THE MILNOR-ROGERS PROOF OF THE BROUWER FIXED POINT THEOREM

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Let $\mathbf{R}^n$ have its usual inner product $\langle \cdot, \cdot \rangle$ and let $\|\cdot\|$ be the indced norm. Let $B^n := \{x \in \mathbf{R}^n : \|x\| < 1\}$ be the open unit ball, $\overline{B}^n := \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ the closed unit ball, and $S^{n-1} := \{x : \|x\| = 1\}$ the unit sphere in $\mathbf{R}^n$.

Theorem 1 (Brouwer Fixed Point Theorem)**.** *Every continuous map* $f\colon \overline{B}^n \to \overline{B}^n$ *has a fixed point. That is there is an* $x \in \overline{B}^n$ *such that* $f(x) = x$.

In [1] J. Milnor gave a proof of this result based on elementary multidimensional integral calculus. In [2] C. A. Rogers simplified Milnor's proof. Here we give an exposition of the Milnor-Rogers proof.

Lemma 1. *There is no* C^1 *map* $f\colon \overline{B}^n \to S^{n-1}$ *such that* $f(x) = x$ *for all* $x \in S^{n-1}$.

Proof. Assume, toward a contradiction, that such an $f\colon \overline{B}^n \to S^{n-1}$ exists. For $t \in [0, 1]$ let

$$f_t(x) = (1 - t)x + tf(x) = x + tg(x)$$

where $g(x) = f(x) - x$. Note that for $x \in \overline{B}^n$ that

$$\|f_t(x)\| \leq (1 - t)\|x\| + t\|f(x)\| \leq (1 - t) + t = 1$$

and therefore $f_t\colon \overline{B}^n \to \overline{B}^n$. Also note that for all $x \in S^{n-1}$

$$f_t(x) = (1 - t)x + tf(x) = (1 - t)x + tx = x,$$

and thus f_t fixes all points on S^{n-1}. As f is C^1, the same is true of g and therefore there is a constant C such that for all $x_1, x_2 \in \overline{B}^n$

$$\|g(x_2) - g(x_1)\| \leq C\|x_2 - x_1\|.$$

Now assume that there are distinct points x_1 and x_2 in $\overline{B}^n$ with $f_t(x_1) = f_t(x_2)$. This implies $x_2 - x_1 = t(g(x_1) - g(x_2))$ and therefore

$$\|x_2 - x_1\| = t\|g(x_1) - g(x_2)\| \leq Ct\|x_2 - x_1\|.$$

As $x_1 \neq x_2$ this implies $Ct \geq 1$. Thus when $t < 1/C$ the function $f_t\colon \overline{B}^n \to \overline{B}^n$ is injective. Let $G_t = f_t[B^n]$ be the image of the open unit ball under

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f_t. The derivative of f_t, viewed as a linear map $f_t'(x)\colon \mathbf{R}^n \to \mathbf{R}^n$, is given by

$$f'(x) = I + tg'(x)$$

where I is the identity map on $\mathbf{R}^n$. As g is C^1 there is a t_0 such that $\det f_t'(x) > 0$ for all $t \in [0, t_0]$. Then by the inverse function theorem G_t is an open set for all $t \in [0, t_0]$. By possibly making t_0 smaller we also have that f_t is injective for all $t \in [0, t_0]$.

We claim that $G_t = B^n$ for all $t \in [0, t_0]$. Assume that this is not the case. Then the boundary ∂G_t will intersect the open ball B^n at some point y_0. As $y_0 \in \partial G_t$ there is a sequence $x_\ell \in B^n$ such that $\lim_{\ell\to\infty} f_t(x_\ell) \to y_0$. By the compactness of $\overline{B}^n$ we can pass to a subsequence and assume that $\lim_{\ell\to\infty} x_\ell = x_0$ for some $x_0 \in \overline{B}^n$. Then, by the continuity of f, we have $f_t(x_0) = y_0$. But, as G_t is open and open sets are disjoint from their boundaries, y_0 is not in $G_t = f[B^n]$, thus $x_0 \in \overline{B}^n \setminus B^n = S^{n-1}$. But for $x_0 \in S^{n-1}$ we have that $f_t(x_0) = x_0$, which implies that $y_0 = f_t(x_0) = x_0 \in S^{n-1}$, which contradicts the assumption that y_0 is in B^n. Therefore for $t \in [0, t_0]$ the map $f_t\colon \overline{B}^n \to \overline{B}^n$ is a bijection.

