The Weierstrass M test.

The following is a standard result and in many cases the easiest and most natural method to show a series is uniformly convergent.

Theorem 1 (Weierstrass M test.)**.** *Let $f_1, f_2, f_3, \ldots X \to \mathbf{R}$ be a sequence of functions form a set X to the real numbers. Assume that there are constants, M_k, such that the two conditions*

$$|f_k(x)| \leq M_k \quad \textit{holds for all } x \in X \textit{ and } k \geq 1$$

and

$$\sum_{k=1}^{\infty} M_k < \infty$$

hold. Then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges absolutely and uniformly on X.

Problem 1. Prove this. *Hint:* First note that $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely on X just by comparison with the series of non-negative terms $\sum_{k=1}^{\infty} M_k$. Let

$$F(x) = \sum_{k=1}^{\infty} f_k(x).$$

So we only need show the partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converge uniformly to $F(x)$. Note

$$|F(x) - S_n(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k$$

and you should be able to take it from here. □

Here is an example of the use of the M-test. Define a function on $\mathbf{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2}.$$

The the k-th term satisfies

$$\frac{1}{x^2 + k^2} \leq \frac{1}{k^2} =: M_k.$$

The series $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} 1/k^2$ converges, there by the M-test the series for $f(x)$ converges uniformly. Moreover we can say more. As each term of the series is continuous and the convergence is uniform the sum function is also continuous. (As the uniform limit of continuous functions is continuous.)

Problem 2. Define a function on the plane $\mathbf{R}^2$ by

$$f(x,y)=\sum_{k=1}^{\infty}\frac{1}{k^{3/2}+\sqrt{x^2+y^4}}.$$

Show this series converges uniformly and that f is continuous on $\mathbf{R}^2$. □

Trigonometric functions.

We can now give definitions of the trigonometric functions. It is enough to define sin and cos as all the others can be defined in terms of these two.

Theorem 2. *The two series*

$$\mathsf{c}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots$$

$$\mathsf{s}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$$

converge absolutely for all $x\in\mathbf{R}$ and therefore these series are absolutely convergent and differentiable for all $x\in\mathbf{R}$. The derivatives satisfy

$$\mathsf{c}'(x)=-\mathsf{s}(x),\qquad \mathsf{s}'(x)=\mathsf{c}(x).$$

The values at $x=0$ are

$$\mathsf{c}(0)=1,\qquad \mathsf{s}(0)=0.$$

Also

$$\mathsf{c}''(x)=-\mathsf{c}(x),\qquad \mathsf{s}''(x)=-\mathsf{s}(x).$$

Problem 3. Prove this. □

Proposition 3. *These functions satisfy*

$$\mathsf{c}(x)^2+\mathsf{s}(x)^2=1.$$

Problem 4. Prove this. *Hint:* Show that $\mathsf{c}(x)^2+\mathsf{s}(x)^2$ is constant by taking its derivative. Note that showing it is constant does not finish the problem, you still have to show the constant is 1. □

Lemma 4. *If g is two times differentiable on $\mathbf{R}$ and*

$$g''=-g,\qquad g(0)=0,\quad g'(0)=0$$

then $g(x)=0$ for all x.

Problem 5. Prove this. *Hint:* Let $E=g^2+(g')^2$ and show $E'=0$. □

Theorem 5. *If f is twice differentiable on $\mathbf{R}$ and*

$$f''=-f$$

then f is a linear combination of c and s. In particular

$$f(x)=f(0)\mathsf{c}(x)+f'(0)\mathsf{s}(x).$$

Problem 6. Prove this. *Hint:* Let $g(x) = f(x) - f(0)\mathsf{c}(x) - f'(0)\mathsf{s}(x)$ and use Lemma 4. $\square$

Theorem 6. *The functions* c *and* s *satisfy*

$$\mathsf{c}(x+a) = \mathsf{c}(a)\mathsf{c}(x) - \mathsf{s}(a)\mathsf{s}(x)$$
$$\mathsf{s}(x+a) = \mathsf{s}(a)\mathsf{c}(x) + \mathsf{c}(a)\mathsf{s}(x).$$

Problem 7. Prove this. *Hint:* For the first one let $f(x) = \mathsf{c}(x+a)$. Then $f''(x) = -f(x)$. Thus, by Theorem 5,

$$f(x) = f(0)\mathsf{c}(x) + f'(0)\mathsf{s}(x).$$

$\square$

Lemma 7. *If for* $0 < x < 6$ *the inequality*

$$\mathsf{s}(x) > x - \frac{x^3}{6} = x\left(1 - \frac{x^2}{6}\right)$$

holds. In particular $\mathsf{s}(x) > 0$ *for* $0 < x < \sqrt{3}$.

Proof. For any x we have

$$\begin{aligned}\mathsf{s}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right) + \frac{x^9}{9!}\left(1 - \frac{x^2}{10\cdot 11}\right) + \frac{x^{13}}{13!}\left(1 - \frac{x^2}{14\cdot 15}\right) + \cdots\end{aligned}$$

If $0 < x < 6$ then $x^2 < 6\cdot 7 < 10\cdot 11 < 14\cdot 15$. Therefore all the terms

$$\frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right), \quad \frac{x^9}{9!}\left(1 - \frac{x^2}{10\cdot 11}\right), \quad \frac{x^{13}}{13!}\left(1 - \frac{x^2}{14\cdot 15}\right), \ldots$$

are positive and the result follows. $\square$

Lemma 8. *If* $0 < x < 7$ *the inequality*

$$\mathsf{c}(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

holds.

