PROCESSES OF FLATS INDUCED BY HIGHER DIMENSIONAL PROCESSES III

Paul Goodey, Ralph Howard and Mark Reeder

Department of Mathematics, University of Oklahoma, Norman, OK 73019
Department of Mathematics, University of South Carolina, Columbia, SC 29208
Department of Mathematics, University of Oklahoma, Norman, OK 73019

Abstract. Stationary processes of k-flats in $\mathbb{E}^d$ can be thought of as point processes on the Grassmannian $\mathcal{L}_k^d$ of k-dimensional subspaces of $\mathbb{E}^d$. If such a process is sampled by a $(d-k+j)$-dimensional space F, it induces a process of j-flats in F. In this work we will investigate the possibility of determining the original k-process from knowledge of the intensity measures of the induced j-processes. We will see that this is impossible precisely when $1 < k < d-1$ and $j = 0, \dots, 2\left[\frac{r}{2}\right] - 1$, where r is the rank of the manifold $\mathcal{L}_k^d$. We will show how the problem is equivalent to the study of the kernel of various integral transforms, these will then be investigated using harmonic analysis on Grassmannian manifolds.

§1. Introduction.

The motivation for this work is a problem posed by Matheron [1974, 1975]. This question asks whether it is possible to determine a k-flat process in $\mathbb{E}^d$ from the densities of its induced point processes in $(d-k)$-dimensional spaces. For background information on geometric stochastic processes the reader is referred to Stoyan, Kendall and Mecke [1987] and Mecke, Schneider, Stoyan and Weil [1990]. Matheron obtained positive results in the cases $k = 1$ and $d-1$ and conjectured that the same would be true for $1 < k < d-1$. This was shown to be false by Goodey and Howard [1990a]. The same authors [1990b] obtained some partial results on the determination of the original process from the intensity measures of its induced j-flat processes for $0 \leqslant j < k \leqslant d-1$. In this work we will obtain complete answers to these questions. Although these problems belong to stochastic geometry, we will see later that they have connections with other branches of geometry. The proofs of our results will employ techniques of harmonic analysis to study the kernels of certain integral transforms on Grassmannian manifolds. In this section we will introduce the basic terminology and discuss Matheron's formulation of the problem.

For $1 \leqslant k \leqslant d-1$, a k-flat process X_k in $\mathbb{E}^d$ is a point process on the homogeneous space $\mathcal{E}_k^d$ of k-flats (k-dimensional affine subspaces) in $\mathbb{E}^d$. This is a random variable with values in the space $\mathbb{M}_k$ of locally finite collections of k-flats. The distribution of X_k is a probability

The research of the first and third authors was supported in part by NSF grants DMS-9207019 and DMS-9304284. The research of the second author was supported in part by NFR contract number R-RA 4873-306 and the Swedish Academy of Sciences.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX measure on $\mathbb{M}_k$. We will consider two processes with the same distribution to be equivalent and our uniqueness statements should be interpreted as meaning uniqueness up to this equivalence. The *intensity measure*, $\tilde{\theta}$, of X_k is the Borel measure on $\mathcal{E}_k^d$ which assigns to each Borel set $S \subset \mathcal{E}_k^d$ the expected number of k-flats of X_k which are in S. We note that equivalent processes have the same intensity measure. X_k is said to be a Poisson process if, for each Borel set $S \subset \mathcal{E}_k^d$, the random variable $\#(X_k \cap S)$ has a Poisson distribution and if, for disjoint Borel sets $S_1, \ldots, S_m$, the random variables $\#(X_k \cap S_1), \ldots, \#(X_k \cap S_m)$ are independent.

We will only consider processes for which the intensity measure is *locally finite*, that is, gives finite measure to compact sets. For each such measure, there is precisely one Poisson process for which it is the intensity measure. A *stationary Poisson process* is one for which the intensity measure is translation invariant. In this paper, all processes will be assumed to be stationary and Poisson.

There is a bijective correspondence between the locally finite translation invariant measures on $\mathcal{E}_k^d$ and the finite measures on $\mathcal{L}_k^d$, the compact Grassmannian of k-dimensional subspaces of $\mathbb{E}^d$. To see this, we note that we can identify $\mathcal{E}_k^d$ with $\{(\eta, x) \in \mathcal{L}_k^d \times \mathbb{E}^d : x \in \eta^\perp\}$ and let $\pi : \mathcal{E}_k^d \to \mathcal{L}_k^d$ be defined by $\pi(\eta, x) = \eta$. For each $\eta \in \mathcal{L}_k^d$, we let $\lambda_{d-k}(\eta^\perp; \cdot)$ denote the $(d-k)$-dimensional Lebesgue measure on $\eta^\perp$. Then any finite measure θ on $\mathcal{L}_k^d$ determines a locally finite measure $\tilde{\theta}$ on $\mathcal{E}_k^d$ by

$$\tilde{\theta}(S) = \int_{\mathcal{L}_k^d} \lambda_{d-k}(\eta^\perp; S \cap \pi^{-1}(\eta))\, \theta(d\eta) \tag{1.1}$$

where S is a Borel subset of $\mathcal{E}_k^d$. The translation invariance of $\tilde{\theta}$ follows from that of Lebesgue measure. Conversely, if $\tilde{\theta}$ is a translation invariant locally finite measure on $\mathcal{E}_k^d$ and σ is a Borel subset of $\mathcal{L}_k^d$, we put