Define a function $F\colon [0,1] \to \mathbf{R}$ by

$$F(t) = \int_{\overline{B}^n} \det f_t'(x)\, dx = \int_{\overline{B}^n} \det(I + tg'(x))\, dx$$

where dx is the volume measure on $\mathbf{R}^n$. This is clearly a polynomial in t. And for $t \in [0, t_0]$ the function $f_t\colon \overline{B}^n \to \overline{B}^n$ is a bijection and so by the change of variable formula for multiple integrals $F(t)$ is just the volume of the image $f_t[\overline{B}^n] = \overline{B}^n$. That is

$$F(t) = \text{Volume}(\overline{B}^n) \qquad \text{for } t \in [0, t_0].$$

But a polynomial that is constant on an interval is constant everywhere. Therefore $F(t) = \text{Volume}(\overline{B}^n)$ for all $t \in [0,1]$ and in particular $F(1) = \text{Volume}(\overline{B}^n) > 0$. But $f_1(x) = f(x) \in S^{n-1}$ for all x and therefore

$$\langle f_1(x), f_1(x)\rangle = \|f_1(x)\|^2 = 1$$

for all x. Thus for any vector $v \in \mathbf{R}^n$

$$2\langle f_1'(x)v, f_1(x)\rangle = \frac{d}{dt}\langle f_1(xt + tv), f_1(x + tv)\rangle\bigg|_{t=0} = \frac{d}{dt} 1 \bigg|_{t=0} = 0.$$

This shows that the range of $f_1'(x)$ is contained in $f(x)^\perp$, the orthogonal compliment of $f(x)$. But then $\operatorname{rank} f_1'(x) \le n - 1$ for all $x \in \overline{B}^n$ and therefore $\det f_1'(x) = 0$ for all $x \in \overline{B}^n$. Whence

$$F(1) = \int_{\overline{B}^n} \det f_1'(x)\, dx = 0.$$

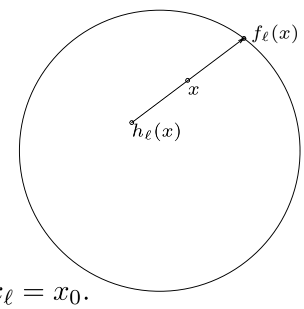
This contradicts that $F(1) > 0$ and completes the proof. $\square$

Proof of the Brouwer Fixed Point Theorem. Let $f \colon \overline{B}^n \to \overline{B}^n$ be a continuous map. Then by the Stone-Weierstrass theorem there is a sequence of C^1 functions $p_\ell \colon \overline{B}^n \to \mathbf{R}^n$ such the $\|f(x) - p_\ell(x)\| \le 1/\ell$ for all $x \in \overline{B}^n$. (In fact we can choose the p_ℓ's to be polynomials.) Then $\|p_\ell(x)\| \le \|f(x)\| + \|p_\ell(x) - f(x)\| \le 1 + 1/\ell$. Therefore if $h_\ell = (1+1/\ell)^{-1}p_\ell$ we have that $h_\ell \colon \overline{B}^n \to \overline{B}^n$ and $h_\ell \to f$ uniformly.

We claim that each h_ℓ has a fixed point in $\overline{B}^n$. For if not, let $f_\ell \colon \overline{B}^n \to S^{n-1}$ be map

$$f_\ell(x) = \text{point where the ray from } h_\ell(x) \text{ to } x \text{ meets } S^{n-1}.$$

If h_ℓ has no fixed point this map is C^1 and has $f_\ell(x) = x$ for all $x \in S^{n-1}$ contradicting Lemma 1.



Let x_ℓ be a fixed point of h_ℓ, that is $h_\ell(x_\ell) = x_\ell$. As $\overline{B}^n$ is compact we can pass to a subsequence and assume that $x_\ell \to x_0$ for some x_0 in $\overline{B}^n$. As $h_\ell \to f$ uniformly this implies

$$f(x_0) = \lim_{\ell\to\infty} h_\ell(x_\ell) = \lim_{\ell\to\infty} x_\ell = x_0.$$

That is f has x_0 as a fixed point. □

We get as a corollay, important enough to be called a theorem, a version of Lemma 1 where f is not required to be C^1.

Theorem 2. *There is no continuous map $f \colon \overline{B}^n \to S^{n-1}$ with $f(x) = x$ for all $x \in S^{n-1}$.*

Proof. Assume that such an $f \colon \overline{B}^n \to S^{n-1}$ existed. Let $g \colon \overline{B}^n \to \overline{B}^n$ be given by $g(x) = -f(x)$. Therefore g also maps $\overline{B}^n$ into S^{n-1}. Therefore if $x = g(x)$ we have $x \in S^{n-1}$. But for $x \in S^{n-1}$, $g(x) = -f(x) = -x \ne x$. Thus g has no fixed point, contradicting Theorem 1. □

References

1. J. Milnor, *Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem*, Amer. Math. Monthly **85** (1978), no. 7, 521–524. MR MR505523 (80m:55001)
2. C. A. Rogers, *A less strange version of Milnor's proof of Brouwer's fixed-point theorem*, Amer. Math. Monthly **87** (1980), no. 7, 525–527. MR MR600910 (82b:55004)