) - \beta \det(h_2 I + \nabla^2 h_2)$. That is, if and only if q is in the kernel of the cosine transform $(Cf)(a) := \int_{\mathbb{S}^{n-1}} f(u)|\langle a, u\rangle|\,du$. But, [8, Theorem C.2.4, p. 381], the kernel of the cosine transform is exactly the set of odd functions on $\mathbb{S}^{n-1}$. □

4. Non-existence of odd functions on $\mathbb{S}^{n-1}$ satisfying certain differential inequalities

Theorem 4.1. *There does not exist any odd $C^{1,1}$ function p on $\mathbb{S}^{n-1}$ such that for some $\delta > 0$*

$$\det(pI + \nabla^2 p) \leqslant -\delta \tag{4.1}$$

holds almost everywhere on $\mathbb{S}^{n-1}$.

Remark 4.2. Trivial changes in the proof show that condition (4.1) can be replaced by the inequality $\det(pI + \nabla^2 p) \geqslant \delta$ almost everywhere.

The proof is based on the following which may be of independent interest.

Proposition 4.3. *Let U be an open neighborhood of the origin in $\mathbf{R}^n$ and $f = (f^1, f^2, \ldots, f^n)\colon U \to \mathbf{R}^n$ a Lipschitz map with $f(0) = 0$ and such that for some $\delta > 0$*

$$\det f'(x) \geqslant \delta$$

for almost all $x \in U$. Then there is a neighborhood W of 0 and a constant $C_o > 0$ such that for all $y \in W$, there is an $x \in U$ with $f(x) = y$ and

$$|x| \leqslant C_o |y|.$$

Remark 4.4. It is not being assumed that the function f is injective. In particular it applies to the map on the plane given in polar coordinates by $f(r,\theta) = (r, k\theta)$ where k is any positive integer. This example rules out proving the proposition by use of some form of the inverse function theorem for Lipschitz functions such as the Clarke inverse function theorem (cf. [5]).

Proof of Proposition 4.3. Let $x \in U$ be a point where $f'(x)$ exists. The *operator norm* of the linear map $f'(x)$ is $\|f'(x)\| := \sup_{|u|=1} |f'(x)u|$. We first claim that for some constant $K \geqslant 1$ that f is *K-quasiregular*, that is

$$\|f'(x)\|^n \leqslant K \det f'(x) \tag{4.2}$$

holds almost everywhere on U. In fact, since f is Lipschitz, there is $C > 0$ such that $\|f'(x)\| \leqslant C$ almost everywhere on U, and therefore the distortion inequality (4.2) holds with $K = C^n/\delta$. By a theorem of Reshetnyak [19] and Martio–Rickman–Väisälä [16] (cf. [20, Theorem 4.3, p. 37]) f has *finite linear distortion* at the origin:

$$\limsup_{r \downarrow 0} \frac{\max_{|x|=r} |f(x)|}{\min_{|x|=r} |f(x)|} =: H < \infty.$$

Thus there is a $r_o > 0$ such that

$$0 < r \leqslant r_o \quad \text{implies} \quad \max_{|x|=r} |f(x)| \leqslant 2H \min_{|x|=r} |f(x)|. \tag{4.3}$$

Let B_r be the ball of radius r centered at the origin. By a standard smoothing argument there is a sequence of C^∞ function $f_\ell = (f_\ell^1, \ldots, f_\ell^n)$, given by a convolution $f_\ell(x) = \int \varphi_\ell(y) f(x-y)\, dy$ with a smooth approximation $\{\varphi_\ell\}_{\ell=1}^\infty$ of the identity, such that $f_\ell \to f$ uniformly on B_{r_o}, $f_\ell' \to f'$ almost everywhere on B_{r_o} and f_ℓ satisfies the same Lipschitz condition as f. Therefore $\|f_\ell'(x)\| \leqslant C$. Hence for $r \leqslant r_o$ by Lebesgue's bounded convergence theorem, Stokes' theorem, the uniform convergence $f_\ell \to f$, and (4.3)