Problem 8. Prove this. $\square$

Theorem 9. *The function* $\mathsf{c}(x)$ *has a unique zero in the interval* $[0, 2]$. *We denote this zero by* $\pi/2$. *This is our* ***official definition of the number*** π.

Proof. As $2 < \sqrt{6}$ we have, by Lemma 7, that $\mathsf{s}(x) > 0$ on the interval $(0, 2)$. Therefore when $0 < x < 2$

$$\mathsf{c}'(x) = -\mathsf{s}(x) < 0.$$

This shows that $\mathsf{c}(x)$ is strictly decreasing on $[0, 2]$. Thus $\mathsf{c}(x)$ can have at most one zero on $[0, 2]$. But

$$\mathsf{c}(0) = 1 > 0$$

and by Lemma 8

$$\mathsf{c}(2) < 1 - \frac{2^2}{2} + \frac{2^4}{24} = \frac{-1}{3} < 0$$

and therefore $\mathsf{c}(x)$ has at least one root in $[0,2]$ by the Intermediate Value Theorem. □

Proposition 10. *The following hold*

$$\begin{aligned} \mathsf{c}(\pi/2) &= 0 & \mathsf{s}(\pi/2) &= 1 \\ \mathsf{c}(\pi) &= -1 & \mathsf{s}(\pi) &= 0 \\ \mathsf{c}(2\pi) &= 1 & \mathsf{s}(2\pi) &= 0 \end{aligned}$$

Problem 9. Prove this. *Hint:* That $\mathsf{c}(\pi/2) = 0$ is the definition of π. Then $\mathsf{c}(\pi/2)^2 + \mathsf{s}(\pi/2)^2 = 1$ implies $\mathsf{s}(\pi/2) = \pm 1$. Use Lemma 7 to rule out $\mathsf{s}(\pi/2) = -1$. The rest should now follow from Theorem 6. □

Theorem 11. *The following hold.*

$$\begin{aligned} \mathsf{c}(x+\pi/2) &= -\mathsf{s}(x) & \mathsf{s}(x+\pi/2) &= \mathsf{c}(x) \\ \mathsf{c}(x+\pi) &= -\mathsf{c}(x) & \mathsf{s}(x+\pi) &= -\mathsf{s}(x) \\ \mathsf{c}(x+2\pi) &= \mathsf{c}(x) & \mathsf{s}(x+2\pi) &= \mathsf{s}(x) \end{aligned}$$

Problem 10. Prove this. □

Definition 12. Our official definition of cos and sin is

$$\cos(x) = \mathsf{c}(x), \qquad \sin(x) = \mathsf{s}(x)$$

where c and s are as in Theorem 2. Then $\tan(x)$ is defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

with the usual formulas for $\sec(x)$ etc. □

Proposition 13. *The tangent satisfies*

$$\tan(x+\pi) = \tan(x), \qquad \frac{d}{dx}\tan(x) = 1 + \tan^2(x).$$

Also the restriction $\tan\colon (-\pi/2, \pi/2) \to \mathbf{R}$ *is a bijection. Let* $\arctan\colon \mathbf{R} \to (-\pi/2, \pi/2)$ *be the inverse of this restriction of* $\tan$. *Then*

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Problem 11. Prove this. □

Remark 14. To compute π we can use the series for the arctan:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

For this to be efficient we wish to use values of x that are close to zero. In 1796 John Machin showed[1]

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}.$$

Using this and the series for $\arctan(x)$ gives

$$\pi = 16 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)5^{2k+1}} - 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(239)^{2k+1}}$$

allows one to compute π to five or six decimals without much trouble. Just using the first five terms in the series gives

$$\pi \approx 3.14159268240440$$

while the correct value to 15 significant digits is

$$\pi = 3.14159265358979\ldots$$

so we are already good to seven decimals. Using nine terms in the series gives you 15 significant digits.

For a less off the wall identity note that if $\theta_1 = \arctan(1/2)$ and $\theta_2 = \arctan(1/3)$, so that $\tan\theta_1 = 1/2$ and $\tan\theta_2$, then using the addition for tan we have

$$\tan(\theta_1 + \theta_2) = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan\theta_1 \tan\theta_2} = \frac{1/2 + 1/3}{1 - (1/2)(1/3)} = 1$$

and therefore

$$\theta_1 + \theta_2 = \arctan(1) = \frac{\pi}{4}.$$

This gives

$$\pi = 4\left(\arctan\frac{1}{2} + \arctan\frac{1}{3}\right) = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}}\right).$$

Using ten terms in this series gives the approximation

$$\pi \approx 3.14159257960635$$

(correct to 7 decimals) which is good enough for any piratical application. To get 15 significant digits using 22 terms in this series is enough.

For a modern method there is the formula found in 1995 by Simon Plouffe:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

A thousand terms of this gives well over a thousand decimal places. □

[1]If you wish to prove this, probably the easiest way is to notice that $(5+i)^4(239-i) = -114244(1+i)$ and use the polar form of complex numbers to get the result.