THE LOEWNER-PU INEQUALITY, ISOSYSTOLIC ESTIMATES, AND QUASI-CONFORMAL GEOMETRY

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1. Introduction

Let M^2 be a compact two dimensional manifold and let g be any smooth Riemannian metric on M^2. Let $\mathrm{Area}(g)$ be the area of this metric and $\mathrm{sys}_1(g)$ the length of the shortest non-contractible closed geodesic in (M^2, g). In 1949 Loewner [3, Note on page 71] proved that when M^2 is the two dimensional torus the remarkable inequality

$$\mathrm{sys}_1(g)^2 \leq \frac{2}{\sqrt{3}}\,\mathrm{Area}(g)$$

holds. The proof is based on the fact any metric on a torus is globally conformal to a flat metric and that by a clever but elementary averaging argument the inequality for general metrics on can be reduced to verifying the inequality for flat metrics. This inequality is sharp and equality holds for the flat torus based on the hexagonal lattice.

A few years latter Pu [3] showed Loewner's method can be extended to any compact Riemannian manifold with a transitive group of isometries to show that for certain "isosytolic" inequalities (defined below) that in a given conformal class the invariant metric will give the extremal inequality (see Theorem 2.1 below). As every smooth metric on the real projective space $\mathbf{RP}^2$ is conformal to a metric with constant curvature $+1$ this implies for any metric g on $\mathbf{RP}^2$ there is the sharp inequality

$$\mathrm{sys}_1(g)^2 \leq \frac{\pi}{2}\,\mathrm{Area}(g).$$

In this note we make the elementary observation that the method of Loewner and Pu works not only for metrics conformal to a an invariant metric, but also for metrics λ-quasi-conformal to an invariant metric. The resulting inequality relates the constant λ to the to isosytolic inequalities. Thus the constants in the isosystolic inequalities can be used to estimate

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the quasi-conformal distance of a metric from an invariant metric. As an application we use recent results of Babenko and Katz [1] to show there are metrics on the three dimensional torus that have arbitrarily large conformal distance from all flat metrics.

2. The Quasi-Conformal Version of the Loewner-Pu Inequality.

Let M be a smooth compact manifold and g_0 and g two Riemannian metrics on M. Then for a constant $\lambda \geq 1$. the metric g is λ ***quasi-conformal to*** g_0 iff there is a diffeomorphism $\varphi : M \to M$ and a positive function $u : M \to \mathbf{R}$ so that

$$u^2 g_0 \leq \varphi^* g \leq \lambda^2 u^2 g_0. \tag{2.1}$$

When $\lambda = 1$ the g_0 and g are ***conformal***.

Let $H_k(M)$ be the k dimensional homology of M with integer coefficients. Then any homology class $\alpha \in H_k(M)$ can be represented by a chain $S = \sum_i a_i s_i$ so that each singular simplex $s_i : \Delta_k \to M$ is a C^1 map (here Δ_k is the standard k dimensional simplex). For a smooth metric g on M can then define the volume of such a chain S by

$$\operatorname{Vol}_k^g(S) = \sum_i |a_i| \operatorname{Vol}_k^g(s_i),$$

where the volume of s_i is the volume the singular Riemannian space $(\Delta_k, s^* g)$. (Thus this volume depends on the Riemannian metric on M.) The volume of a homology class $\alpha \in H_k(M)$ is then

$$\operatorname{Vol}_k^g(\alpha) := \inf_{S \in \alpha} \operatorname{Vol}_k^g(S)$$

where the infimum is over all C^1 chains representing α. Also for any smooth metric g on M define, following Berger [2], the k-systole of g by

$$\operatorname{sys}_k(g) := \inf_{0 \neq \alpha \in H_k(M)} \operatorname{Vol}_k^g(\alpha).$$

The following is generalization of a result of Pu [3, Theorem 2 p. 62] form conformally related to quasi-conformally related metrics. The proof here only involves trivial changes form that given in [3].

Theorem 2.1. *Let $M = G/K$, $\dim M = n$, be a homogeneous space with G a compact Lie group and K a closed subgroup. Let g_0 be a Riemannian metric on M that is invariant under the action of G and let g be any Riemannian metric on M that is λ quasi-conformal to g_0. Then*

$$\frac{\operatorname{sys}_k(g)}{\operatorname{Vol}_n(g)^{\frac{k}{n}}} \leq \lambda^k \frac{\operatorname{sys}_k(g_0)}{\operatorname{Vol}_n(g_0)^{\frac{k}{n}}}$$

for $1 \leq k \leq n-1$.

Corollary 2.2. *With the same hypothesis*

$$\frac{\mathrm{Vol}_n(g)}{\mathrm{sys}_k(g)\,\mathrm{sys}_{n-k}(g)} \geq \frac{\mathrm{Vol}_n(g_0)}{\lambda^n\,\mathrm{sys}_k(g_0)\,\mathrm{sys}_{n-k}(g_0)}.$$

Proof. If g is λ quasi-conformal to g_0 then for some diffeomorphism $\varphi : M \to M$ the inequalities (2.1) hold. By replacing g by $\varphi^* g$ we can assume

$$u^2 g_0 \leq g \leq \lambda^2 u^2 g_0.$$

We denote by dV_n^g the volume element of the metric g and the volume element induced on k dimensional submanifolds by g will be denoted by dV_k^g with similar notation for volumes induced by g_0. The inequalities between g and g_0 imply

$$u^k\, dV_k^{g_0} \leq dV_k^g \leq \lambda^k u^k\, dV_k^{g_0}.$$

Thus

$$\int_M u(z)^n\, dV_k^{g_0}(z) \leq \int_M 1\, dV_n^g(z) = \mathrm{Vol}_n^g(M).$$

and for and k dimensional submanifold S

$$\mathrm{Vol}_k^g(S) \leq \lambda^k \int_S u(x)^k\, dV_k^{g_0}(x).$$

We use the normalized measure Haar measure $d\xi$ on G so that $\int_G 1\, d\xi = 1$. Then for any point $z \in M$ using the invariance of the metric g_0 we have for any function $f : M \to \mathbf{R}$ and $z \in M$

$$\int_G f(\xi z)\, d\xi = \mathrm{Vol}_n^{g_0}(M)^{-1} \int_M f(y)\, dV_n^{g_0}(y).$$