$$\theta(\sigma) = \omega_{d-k}^{-1} \tilde{\theta}(B_k^d \cap \pi^{-1}(\sigma)), \tag{1.2}$$

where ω_{d-k} is the volume of the unit $(d-k)$-ball and B_k^d denotes those elements of $\mathcal{E}_k^d$ which intersect the unit ball of $\mathbb{E}^d$. The fact that (1.2) inverts (1.1) follows from the characteristic properties of Lebesgue measure. If K is a compact subset of $\mathbb{E}^d$ we put

$$\psi(K) = \tilde{\theta}(\{E \in \mathcal{E}_k^d : E \cap K \neq \emptyset\})$$

and note that, as a consequence of (1.1), we have

$$\psi(K) = \int_{\mathcal{L}_k^d} \lambda_{d-k}(\eta^\perp; \Pi_{\eta^\perp} K)\, \theta(d\eta), \tag{1.3}$$

where $\Pi_{\eta^\perp}$ denotes the orthogonal projection onto $\eta^\perp$. If $\tilde{\theta}$ is the intensity measure of the stationary k-flat process X_k we will say that θ is the *measure associated with* the process X_k. The comments above show that X_k is uniquely determined by θ.

In order to motivate our problem, we will repeat some of the discussion given in Matheron [1975], but with slightly different notation. If $0 \leqslant j < k < d$ then, for almost all

$E \in \mathcal{E}^d_{d-k+j}$, the process $X_k \cap E$ is a stationary j-flat process in E which depends only on the subspace $\zeta \in \mathcal{L}^d_{d-k+j}$ parallel to E. Our objective is to determine those triples (d, k, j) for which the process X_k is determined by knowledge of the processes $X_k \cap E$ for all $E \in \mathcal{E}^d_{d-k+j}$. For $1 \leqslant m, n < d$, $E \in \mathcal{E}^d_m$ and $\zeta \in \mathcal{L}^d_m$, we put

$$\mathcal{E}^d_n(E) = \{F \in \mathcal{E}^d_n : F \subset E\} \qquad \text{and} \qquad \mathcal{L}^d_n(\zeta) = \{\eta \in \mathcal{L}^d_n : \eta \subset \zeta\} \qquad \text{if } n < m$$

and

$$\mathcal{E}^d_n(E) = \{F \in \mathcal{E}^d_n : F \supset E\} \qquad \text{and} \qquad \mathcal{L}^d_n(\zeta) = \{\eta \in \mathcal{L}^d_n : \eta \supset \zeta\} \qquad \text{if } n > m.$$

If $E \in \mathcal{L}^d_{d-k+j}$ and $\zeta \in \mathcal{L}^d_{d-k+j}$ is parallel to E, we will denote by θ^ζ the measure on $\mathcal{L}^d_j(\zeta)$ associated with the process $X_k \cap E$. We now want to formulate the relationship between the measures θ and θ^ζ for $\zeta \in \mathcal{L}^d_{d-k+j}$. For the moment we will assume that X_k is a process of flats parallel to $\xi_0 \in \mathcal{L}^d_k$. Then θ is an atomic measure concentrated on ξ_0. We choose $E \in \mathcal{L}^d_{d-k+j}$ so that $\dim(\xi_0 \cap E) = j$ and put $\sigma_0 = \xi_0 \cap \zeta$ where $\zeta \in \mathcal{L}^d_{d-k+j}$ is parallel to E. Then it is clear that θ^ζ is concentrated on σ_0. Now let K be any compact subset of E. We will use (1.3) to calculate $\psi(K)$ first in terms of θ and then in terms of θ^ζ. This gives

$$\psi(K) = \lambda_{d-k}(\xi_0^\perp; \Pi_{\xi_0^\perp} K)\theta(\xi_0) = \lambda_{d-k}(\sigma_0^\perp; \Pi'_{\sigma_0^\perp} K)\theta^\zeta(\sigma_0), \tag{1.4}$$

where $\Pi'_{\sigma_0^\perp}$ denotes the orthogonal projection in ζ onto $\sigma_0^\perp$, the $(d-k)$-dimensional subspace of ζ orthogonal to σ_0. If $u_1, \dots, u_j$ is an orthonormal basis of σ_0, we first extend it to an orthonormal basis of ξ_0, introducing the vectors $v_1, \dots, v_{k-j}$ and then to an orthonormal basis of ζ introducing the vectors $w_1, \dots, w_{d-k}$. We denote by $[\xi_0, \zeta]$ the volume of the parallelepiped generated by the vectors $u_1, \dots, u_j, v_1, \dots, v_{k-j}, w_1, \dots, w_{d-k}$. Then we have

$$\lambda_{d-k}(\xi_0^\perp; \Pi_{\xi_0^\perp} K) = [\xi_0, \zeta]\lambda_{d-k}(\sigma_0^\perp; \Pi'_{\sigma_0^\perp} K).$$

So it follows from (1.4) that, if $g \in C(\mathcal{L}^d_j(\zeta))$, then

$$\int_{\mathcal{L}^d_j(\zeta)} g(\sigma)\, \theta^\zeta(d\sigma) = g(\sigma_0)\theta^\zeta(\sigma_0) = [\xi_0, \zeta] g(\xi_0 \cap \zeta)\theta(\xi_0).$$

Since all measures can be approximated by linear combinations of atomic measures, standard measure-theoretic results now show that for any k-flat process X_k and almost all $\zeta \in \mathcal{L}^d_{d-k+j}$, we have

$$\int_{\mathcal{L}^d_j(\zeta)} g(\sigma)\, \theta^\zeta(d\sigma) = \int_{\mathcal{L}^d_k} [\xi, \zeta] g(\xi \cap \zeta)\, \theta(d\xi),$$

for all $g \in C(\mathcal{L}^d_j)$.

