

Series.

CONTENTS

1. Basic definitions and results about series.	1
2. Series with positive terms.	4
3. Tests for the convergence of series with monotone terms.	5
4. Comparison tests.	8
5. The root and ratio tests	10
6. Absolutely and conditional convergent series.	11
7. Power series.	13

The material here corresponds to parts of Chapter VII Rosenlicht.

1. BASIC DEFINITIONS AND RESULTS ABOUT SERIES.

We now wish to make sense out of infinite sums

$$\sum_{k=1}^{\infty} = a_1 + a_2 + a_3 + \cdots$$

Definition 1. Let $\langle a_k \rangle_{k=n_0}^{\infty}$ be a sequence of real numbers. The corresponding *infinite series* is (or just *series*) is the sum

$$\sum_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots .$$

The n -th *partial sum* of the series is

$$A_n = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots + a_{n-1} + a_n = \sum_{k=n_0}^n a_k.$$

We say the series *converges* and has sum A iff

$$\lim_{n \rightarrow \infty} A_n = A.$$

If $\sum_{k=1}^{\infty} a_k$ does not converge, it *diverges*. □

To make notation easier, when proving results about series we will usually let $n_0 = 0$ or $n_0 = 1$.

Here is a result that follows at once from the facts about limits of sequences.

Theorem 2. If $\sum_{n=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then for any constants c_1 and c_2 the series $\sum_{k=1}^{\infty} (c_1 a_k + c_2 b_k)$ also converges and

$$\sum_{k=1}^{\infty} (c_1 a_k + c_2 b_k) = c_1 \sum_{n=1}^{\infty} a_k + c_2 \sum_{n=1}^{\infty} b_k$$

Proof. Let

$$\begin{aligned} A_n &= (a_1 + \cdots + a_n) \\ B_n &= (b_1 + \cdots + b_n) \\ C_n &= ((c_1 a_1 + c_2 b_1) + \cdots + (c_1 a_n + c_2 b_n)) \end{aligned}$$

be the partial sums of the series. We are given that

$$\lim_{n \rightarrow \infty} A_n = A, \quad \lim_{n \rightarrow \infty} B_n = B$$

exist and want to show $\lim_{n \rightarrow \infty} C_n = c_1 A + c_2 B$. Note

$$\begin{aligned} C_n &= ((c_1 a_1 + c_2 b_1) + \cdots + (c_1 a_n + c_2 b_n)) \\ &= c_1 (a_1 + \cdots + a_n) + c_2 (b_1 + \cdots + b_n) \\ &= c_1 A_n + c_2 B_n \end{aligned}$$

and therefore

$$\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} (c_1 A_n + c_2 B_n) = c_1 A + c_2 B$$

as required. \square

Before going on we note that for any series $\sum_{k=1}^{\infty} a_k$ with partial sums $A_n = \sum_{k=1}^n a_k$ we have the elementary relation

$$A_n = A_{n-1} + a_n,$$

or equivalently

$$a_n = A_n - A_{n-1}.$$

This will come up several times in what follows starting with the following:

Theorem 3. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then*

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Proof. If $A_n = \sum_{k=1}^n a_k$ then $\lim_{n \rightarrow \infty} A_n = A$ exists as the series converges. But then also $\lim_{n \rightarrow \infty} A_{n-1} = A$ and so

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (A_n - A_{n-1}) = A - A = 0. \quad \square$$

Remark 4. Often the previous theorem is used in its contrapositive form: If $\lim_{k \rightarrow \infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges. From this it is not hard to give lots of examples of series that do not converge. For example none of the following converge

$$\sum_{k=1}^{\infty} (-1)^k, \quad \sum_{k=1}^{\infty} \sin(k), \quad \sum_{n=1}^{\infty} \frac{n^2 - 2}{2n^2 + 5}. \quad \square$$

Proposition 5. *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\varepsilon > 0$ there is a N such that*

$$N \leq m < n \quad \implies \quad |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

Problem 1. Prove this. *Hint:* What is the Cauchy condition for the sequence $\langle A_n \rangle_{n=1}^{\infty}$ of partial sums? \square

Proposition 6. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series such that $a_k = b_k$ except for a finite number of values k . Then either they both converge or both diverge. (An informal way to state this is that changing a finite number of terms of a series does not effect whether it converges or diverges.)