We now assume S is a k dimensional submanifold of M so that $\mathrm{Vol}_k^{g_0}(S) = \mathrm{sys}_k(g_0)$. (Here we are cheating a bit. By a basic result of Federer and Fleming there is an integral currant S with $\mathrm{Vol}_k^{g_0}(S) = \mathrm{sys}_k(g_0)$ and by a regularity theorem due to Almgren this current is regular almost everywhere. The use of these results can be avoided in the following argument by working with a minimizing sequences of singular chains, at the expense of making the proof slightly more complicated notationally.) Note for any $\xi \in G$ we have $\mathrm{sys}_k(g) \leq \mathrm{Vol}_k^g(\xi S)$. Therefore, using the inequalities above and Hölder's inequality,

$$\begin{aligned}
\operatorname{sys}_k(g) &\le \int_G \operatorname{Vol}_k^g(\xi S)\, d\xi \\
&\le \int_G \lambda^k \int_S u(\xi x)^k \, dV_k^{g_0}(x)\, d\xi \\
&= \lambda^k \int_S \int_G u(\xi x)^k \, d\xi\, dV_k^{g_0}(x) \\
&= \lambda^k \operatorname{sys}_k(g_0) \int_G u(\xi x_0)^k \, d\xi \qquad \text{(This is independent of choice of } x_0\text{)} \\
&\le \lambda^k \operatorname{sys}_k(g_0) \left(\int_G u(\xi x_0)^n \, d\xi \right)^{\frac{k}{n}} \\
&= \lambda^k \operatorname{sys}_k(g_0) \operatorname{Vol}_n^{g_0}(M)^{-\frac{k}{n}} \left(\int_M u(z)^n \, dV_n^{g_0}(z) \right)^{\frac{k}{n}} \\
&\le \lambda^k \operatorname{sys}_k(g_0) \operatorname{Vol}_n^{g_0}(M)^{-\frac{k}{n}} (\operatorname{Vol}_n^g(M))^{\frac{k}{n}}
\end{aligned}$$

which is equivalent to the required inequality. The proof of the corollary is straightforward. □

3. Application to Quasi-Conformal Geometry.

Let g_0, g be two smooth Riemannian metrics on the compact Riemannian manifold M^n. Then define the ***quasi-conformal distance*** between g_0 and g by

$$\operatorname{Conf\,Dist}(g, g_0) := \inf\{\lambda : g \text{ is } \lambda \text{ quasi-conformal to } g_0\}.$$

A little work then shows

$$\operatorname{Conf\,Dist}(g, g_0) = \operatorname{Conf\,Dist}(g_0, g)$$

and if g_1 is anther smooth metric then

$$\operatorname{Conf\,Dist}(g_1, g_00) \le \operatorname{Conf\,Dist}(g_1, g) \operatorname{Conf\,Dist}(g, g_0).$$

Thus the function $d(g, g_0) := \ln \operatorname{Conf\,Dist}(g, g_0)$ makes the space of all conformal classes of metrics on M^n into a metric space. Corollary 2.2 implies

$$\operatorname{Conf\,Dist}(g, g_0)^n \ge \frac{\operatorname{Vol}_n(g_0) \operatorname{sys}_k(g) \operatorname{sys}_{n-k}(g)}{\operatorname{Vol}_n(g) \operatorname{sys}_k(g_0) \operatorname{sys}_{n-k}(g_0)}. \tag{3.1}$$

Lemma 3.1. *There is a constant $c(n)$ so that for any flat metric g_0 on the n dimensional torus T^n the inequality*

$$\frac{\operatorname{Vol}_n(g_0)}{\operatorname{sys}_k(g_0) \operatorname{sys}_{n-k}(g_0)} \ge c(n)$$

holds for $1 \le k \le n-1$.

Proof. A (non-trivial) exercise. □

Combining this with (3.1) implies that for any smooth metric g and any flat metric on T^n that

$$\operatorname{Conf\,Dist}(g, g_0)^n \geq c(n) \frac{\operatorname{sys}_k(g) \operatorname{sys}_{n-k}(g)}{\operatorname{Vol}_n(g)}. \tag{3.2}$$

Theorem 3.2. *For each $n \geq 3$ and $D_0 > 0$ there is smooth metric g on the n dimensional torus T^n so that for all flat metrics g_0 on T^n* $\operatorname{Conf\,Dist}(g, g_0) \geq D_0$.

Proof. By a recent result of Babenko and Katz [1] there is a sequence of smooth metrics g_ℓ on T^n so that

$$\lim_{\ell \to \infty} \frac{\operatorname{sys}_1(g_\ell) \operatorname{sys}_{n-1}(g_\ell)}{\operatorname{Vol}_n(g_\ell)} = \infty.$$

This together with (3.2) completes the proof. □

References

1. Ivan Babenko and Mikhail Katz, *Systolic freedom of orientable manifolds*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 6, 787–809.
2. Marcel Berger, *Du côté de chez Pu*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 1–44.
3. P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*, Pacific J. Math. **2** (1952), 55–71.