So our problem is to determine those triples (d,k,j) with the property that if ϕ is a signed measure on $\mathcal{L}_k^d$ with

$$\int_{\mathcal{L}_k^d} [\xi,\zeta] g(\xi\cap\zeta)\,\phi(d\xi)=0$$

for all $g\in C(\mathcal{L}_j^d)$ and all $\zeta\in\mathcal{L}_{d-k+j}^d$, then ϕ is identically zero. In order to summarize our results, we let $M(\mathcal{L}_k^d)$ denote the signed Borel measures on $\mathcal{L}_k^d$ and put

$$\mathfrak{M}(d,k,j)=\Big\{\phi\in M(\mathcal{L}_k^d):\int_{\mathcal{L}_k^d}[\xi,\zeta]g(\xi\cap\zeta)\,\phi(d\xi)=0 \quad \text{for all } \zeta\in\mathcal{L}_{d-k+j} \text{ and all } g\in C(\mathcal{L}_j^d)\Big\}.$$

We will show that, for $2\leqslant k\leqslant d-2$,

$$\mathfrak{M}(d,k,j)\neq\{0\}\iff j=0,\dots,2\left[\frac{r}{2}\right]-1, \tag{1.5}$$

where $r=\min\{k,d-k\}$. The case $j=0$ deserves special mention. In this case our problem reduces to determining the pairs (d,k) with the property that if ϕ is a signed measure on $\mathcal{L}_k^d$ such that

$$\int_{\mathcal{L}_k^d}|\langle\zeta,\xi^\perp\rangle|\,\phi(d\xi)=0$$

for all $\zeta\in\mathcal{L}_{d-k}^d$, then ϕ is identically zero; here $|\langle\zeta,\xi^\perp\rangle|$ denotes the absolute value of the determinant of the projection of ζ onto $\xi^\perp$. In fact this was the problem originally considered by Matheron. He [1974, 1975] gave an affirmative answer in the equivalent cases $k=1,\ d-1$ whereas Goodey and Howard [1990a] gave negative answers in all the cases $1<k<d-1$. In the above notation, Matheron's results give

$$\mathfrak{M}(d,d-1,j)=\mathfrak{M}(d,1,0)=\{0\}\qquad\text{for } 0\leqslant j\leqslant d-2.$$

Partial progress towards (1.5) was made by Goodey and Howard [1990a,b] using a special representation for $\mathcal{L}_2^4$ and various embedding arguments. In this work we use rather different methods and are able to shed some new light on these previous results.

§2. Statement of results.

The space $\mathfrak{M}(d,k,j)$ is a weakly closed subspace of $M(\mathcal{L}_k^d)$ which is invariant under the action of the group $SO(d)$. It follows (see Loomis [1953, Theorem 31F], for example) that $\mathfrak{M}(d,k,j)$ forms an ideal with respect to convolution. But convolution of functions in $C^\infty(\mathcal{L}_k^d)$ with measures in $M(\mathcal{L}_k^d)$ produces functions in $C^\infty(\mathcal{L}_k^d)$. So if

$$\mathfrak{F}(d,k,j)=\Big\{f\in C^\infty(\mathcal{L}_k^d):\int_{\mathcal{L}_k^d}[\xi,\zeta]g(\xi\cap\zeta)f(\xi)\,\nu_k(d\xi)=0 \quad \text{for all } \zeta\in\mathcal{L}_{d-k+j}^d \text{ and all } g\in C(\mathcal{L}_j^d)\Big\},$$

where ν_k denotes the invariant probability measure on $\mathcal{L}_k^d$, then

$$\mathfrak{F}(d,k,j)\neq\{0\}\iff\mathfrak{M}(d,k,j)\neq\{0\}.$$

From now on we will denote the rank $\min\{k,d-k\}$ of the Grassmannian $\mathcal{L}_k^d$ by r. Our main result is

Theorem. *If $2 \leqslant k \leqslant d-2$ and $0 \leqslant j < 2[\frac{r}{2}]$, then $\mathfrak{F}(d,k,j)$ is infinite dimensional. In all other cases $\mathfrak{F}(d,k,j)$ is trivial.*

Part of this was established in Goodey and Howard [1990b] where it was shown that $\mathfrak{F}(d,k,j)$ is trivial in the following cases:

(1) $d-k \leqslant j < k$;
(2) $k = j+1$ and k is odd;
(3) $j = d-k-1$, $d-k < k$ and $d-k$ is odd.

For $2 \leqslant k \leqslant d-2$, this is exactly the set of j where the relation $0 \leqslant j < 2[\frac{r}{2}]$ does not hold. Thus, what remains is to show that $\mathfrak{F}(d,k,j)$ is infinite dimensional if $j = 0,\dots,2[\frac{r}{2}]-1$. This result, for $j = 0,1$ was given in Goodey and Howard [1990a] but will be repeated here with a different proof, for completeness.