$$\begin{aligned}
\delta V_n(B_r) &\leqslant \int_{B_r} \det f'(x)\, dx = \lim_{\ell\to\infty} \int_{B_r} \det f_\ell'(x)\, dx \\
&= \lim_{\ell\to\infty} \int_{B_r} d(f_\ell^1\, df_\ell^2 \wedge \cdots \wedge df_\ell^n) = \lim_{\ell\to\infty} \int_{\partial B_r} f_\ell^1\, df_\ell^2 \wedge \cdots \wedge df_\ell^n \\
&\leqslant V_{n-1}(\partial B_r) \lim_{\ell\to\infty} \left(\max_{|x|=r} \|f_\ell'(x)\|^{n-1} \max_{|x|=r} |f_\ell(x)| \right) \\
&\leqslant V_{n-1}(\partial B_r) C^{n-1} \max_{|x|=r} |f(x)| \\
&\leqslant 2H V_{n-1}(\partial B_r) C^{n-1} \min_{|x|=r} |f(x)|.
\end{aligned}$$

Using this and that $rV_{n-1}(\partial B_r) = nV_n(B_r)$ yields that when $0 < |x| = r \leqslant r_o$

$$|f(x)| \geqslant \min_{|x|=r} |f(x)| \geqslant \frac{\delta V_n(B_r)}{2HV_{n-1}(\partial B_r)C^{n-1}} = \frac{\delta r}{2nHC^{n-1}} = \frac{\delta}{2nHC^{n-1}}|x|.$$

Hence if $C_o := 2nHC^{n-1}/\delta$ then

$$x \in B_{r_o} \quad \text{implies} \quad |x| \leqslant C_o|f(x)|.$$

But, by a Theorem of Reshetnyak [18] (cf. [20, Theorem 4.1, p. 16]), non-constant quasiregular maps are open, that is they map open sets to open sets. Hence $W = f[B_{r_o}]$ is a neighborhood of $f(0) = 0$ with the required properties. $\square$

Corollary 4.5. *Let U be an open neighborhood of the origin in $\mathbf{R}^n$ and $\psi: U \to \mathbf{R}^n$ a Lipschitz map such that for some constant $\delta > 0$ the inequality $\det \psi'(x) \leqslant -\delta$ holds for almost all $x \in U$. Then there is a constant $C_o > 0$ and a neighborhood W of $\psi(0)$ such that for all $y \in W$, there is an $x \in U$ with $\psi(x) = y$ and*

$$|x| \leqslant C_o|y - \psi(0)|.$$

Proof. Let a be a unit vector in $\mathbf{R}^n$ and $Rx := x - 2\langle x, a\rangle a$ the reflection in the hyperplane $a^\perp$. Then $\det R = -1$ and therefore the corollary follows from Proposition 4.3 applied to the map $f(x) := R(\psi(x) - \psi(0))$. $\square$

Lemma 4.6. *Let $p: \mathbb{S}^{n-1} \to \mathbf{R}$ be a $C^{1,1}$ function such that for some $\delta_0 > 0$ the inequality $\det(pI + \nabla^2 p) < -\delta_0$ holds almost everywhere on $\mathbb{S}^{n-1}$. Let $\varphi(u) := p(u)u + \nabla p(u)$. Then for any unit vector, a, the height function $H_a(u) := \langle \varphi(u), a\rangle$ can only have a local maximum or minimum at $u = a$ or $u = -a$.*

Proof. Let a be a unit vector and let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbf{R}^n$. By a rotation we can assume that the height function H_a has a local maximum or minimum at e_n. We then need to show that $a = \pm e_n$. Let B^{n-1} be the open unit ball in $\mathbf{R}^{n-1}$ and parameterize the upper hemisphere $\mathbb{S}^{n-1}_+$ of $\mathbb{S}^{n-1}$ by $u: B^{n-1} \to \mathbb{S}^{n-1}_+$ given by