Proof. By the hypothesis there is an n_0 such that

$$a_k = b_k \quad \text{for all} \quad k \geq n_0.$$

If $n \geq n_0$ then

$$\begin{aligned} B_n &= B_{n_0} + \sum_{k=n_0+1}^n b_k \\ &= B_{n_0} + \sum_{k=n_0+1}^n a_k && \text{(as } a_k = b_k \text{ when } k \geq n_0) \\ &= B_{n_0} - A_{n_0} + A_{n_0} + \sum_{k=n_0+1}^n a_k \\ &= (B_{n_0} - A_{n_0}) + A_n. \end{aligned}$$

Letting $c = B_{n_0} - A_{n_0}$, which is a constant, we have that $B_n = A_n + c$ for $n \geq n_0$. Thus the sequences $\langle A_n \rangle_{n=1}^{\infty}$ and $\langle B_n \rangle_{n=1}^{\infty}$ either both converge or both diverge. \square

Lemma 7. If $|r| \neq 1$ then

$$a + ar + ar^2 + \cdots + ar^n = \sum_{k=0}^n ar^k = \frac{a - ar^{n+1}}{1 - r}.$$

Proof. Let $S_n = a + ar + ar^2 + \cdots + ar^n$. Then

$$\begin{aligned} (1 - r)S_n &= a + ar + ar^2 + \cdots + ar^n - r(a + ar + ar^2 + \cdots + ar^n) \\ &= a + ar + ar^2 + \cdots + ar^n - ar - ar^2 - \cdots - ar^n - ar^{n+1} \\ &= a - ar^{n+1}. \end{aligned}$$

As $r \neq 1$ we can divide by $(1 - r)$ to get the desired result. \square

Lemma 8. If $|r| < 1$ then

$$\lim_{n \rightarrow \infty} |r|^n = 0.$$

Proof. Let $\varepsilon > 0$ and set $N = \ln(\varepsilon)/\ln(|r|)$. Then if $n > N$ it is not hard to check $||r|^n - 0| = |r|^n < \varepsilon$. \square

Here one of the most basic examples of series. Many results about series involve comparison to a geometric series.

Theorem 9 (Infinite Geometric Series). *Let a, r be real numbers with $a \neq 0$. Then the series*

$$a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k$$

converges if and only if $|r| < 1$ in which case its sum is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Proof. If $|r| \geq 1$ then the n -th term ar^n satisfies $|ar^n| \geq |a| > 0$ and so $\lim_{n \rightarrow \infty} ar^n \neq 0$ and thus the series diverges.

Now assume $|r| < 1$. We have seen in Lemma 7 that the n th partial sum is

$$S_n = \frac{a - ar^{n+1}}{1-r}.$$

Now by the last lemma,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a - ar^{n+1}}{1-r} = \frac{a - a \cdot 0}{1-r} = \frac{a}{1-r}$$

as required. \square

2. SERIES WITH POSITIVE TERMS.

Theorem 10. *Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$ for all k . Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence, $\langle A_n \rangle_{n=1}^{\infty}$ (with $A_n = a_1 + \cdots + a_n$) of partial sums is bounded.*

Proof. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n \rightarrow \infty} A_n = A$ exists by definition. But a convergent sequence is bounded. If $\langle A_n \rangle_{n=1}^{\infty}$ is bounded, then $A_{n+1} = A_n + a_{n+1} \geq A_n$ so the series is monotone increasing. But a bounded monotone sequence is convergent. \square

Remark 11. When talking about series, $\sum_{k=1}^{\infty} a_k$, of non-negative terms we will use the following suggestive notation.