Each $\mathfrak{F}(d,k,j)$ is invariant under the action of $SO(d)$ and so, by the Peter-Weyl theorem (see Helgason [1984], for example), must be a direct sum of irreducible $SO(d)$-representations occuring in the harmonic expansion of $L^2(\mathcal{L}_k^d)$. We will show that, in the cases mentioned in the first part of the theorem, this sum involves infinitely many such representations.

§3. Reduction to the special cases $\mathfrak{F}(d,2n,2n-1)$.

In this section we will see that it suffices to prove that $\mathfrak{F}(d,2n,2n-1)$ is infinite dimensional for $4 \leqslant 4n \leqslant d$, and then give an alternative formulation for these cases.

Our first observation is that, for $1 \leqslant j < k$, we have $\mathfrak{F}(d,k,j) \subset \mathfrak{F}(d,k,j-1)$. This is most easily seen from the stochastic interpretation. If $f \in \mathfrak{F}(d,k,j)$ then the positive and negative parts of f can each be thought of as the density of the measure θ on $\mathcal{L}_k^d$ associated with a stationary k-flat process. These two processes must induce the same j-flat process on all members of $\mathcal{L}_{d-k+j}^d$. If $E \in \mathcal{E}_{d-k+j-1}^d$ and $F \in \mathcal{E}_{d-k+j}^d(E)$, the two processes we have constructed from f induce the same j-flat process in F. The $(j-1)$-flat processes they induce on E are, of course, precisely those induced by the above j-flat process in F and so must be the same. Consequently $f \in \mathfrak{F}(d,k,j-1)$, as claimed. It follows that we need only show that $\mathfrak{F}(d,k,2[\frac{r}{2}]-1)$ is infinite dimensional, for $2 \leqslant k \leqslant d-2$.

Proposition 3.1. *Assume $2 \leqslant k \leqslant d-2$ and $0 \leqslant j < k$. Then, if $\mathfrak{F}(d,k,j)$ is infinite dimensional, so is $\mathfrak{F}(d+1,k+1,j)$.*

Proof. We think of $\mathbb{E}^d$ being embedded in $\mathbb{E}^{d+1}$ and orthogonal to the final basis vector e_{d+1}, and let $\ell \in \mathcal{L}_1^{d+1}$ be the line parallel to e_{d+1}. If $f \in \mathfrak{F}(d,k,j)$ is non-trivial then, as above, its positive and negative parts give rise to different k-flat processes X_k, Y_k in $\mathbb{E}^d$ which induce the same j-flat processes on the members of $\mathcal{L}_{d-k+j}^d$. As we just noticed, they also induce the same $(j-1)$-flat processes on the members of $\mathcal{L}_{d-k+j-1}^d$. We denote by X_{k+1} the $(k+1)$-flat process in $\mathbb{E}^{d+1}$ whose realisations are the vector sum of ℓ with those of X_k. We shall denote this by writing $X_{k+1} = X_k + \ell$. For Borel sets $\eta \subset \mathcal{L}_k^d$, we will write $\eta + \ell$ to denote the set $\{E + \ell : E \in \eta\}$. In terms of the measures θ and ϕ associated with X_k and X_{k+1}, respectively, we have

$$\phi(\eta + \ell) = \theta(\eta)$$

for all Borel sets $\eta \subset \mathcal{L}_k^d$, and $\phi(\xi) = 0$ for all Borel sets $\xi \subset \mathcal{L}_{k+1}^{d+1}$ which are disjoint from $\mathcal{L}_k^d + \ell$. We now assume $\zeta \in \mathcal{L}_{d-k+j}^{d+1}$. If $\ell \subset \zeta$ then $\zeta = \zeta' + \ell$ for some $\zeta' \in \mathcal{L}_{d-k+j-1}^d$ and, denoting the j-flat process induced on ζ by $X_{k+1} \cap \zeta$, we have $X_{k+1} \cap \zeta = (X_k \cap \zeta') + \ell$. It follows immediately that X_{k+1} and Y_{k+1} induce the same j-flat process on ζ. If ζ does not contain ℓ then $\zeta + \ell = \zeta' + \ell \in \mathcal{L}_{d-k+j+1}^{d+1}$ for some $\zeta' \in \mathcal{L}_{d-k+j}^d$. But then, as previously,

$$X_{k+1} \cap (\zeta + \ell) = (X_k \cap \zeta') + \ell.$$

So X_{k+1} and Y_{k+1} induce the same $(j+1)$-flat process on $\zeta + \ell$. It follows that they must also induce the same j-flat process on ζ. Since $X_{k+1} \neq Y_{k+1}$ we see that $\mathfrak{M}(d+1, k+1, j)$ is non-trivial. The same is therefore true of $\mathfrak{F}(d+1, k+1, j)$ and, furthermore the relationship between X_{k+1} and X_k shows that $\mathfrak{F}(d+1, k+1, j)$ must be infinite dimensional.

In the case $r = d - k$, repeated application of Proposition 3.1 shows that it suffices to prove that $\mathfrak{F}(2r, r, 2[\frac{r}{2}] - 1)$ is infinite dimensional. It follows that, in general, it suffices to prove that $\mathfrak{F}(d, m, 2[\frac{m}{2}] - 1)$ is infinite dimensional for $d \geq 2m \geq 4$. If m is odd, Proposition 3.1 shows that the latter is a consequence of the infinite dimensionality of $\mathfrak{F}(d-1, m-1, m-2)$. Collecting these observations together, we see that it suffices to show that $\mathfrak{F}(d, 2n, 2n-1)$ is infinite dimensional for $d \geq 4n \geq 4$.