$$u(x) = (x, \sqrt{1 - |x|^2}).$$

We can view the restriction of p to the upper hemisphere $\mathbb{S}^{n-1}_+$ as a function of $x \in B^{n-1}$. Let $\partial p := \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \ldots, \frac{\partial p}{\partial x_{n-1}}\right)$. Then the spherical gradient is given by

$$\begin{aligned}\nabla p(x) &= (\partial p(x), 0) - \langle(\partial p(x), 0), u(x)\rangle u(x) \\ &= (\partial p(x), 0) - \langle x, \partial p(x)\rangle(x, \sqrt{1 - |x|^2})\end{aligned}$$

and thus

$$\begin{aligned}\varphi(u) &= p(x)u(x) + \nabla p(x)\\ &= (p(x)x, p(x)\sqrt{1-|x|^2}\,) + (\partial p(x), 0) - \langle x, \partial p(x)\rangle(x, \sqrt{1-|x|^2}\,).\end{aligned}$$

Write this as

$$\varphi(x) = (\psi(x), \langle\varphi(x), e_n\rangle),$$

where

$$\psi(x) = \partial p(x) + (p(x) - \langle x, \partial p(x)\rangle)x$$

is the orthogonal projection of $\varphi(x)$ onto $e_n^\perp = \mathbf{R}^{n-1}$. This is clearly Lipschitz in a neighborhood of the origin. For $x \in B^{n-1}$ the tangent space to $\mathbb{S}^{n-1}$ at $u(x)$ is $u(x)^\perp$ and the orientation of this tangent space is so that the projection onto $\mathbf{R}^{n-1}$ is orientation preserving. (This because $u(x)$ is in the upper hemisphere). By Proposition 3.2 $\varphi'(x) = p(x)I + \nabla^2 p(x)$ almost everywhere and so by hypothesis $\det\varphi'(x) \leqslant -\delta_0$. The projection, π, of the tangent space $T_u\mathbb{S}^{n-1}$ onto $\mathbf{R}^{n-1}$ has Jacobian $J(\pi) = \langle u, e_n\rangle$. As $\psi = \pi \circ \varphi$

$$J(\psi) = J(\pi)J(\varphi) = \langle u, e_n\rangle \det\varphi'(x) \leqslant -\langle u, e_n\rangle\delta_0.$$

But $\langle u(x), e_n\rangle = \sqrt{1-|x|^2}$ so if $x \in U := \{x \in B^{n-1} : |x| < \sqrt{3}/2\}$, then $\langle u(x), e_n\rangle > 1/2$ for $x \in U$. Thus for $x \in U$ the inequality $J(\psi) < -\delta$ holds where $\delta = \frac{1}{2}\delta_0$. Therefore ψ and U satisfy the hypothesis of Corollary 4.5. Hence $\psi(0)$ has a neighborhood W such that

$$\text{For } y \in W \text{ there is an } x \in U \text{ with } |x| \leqslant C_o|y - \psi(0)|. \tag{4.4}$$

Write $a = (\tilde{a}, a_n)$ where $\tilde{a} \in \mathbf{R}^{n-1}$. Then the height function H_a is given by

$$H_a(u(x)) = \langle\psi(x), \tilde{a}\rangle + a_n\langle\varphi(x), e_n\rangle. \tag{4.5}$$

But $\langle\varphi(x), e_n\rangle = (p(x) - \langle x, \partial p(x)\rangle)\sqrt{1-|x|^2}$. Taylor's theorem implies $p(x) = p(0) + \langle x, \partial p(0)\rangle + O(|x|^2)$, $\langle x, \partial p(x)\rangle = \langle x, \partial p(0)\rangle + O(|x|^2)$, and $\sqrt{1-|x|^2} = 1 + O(|x|^2)$. Therefore $\langle\varphi(x), e_n\rangle = p(0) + O(|x|^2)$. Using this in (4.5) gives

$$H_a(u(x)) = \langle\psi(x), \tilde{a}\rangle + a_n p(0) + O(|x|^2). \tag{4.6}$$