$$\begin{aligned} \sum_{k=1}^{\infty} a_k < \infty &\iff \text{The series converges} \\ \sum_{k=1}^{\infty} a_k = \infty &\iff \text{The series series diverges.} \end{aligned}$$

This notation is not appropriate when talking about series with terms of mixed signs. For example the series $\sum_{k=1}^{\infty} (-1)^{k+1}$ has bounded partial sums, but is not convergent. \square

3. TESTS FOR THE CONVERGENCE OF SERIES WITH MONOTONE TERMS.

In general it is easier to understand the convergence of series with monotone decreasing terms. As a first example.

Theorem 12 (Cauchy Condensation Test). *If $\langle a_k \rangle_{k=1}^{\infty}$ is a sequence of non-negative numbers that are monotone decreasing, then*

$$\sum_{k=1}^{\infty} a_k < \infty$$

if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty.$$

Proof. Let the partial sums of the two series be

$$A_n = \sum_{k=1}^n a_k, \quad B_n = \sum_{k=0}^n 2^k a_{2^k}.$$

We will show

$$(1) \quad A_{2^{n+1}-1} \leq B_n$$

$$(2) \quad B_n \leq 2A_{2^n}.$$

If these hold the result is easy. If $\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$ then for any positive integer m choose n such that $m \leq 2^{n+1} - 1$. By (1),

$$A_m \leq A_{2^{n+1}-1} \leq B_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$$

and therefore the partial sums of $\sum_{k=1}^{\infty} a_k$ are bounded above and thus $\sum_{k=0}^{\infty} a_k < \infty$.

Conversely if $\sum_{k=1}^{\infty} a_k < \infty$ then for any positive integer n we use (2) to get

$$B_n \leq 2A_{2^n} \leq 2 \sum_{k=1}^{\infty} a_k < \infty$$

which shows the partial sums of $\sum_{k=0}^{\infty} 2^k a_{2^k}$ are bounded above and thus $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

We now prove (1). Using that the terms are monotone decreasing,

$$\begin{aligned} A_{2^{n+1}-1} &= a_1 + \underbrace{(a_2 + a_3)}_{2^1 \text{ terms}} + \underbrace{(a_4 + \cdots + a_7)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n} + \cdots + a_{2^{n+1}-1})}_{2^n \text{ terms}} \\ &\leq a_1 + \underbrace{(a_2 + a_2)}_{2^1 \text{ terms}} + \underbrace{(a_4 + \cdots + a_4)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n} + \cdots + a_{2^n})}_{2^n \text{ terms}} \\ &= a_1 + 2^2 a_2 + 2^3 a_3 + \cdots + 2^n a_{2^n} \\ &= B_n. \end{aligned}$$

The proof (2) is similar

$$\begin{aligned}
A_{2^n} &= a_1 + a_2 + \underbrace{(a_3 + a_4)}_{2^1 \text{ terms}} + \underbrace{(a_5 + \cdots + a_8)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^{n-1}+1} + \cdots + a_{2^n})}_{2^{n-1} \text{ terms}} \\
&\geq a_1 + a_2 + \underbrace{(a_4 + a_4)}_{2^1 \text{ terms}} + \underbrace{(a_8 + \cdots + a_8)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n} + \cdots + a_{2^n})}_{2^{n-1} \text{ terms}} \\
&= a_1 + a_2 + 2^1 a_{2^2} + 2^2 a_{2^3} + \cdots + 2^{n-1} a_{2^n} \\
&= 2^{-1} a_1 + 2^{-1} a_1 + a_2 + 2^1 a_{2^2} + 2^2 a_{2^3} + \cdots + 2^{n-1} a_{2^n} \\
&= 2^{-1} a_1 + 2^{-1} (2^0 a_1 + 2^1 a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots + 2^n a_{2^n}) \\
&= 2^{-1} a_1 + 2^{-1} B_n \\
&\geq \frac{1}{2} B_n.
\end{aligned}$$