Our objective now is to find an alternative formulation of the above observation. For this, we will use a result of Goodey and Howard [1990b]. If $1 \leqslant j < k \leqslant d-1$ and if $\sigma \in \mathcal{L}_j^d$, we put

$$\mathcal{L}_{j,k}^d = \big\{(\sigma, \xi) \in \mathcal{L}_j^d \times \mathcal{L}_k^d : \xi \in \mathcal{L}_k^d(\sigma)\big\}.$$

The transform $T : C^\infty(\mathcal{L}_k^d) \to C^\infty(\mathcal{L}_{j,k}^d)$ is defined by

$$Tf(\sigma, \xi) = \int_{\mathcal{L}_k^d(\sigma)} |\langle \eta, \xi \rangle|^{j+1} f(\eta)\, \nu_k(\sigma; d\eta),$$

where $\nu_k(\sigma; \cdot)$ is the invariant probability measure on the compact manifold $\mathcal{L}_k^d(\sigma)$.

Proposition 3.2 (Goodey and Howard)**.** *$\mathfrak{F}(d, k, j) = \ker T$.*

Consequently, we are interested in the kernel of the transform

$$T : C^\infty(\mathcal{L}_{2n}^d) \to C^\infty(\mathcal{L}_{2n-1,2n})$$

where, for $(\sigma, \xi) \in \mathcal{L}_{2n-1,2n}^d$

$$Tf(\sigma, \xi) = \int_{\mathcal{L}_{2n}^d(\sigma)} \langle \eta, \xi \rangle^{2n} f(\eta)\, \nu_k(\sigma; d\eta). \tag{3.1}$$

We notice that, if $\sigma \in \mathcal{L}_{2n-1}^d$ then $\mathcal{L}_{2n}^d(\sigma)$ is isomorphic to $\mathcal{L}_1^{d-2n+1}$. For our purposes it is convenient to think of the latter as the sphere S^{d-2n} with antipodal points identified. If $\xi, \eta \in \mathcal{L}_{2n}^d(\sigma)$ and if u, v are the corresponding points of S^{d-2n} under this identification, we have $\langle \xi, \eta \rangle = \langle u, v \rangle$. Furthermore, this identification allows us to use techniques from the theory of spherical harmonics. We will denote by $\rho : L^2(\mathcal{L}_{2n}^d) \to L^2(\mathcal{L}_{2n}^d(\sigma))$ the restriction map and by $\mathcal{U}_{2n}$ the span of the spherical harmonics on S^{d-2n} of degrees $0, 2, \dots, 2n$.

Proposition 3.3. *For $4 \leqslant 4n \leqslant d$ and $\sigma \in \mathcal{L}^d_{2n-1}$ the space $\mathfrak{F}(d,2n,2n-1)$ is the space spanned by those irreducible $SO(d)$-representations W with $\rho(W)$ contained in $\mathcal{U}_{2n}^{\perp}$, the orthogonal complement of $\mathcal{U}_{2n}$ in $L^2(\mathcal{L}^d_{2n}(\sigma))$.*

Proof. Since we are viewing S^{d-2n} as the unit sphere in $\sigma^{\perp}$, the space $\mathcal{U}_{2n}$ is spanned by the restriction to S^{d-2n} of the homogeneous polynomials of degree $2n$ on $\sigma^{\perp}$. These polynomials are, in turn, spanned by the functions of the form $\langle u, \cdot\rangle^{2n}$ for $u \in S^{d-2n}$. Now, since $SO(d)$ acts transitively on $\mathcal{L}^d_{2n-1}$, the requirement that f be in $\mathfrak{F}(d,2n,2n-1)$ may be expressed as

$$\int_{S^{d-2n}} \langle u,v\rangle^{2n} f'(v)\,\lambda_{d-2n}(dv) = 0$$

for all $u \in S^{d-2n}$ and all $SO(d)$-rotations f' of f; here λ_{d-2n} denotes spherical Lebesgue measure on S^{d-2n}. If $W \subset L^2(\mathcal{L}^d_{2n})$ is an irreducible $SO(d)$-representation, then it is spanned by the set of all f' for any fixed non-trivial $f \in W$. It follows that $W \subset \mathfrak{F}(d,2n,2n-1)$ if and only if $\rho(W)$ is orthogonal to $\mathcal{U}_{2n}$, as required.

Proposition 3.3 allows us to compare $\mathcal{F}(d,2n,2n-1)$ with the kernel of the Radon transform $R_{2n,2n-1}$. We recall that, for $0 < i < j < d$, the Radon transform $R_{j,i}$ maps $L^2(\mathcal{L}^d_j)$ to $L^2(\mathcal{L}^d_i)$ and is defined by

$$(R_{j,i}f)(\sigma) = \int_{\mathcal{L}^d_j(\sigma)} f(\xi)\,\nu_j(\sigma;d\xi) \qquad \text{for } \sigma \in \mathcal{L}^d_i \text{ and } f \in L^2(\mathcal{L}^d_j),$$

see Helgason [1980 and 1984], for example. If $i < j$, the transform $R_{j,i}$ is injective if and only if $i+j \geq d$, see Gelfand, Graev and Roşu [1984] and Grinberg [1985 and 1986]. So, for $d \geq 4n \geq 4$, we see that $R_{2n,2n-1}$ is not injective. In fact, its kernel is the space spanned by those irreducible $SO(d)$-representations W with $\rho(W)$ contained in $\mathcal{U}_0^{\perp}$. Consequently, $\mathcal{F}(d,2n,2n-1)$ is a proper subspace of $\ker R_{2n,2n-1}$.