For real t with $|t|$ small the point $y_t := \psi(0)+t\tilde{a}$ will be in the neighborhood W of $\psi(0)$. By implication (4.4) there is an $x_t \in U$ with $\psi(x_t) = y_t$ and $|x_t| \leqslant C_o|y_t - \psi(0)| = C_o|t|$. Using this in (4.6) gives

$$H_a(u(x_t)) = \langle y_t, \tilde{a}\rangle + a_n p(0) + O(t^2) = \big(\langle \psi(0), \tilde{a}\rangle + a_n p(0)\big) + t|\tilde{a}| + O(t^2).$$

This has a local maximum or minimum at $t = 0$ which is only possible if $\tilde{a} = 0$. As a is a unit vector this implies $a_n = \pm 1$. That is $a = \pm e_n$, which completes the proof. $\square$

Proof of Theorem 4.1. Assume that there is an odd $C^{1,1}$ function p on $\mathbb{S}^{n-1}$ such that for some $\delta > 0$ the inequality $\det(pI + \nabla^2 p) \leqslant -\delta$ holds almost everywhere on $\mathbb{S}^{n-1}$. Let $\varphi(u) = p(u)u + \nabla p(u)$. By Lemma 3.5 the function $u \mapsto \nabla p(u)$ is even on $\mathbb{S}^{n-1}$. Therefore $\varphi(-u) = \varphi(u)$ for all $u \in \mathbb{S}^{n-1}$. Let a be any unit vector in $\mathbf{R}^n$. Then the height function $H_a(u) = \langle \varphi(u), a\rangle$ will also satisfy $H_a(-u) = H_a(u)$. As $\mathbb{S}^{n-1}$ is compact H_a will have both a global maximum and a global minimum. By Lemma 4.6 these maximizers can only occur at a and $-a$. But then $H_a(-a) = H_a(a)$ implies that the maximum and minimum values of H_a are the same and therefore H_a is constant. Hence all the height functions H_a are constant which implies φ is constant and thus $\varphi'(u) = 0$ for all u. But by Proposition 3.2 $\det(\varphi'(u)) = \det(pI + \nabla^2 p) \leqslant -\delta < 0$ for almost all u. This contradiction completes the proof. $\square$

5. Three-dimensional bodies of constant width and brightness

To prove Theorem 1.2 we let K and K_0 be convex bodies in $\mathbf{R}^3$ such that K_0 is centrally symmetric about the origin and that there are constants α and β such that $w_K(u) = \alpha w_{K_0}(u)$ and $(K|u^\perp) = \beta V_2(K_0|y^\perp)$ for all unit vectors u. By rescaling K by a factor of $1/\alpha$ we can assume that $\alpha = 1$, that is K and K_0 have same width in all directions. Then K_0 being centrally symmetric about the origin implies that K_0 is the central symmetral $\frac{1}{2}(K - K)$ of K. Therefore to prove Theorems 1.1 and 1.2 it is enough to prove:

Theorem 5.1. *Let K be a convex body in $\mathbf{R}^3$ such that its central symmetral $K_0 = \frac{1}{2}(K - K)$ is a regular gauge and for some constant β*

$$V_2(K|u^\perp) = \beta V_2(K_0|u^\perp) \quad \textit{for all } u \in \mathbb{S}^2. \tag{5.1}$$

Then K is a translate of K_0.

Lemma 5.2. *If* (5.1) *holds, then $\beta \leqslant 1$ and if $\beta = 1$, then K is a translate of K_0.*

Proof. Let $u \in \mathbb{S}^2$. Then $K_0|u^\perp$ is centrally symmetric about the origin and, viewed as convex bodies in the two-dimensional space $u^\perp$, the sets $K_0|u^\perp$ and $K|u^\perp$ have the same width function. Therefore $K_0|u^\perp$ is the central symmetral of $K|u^\perp$. By Proposition 2.1 this implies $V_2(K_0|u^\perp) \geqslant V_2(K|u^\perp)$ with equality if and only if $K|u^\perp$ is a translate of $K_0|u^\perp$. As $V_2(K|u^\perp) = \beta V_2(K_0|u^\perp)$ this yields that $\beta \leqslant 1$. If $\beta = 1$, then for all $u \in \mathbb{S}^2$ the set $K|u^\perp$ is a translate of $K_0|u^\perp$. This implies, [8, Theorem 3.1.3, p. 93], that K is a translate of K_0. $\square$

From now on we assume K and K_0 satisfy the hypothesis of Theorem 5.1 and that h and h_0 are the support functions of K and K_0, respectively. By Lemma 5.2 if $\beta = 1$, Theorem 5.1 holds, so, towards a contradiction, assume $\beta < 1$.