§4. Non-trivial functions in $\mathfrak{F}(d,2n,2n-1)$.

In this section we will complete the proof of the theorem by finding an infinite dimensional subspace of $\mathfrak{F}(d,2n,2n-1)$ in the case $d \geq 4n$. The key to this construction will be the branching theorem for the irreducible representations of $SO(d)$, see Boerner [1963, Theorem 12.1], for example. The symmetric space $SO(d)/(SO(d-2n)\times SO(2n))$ is the manifold $\hat{\mathcal{L}}^d_{2n}$ of oriented $2n$-dimensional subspaces of $\mathbb{E}^d$. As representations of $SO(d)$, $L^2(\hat{\mathcal{L}}^d_{2n})$ is a Hilbert direct sum of irreducible representations. These, in turn, are characterised by their highest weights; if $d = 2p$ is even these are the integer p-tuples of the form ${}^d(m_1,\ldots,m_p)$ with $m_1 \geq m_2 \geq \cdots \geq m_{p-1} \geq |m_p|$, whereas if $d = 2p+1$ is odd they are again integer p-tuples but now those which satisfy $m_1 \geq m_2 \geq \cdots \geq m_p \geq 0$. In this context the spherical harmonics of degree $2m$ on S^{d-1} correspond to the highest weights ${}^d(2m,0,\ldots,0)$ for $m = 0,1,\ldots$. The highest weights of the irreducible representations of $SO(d)$ which appear in $L^2(\hat{\mathcal{L}}^d_{2n})$, where $d \geq 4n$, are precisely those p-tuples ${}^d(m_1,\ldots,m_p)$ where $p = [\frac{d}{2}]$ which satisfy the the following two conditions:

(a) $m_i = 0$ for all $i > 2n$,
(b) the integers $m_1,\ldots,m_p$ all have the same parity;

see Strichartz [1975], Sugiura [1962] and Takeuchi [1973]. Consequently, we have a Hilbert direct sum

$$L^2(\hat{\mathcal{L}}^d_{2n}) = \bigoplus \mathcal{V}^d_{m_1,\dots,m_p}$$

where $\mathcal{V}^d_{m_1,\dots,m_p}$ is isomorphic to the $SO(d)$-representation corresponding to the highest weight ${}^d(m_1,\dots,m_p)$; here the summation is over all p-tuples satisfying (a) and (b) above. The functions of $L^2(\mathcal{L}^d_{2n})$ are those of $L^2(\hat{\mathcal{L}}^d_{2n})$ which are independent of orientation and so

$$L^2(\mathcal{L}^d_{2n}) = \bigoplus \mathcal{V}^d_{2m_1,\dots,2m_{2n},0,\dots,0}. \tag{4.1}$$

Proposition 4.1. *If $d \geq 4n \geq 4$ then*

$$\mathfrak{F}(d, 2n, 2n-1) \supseteq \bigoplus_{|m_{2n}|>n} \mathcal{V}^d_{2m_1,\dots,2m_{2n},0,\dots,0} \tag{4.2}$$

and is therefore infinite dimensional.

Proof. The branching theorem explains how to express the restriction of an irreducible representation of $SO(d)$ to $SO(d-1)$ as a sum of irreducible representations of $SO(d-1)$. In terms of the expansion in (4.2), it can be stated as follows

$$\mathcal{V}^{2p}_{m_1,\dots,m_p}\big|_{SO(2p-1)} \simeq \bigoplus_{m_1 \geq m'_1 \geq \dots \geq m_{p-1} \geq m'_{p-1} \geq |m_p|} \mathcal{V}^{2p-1}_{m'_1,\dots,m'_{p-1}}$$

and

$$\mathcal{V}^{2p+1}_{m_1,\dots,m_p}\big|_{SO(2p)} \simeq \bigoplus_{m_1 \geq m'_1 \geq \dots \geq m_{p-1} \geq m'_{p-1} \geq m_p \geq |m'_p|} \mathcal{V}^{2p}_{m'_1,\dots,m'_p}.$$

According to Proposition 3.3, it suffices to show that the representations W occurring in (4.2) do not contain any $SO(d-2n+1)$ subrepresentations isomorphic to $\mathcal{V}^{d-2n+1}_{2i,0,\dots,0}$ for $i < 2n$. This, in turn, is a consequence of

$$\mathcal{V}^d_{2m_1,\dots,2m_{2n},0,\dots,0}\big|_{SO(d-2n+1)} \begin{cases} \supset \mathcal{V}^{d-2n+1}_{2m_{2n},0,\dots,0}, \\ \not\supset \mathcal{V}^{d-2n+1}_{2i,0,\dots,0}, \quad \text{for } i < m_{2n}. \end{cases} \tag{4.3}$$

We will establish (4.3) by $2n-1$ applications of the branching theorem. We first consider the number of zeros in the subscripts of $SO(d)$-representations. On the left side of (4.3) there are $p-2n$ zeros, whereas on the right side there are $p-n-1$ zeros if $d = 2p$ and $p-n$ if $d = 2p+1$. The branching theorem shows that we can only gain a zero when we restrict from an odd dimension to an even dimension and that is achieved only by the term corresponding to $m'_k = 0$ where k is the largest index such that $m_k \neq 0$. In view of the number of zeros which have to be gained, we must make this choice at each opportunity. Furthermore, on making this choice, it is clear that, in the lexicographic ordering of highest weights, the least highest weight of the form ${}^{d-1}(k_1,\dots,k_t)$ which occurs in the restriction of ${}^d(2m_1,\dots,2m_{2n},0,\dots,0)$ to $SO(d-1)$ is the weight ${}^{d-1}(2m_2,\dots,2m_{2n},0,\dots,0)$. Repeated application of these observations now yields (4.3).

§5. Remarks.

The case $d = 4$ of Proposition 4.1 was obtained by Goodey and Howard [1990a] using a rather different approach. They used the fact that $\hat{\mathcal{L}}_2^4 = S^2 \times S^2$ (see Gluck and Warner [1983]) to write

$$L^2(\mathcal{L}_2^4) = \bigoplus_{n-m \text{ even}} S_{n,m} \tag{5.1}$$

where

$$S_{n,m} = \{f \in L^2(S^2 \times S^2) : \Delta_1 f + n(n+1)f = \Delta_2 f + m(m+1)f = 0\}.$$

Here, Δ_1 and Δ_2 are the Laplace-Beltrami operators on the two factors of $S^2 \times S^2$. So the functions of $S_{n,m}$ are spherical harmonics of degree n in the first factor, and of degree m in the second. We will now use this representation to show that, in case $d = 4,\ n = 1$ there is equality in (4.2). That is

$$\mathfrak{F}(4, 2, 0) = \mathfrak{F}(4, 2, 1) = \bigoplus_{|j|>1} \mathcal{V}_{2i,2j}^4. \tag{5.2}$$

The former equality is established in Goodey and Howard [1990a] and so we will concentrate on the latter. The first step is to establish the relationship between the two representations, (4.1) and (5.1), for $L^2(\mathcal{L}_2^4)$. This will be achieved by showing that

$$\mathcal{V}_{2i,\pm 2j}^4 \simeq S_{i\mp j, i\pm j} \qquad (i \geq j \geq 0). \tag{5.3}$$

In the $S^2 \times S^2$ representation, the members of $\mathcal{L}_2^4$ which lie in a fixed 3-dimensional subspace of $\mathbb{E}^4$ comprise a set isomorphic to the diagonal of $S^2 \times S^2$. We will use this to find isomorphic copies of certain $SO(3)$ subrepresentations in $S_{n,m}$.

We use coordinates (x, θ) where $-1 \leqslant x \leqslant 1$ and $0 \leqslant \theta \leqslant 2\pi$ to parameterize the points of each S^2 factor. The spherical harmonics of degree n on S^2 which are independent of θ are multiples of the Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

In fact there are points $u_1, \ldots, u_{2n+1} \in S^2$ such that the functions $P_n(\langle u_i, \cdot \rangle)$, for $i = 1, \ldots, 2n+1$, form a basis of the space of spherical harmonics of degree n. In any case, the spaces $S_{n,m}$ contain functions of the form $P_n P_m$. In looking for $SO(3)$ subrepresentations of these spaces, we note that

$$P_n(x)P_m(x) = \sum_{j=n-m}^{n+m} \gamma_j^{n,m} P_j(x) \qquad (n \geq m \geq 0), \tag{5.4}$$

where $\gamma_j^{n,m} \neq 0$ for all j of the same parity as $n-m$; see, for example, Gradshteyn and Ryzhik [1994, p.1046] or Adams and Hippisley [1922]. We know that, given $i \geq j \geq 0$, there are $n \geq m$ such that $\mathcal{V}^4_{2i,2j}$ is isomorphic to some $S_{n,m}$. The branching Theorem shows that the former contains $SO(3)$ subrepresentations isomorphic to $\mathcal{V}^3_k$ (which has dimension $2k+1$) for all $k = 2j, \dots, 2i$. Furthermore, the dimension of $\mathcal{V}^4_{2i,2j}$ is the sum of the dimensions of these $\mathcal{V}^3_k$, namely $4(i^2 - j^2 + i) + 1$. From (5.4) and the fact $\gamma^{n,m}_{n+m}$ and $\gamma^{n,m}_{n-m}$ are non-zero, we see that $S_{n,m}$ contains isomorphic copies of $\mathcal{V}^3_{n-m}$ and $\mathcal{V}^3_{n+m}$. So $n-m \geq 2j$ and $n+m \leqslant 2i$. If either of these inequalities were strict the dimension of $S_{n,m}$, namely $(2n+1)(2m+1)$ would be strictly less than that of $\mathcal{V}^4_{2i,2j}$. So we must have equality in both cases, which gives (5.3). We note that (5.3) could also be obtained using quaternions.