As K and K_0 have the same width function, $h(u)+h(-u) = h_0(u)+h_0(-u) = 2h_0(u)$ because $h_0(-u) = h_0(u)$ from the central symmetry of K_0 about the origin. Therefore

$$h(u) = \frac{1}{2}(h(u) + h(-u)) + \frac{1}{2}(h(u) - h(-u)) = h_0(u) + p(u),$$

where $p(u) := \frac{1}{2}(h(u) - h(-u))$ is clearly an odd function. As K_0 is a regular gauge it slides freely inside of some Euclidean ball and thus by Proposition 2.3 h_0 is $C^{1,1}$. Then Corollary 2.6 implies h is $C^{1,1}$ and the formula $p(u) = \frac{1}{2}(h(u) - h(-u))$ shows that p is also $C^{1,1}$. Proposition 3.9 implies there is an odd function q on $\mathbb{S}^2$ such that

$$\det(hI + \nabla^2 h) = \beta \det(h_0 I + \nabla^2 h_0) + q \tag{5.2}$$

holds almost everywhere on $\mathbb{S}^2$. The equality $h = h_0 + p$ implies

$$\det(hI + \nabla^2 h) = \det\Big((pI + \nabla^2 p) + (h_0 I + \nabla^2 h_0)\Big). \tag{5.3}$$

For any 2×2 matrix $\mathrm{tr}(A)^2 - \mathrm{tr}(A^2) = 2\det(A)$, where $\mathrm{tr}(A)$ is the trace of A. Define $\sigma(A, B)$ on pairs of 2×2 matrices by $\sigma(A, B) = \frac{1}{2}(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))$. Then $\sigma(\,,\,)$ is a symmetric bilinear form and $\sigma(A, A) = \det(A)$. Hence $\det(A + B) = \det(A) + 2\sigma(A, B) + \det(B)$. Using this in (5.3) gives

$$\begin{aligned}\det(hI + \nabla^2 h) &= \det(pI + \nabla^2 p) + 2\sigma(pI + \nabla^2 p, h_0 I + \nabla^2 h_0) \\ &\quad + \det(h_0 I + \nabla^2 h_0).\end{aligned} \tag{5.4}$$

The function h_0 is even on $\mathbb{S}^2$ and Lemma 3.5 implies $\nabla^2 h_0$ is also even. Therefore $h_0 I + \nabla^2 h_0$ is even. Likewise Lemma 3.5 applied to the odd function p implies $pI + \nabla^2 p$ is odd. But $\det(-A) = \det(A)$ for 2×2 matrices, so the function $\det(pI + \nabla^2 p)$ is even. The function $\sigma(pI + \nabla^2 p, h_0 I + \nabla^2 h_0)$ is odd as a function of the first argument and even as a function of the second argument, therefore $\sigma(pI + \nabla^2 p, h_0 I + \nabla^2 h_0)$ is an odd function. Comparing the two formulas (5.2) and (5.4) for $\det(hI + \nabla^2 h)$, equating the even parts, and rearranging gives

$$\det(pI + \nabla^2 p) = -(1 - \beta) \det(h_0 I + \nabla^2 h_0).$$

By Proposition 3.7 and the assumption that a Euclidean ball slides freely inside of K_0, there is a constant $C_1 > 0$ such that $\det(h_0 I + \nabla^2 h_0) \geqslant C_1$ and therefore $\det(pI + \nabla^2 p) \leqslant -\delta$ holds almost everywhere with $\delta = (1-\beta)C_1 > 0$. This contradicts Theorem 4.1 and completes the proof.

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