It follows from (5.3) and the comments above that (5.2) is equivalent to

$$\mathcal{F}(4,2,1) = \bigoplus_{|n-m|>2} S_{n,m} \tag{5.5}$$

Since $\mathcal{F}(4,2,1)$ is invariant under the action of $SO(4)$ and since the $S_{n,m}$ are the irreducible spaces in $L^2(\mathcal{L}^4_2)$, it suffices to prove that, for each pair n, m, there is an $f \in S_{n,m}$ with $f \in \mathcal{F}(4,2,1)$ if and only if $|n-m| > 2$. As mentioned previously, a natural choice for such a function is the product of Legendre polynomials. Now (5.4) shows that the restriction of $P_n P_m \in S_{n,m}$ to planes containing a given line has non-trivial projection onto $S_{|n-m|}$, the space of spherical harmonics on S^2 of degree $|n-m|$. It follows from Proposition 3.3 that $P_n P_m \in \mathcal{F}(4,2,1)$ if and only if $|n-m| > 2$, as required. Of course, it is tempting to conjecture that (4.2) is always an equality, but we have been unable to prove this.

We conclude this work by noting some connections with other branches of geometry. If K is a convex body in $\mathbb{E}^d$ and if $E \in \mathcal{L}^d_k$, we denote the orthogonal projection of K onto E by $K|E$. If K is sufficiently smooth and centrally symmetric, there is a continuous function $f_K \in C(\mathcal{L}^d_k)$ such that

$$V_k(K|E) = \int_{\mathcal{L}^d_k} [E,F] f_K(F)\, \nu_k(dF) \qquad \text{for all } E \in \mathcal{L}^d_k,$$

see Schneider and Weil [1983] or Goodey and Weil [1993], for example. This representation for volumes of projections leads to similar integral formulas for functionals which arise naturally in translative integral geometry, see Goodey and Weil [1987]. More recently Schneider and Wieacker [1994] have shown that such integral equations also play an important role in the integral geometry of Minkowski spaces. In case $k = 1$ or $d-1$, functions of the form $V_k(K|\cdot)$ span a dense subspace of $C(\mathcal{L}^d_k)$. In fact, any sufficiently differentiable member of $C(\mathcal{L}^d_k)$ is a difference $V_k(K|\cdot) - V_k(L|\cdot)$ for some convex bodies K, L. The fact that $\mathcal{M}(d,k,0) \neq \{0\}$ for all $2 \leqslant k \leqslant d-2$ shows that, for these values of k, the projection functions of centrally symmetric convex bodies do not span a dense subspace of $C(\mathcal{L}^d_k)$. It is not known whether this is also true for projection functions of arbitrary convex bodies.

References

Adams, E. P. and Hippisley, R. L.
1922 *Smithsonian mathematical formulae and tables of elliptic functions*, Smithsonian Institute, Washington D. C., 1922.

Boerner, H.
1963 *Representations of groups*, North Holland, Amsterdam 1963.

Gelfand, I. M., Graev, M. I. and Roşu, R.
1986 The problem of integral geometry and intertwining operators for a pair of real Grassmannian manifolds, *Jour. Operator Theory* **12** (1986), 359–383.

Gluck, H. and Warner, F. W.
1983 Great circle fibrations of the three-sphere, *Duke Math. J.* **50** (1983), 107–132.

Goodey, P. R. and Howard, R.
1990a Processes of flats induced by higher dimensional processes, *Adv. in Math.* **80** (1990), 92–109.
1990b Processes of flats induced by higher dimensional processes II, *Contemporary Math.* **113** (1990), 111–119.

Goodey, P. R. and Weil, W.
1987 Translative integral formulae for convex bodies, *Aequationes Math.* **34** (1987), 64–77.
1993 Zonoids and generalisations, in *Handbook of Convex Geometry* eds. P. M. Gruber and J. M. Wills, North-Holland, Amsterdam 1993, 1297–1326.

Gradshteyn, I. S. and Ryzhik, I. M.
1994 *Tables of integrals, series and products*, Academic Press, San Diego 1994.

Grinberg, E.
1985 On images of Radon transforms, *Duke. Math. Jour.* **52** (1985), 939–972.
1986 Radon transforms on higher rank Grassmannians, *Jour. Differential Geometry* **24** (1986), 53–68.

Helgason, S.
1980 *The Radon Transform*, Birkhäuser, Boston 1980.
1984 *Groups and Geometric Analysis*, Academic Press, Orlando 1984.

Loomis, L.H.
1953 *An introduction to abstract harmonic analysis*, van Nostrand, New York 1953.

Matheron, G.
1974 Un théorèm d'unicité pour les hyperplans poissoniens, *J. Appl. Probability* **11** (1974), 184–189.
1975 *Random sets and integral geometry,* Wiley, New York 1975.

Mecke, J., Schneider, R., Stoyan, D. and Weil, W.
1990 *Stochastische Geometrie*, Birkhäuser, Berlin 1990.

Schneider, R. and Weil,W.
1983 Zonoids and related topics, in *Convexity and its Applications,* eds P. M. Gruber and J. M. Wills, Birkhäuser, Basel 1983, 296–317.

Schneider, R. and Wieacker, J.
1994 Integral geometry in Minkowski spaces, *to appear.*

Stoyan, D., Kendall, W.S. and Mecke, J.
1987 *Stochastic geometry and its applications*, Akademie-Verlag, Berlin 1987.

Strichartz, R.
1975 The explicit Fourier decomposition of $L^2(SO(n)/SO(n-m))$, *Can. J. Math.* **27** (1975), 294–310.

Sugiura, M.
1962 Representations of compact groups realized by spherical functions on symmetric spaces, *Proc. Japan Acad.* **38** (1962), 111–113.

Takeuchi, M.
1973 Polynomial representations associated with symmetric bounded domains, *Osaka J. Math.* **10** (1973), 441–475.