Multiplication by 2 completes the proof. \square

Theorem 13. For any real number $p > 0$ the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Proof. We use the Cauchy-Condensation Test, which applies as the terms of the series are decreasing. The given series converges if and only if

$$\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \left(\frac{2}{2^p}\right)^k$$

converges. This is a geometric series with ratio

$$r = \frac{2}{2^p}.$$

Therefore the series converges if and only if $r = 2/2^p < 1$, that is if and only if $p > 1$. \square

Another method of dealing with series with monotone terms is by comparison with an integral. Let us start with an example. Let $f(x)$ be monotone decreasing on the interval $[0, 6]$ and let

$$a_k = f(k) \quad \text{for} \quad 1 \leq k \leq 6$$

and

$$A_n = a_1 + \cdots + a_n = f(1) + \cdots + f(n).$$

Then, see Figure 1, we can compare the integral $\int_1^6 f(x) dx$ with some of the Riemann sums for the partition $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$ to get

$$\int_1^6 f(x) dx \leq A_5 \leq A_6 \leq f(1) + \int_1^6 f(x) dx.$$

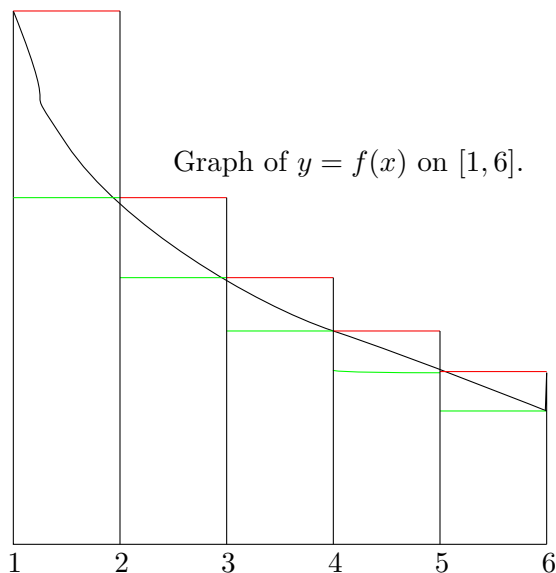


FIGURE 1. The area under the tall (with red tops) rectangles is $A_5 = f(1) + f(2) + f(3) + f(4) + f(5)$. The area under the short (with green tops) rectangles is $A_6 - f(1) = f(2) + f(3) + f(4) + f(5) + f(6)$. The area of the integral is clearly in between these two areas and therefore

$$A_6 - f(1) \leq \int_1^6 f(x) dx \leq A_5.$$

This can be rearranged to give

$$\int_1^6 f(x) dx \leq A_5 \leq A_6 \leq f(1) + \int_1^6 f(x) dx = a_1 + \int_1^6 f(x) dx$$

which is a bit more aesthetic.

We could, and since this is a mathematics class, should be a bit more formal. Note that on any interval $[k, k+1]$ we have, because f is decreasing, that

$$f(k) \geq f(x) \geq f(k+1).$$

Then integration over $[k, k+1]$ and using that $\int_k^{k+1} f(k) dx = f(k)$ and $\int_k^{k+1} f(k+1) dx = f(k+1)$

$$f(k) \geq \int_k^{k+1} f(x) dx \geq f(k+1).$$

This can be summed it two ways to get

$$\int_1^6 f(x) dx = \sum_{k=1}^5 \int_k^{k+1} f(x) dx \leq \sum_{k=1}^5 f(k) = A_5$$

and

$$A_6 - a_1 = \sum_{k=2}^6 f(k) \leq \sum_{k=1}^5 \int_k^{k+1} f(x) dx = \int_1^6 f(x) dx.$$

Of course there is nothing special about $n = 6$ in this argument.

Proposition 14. *Let $f: [1, \infty) \rightarrow [0, \infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let*

$$A_n = \sum_{k=1}^n a_k$$

be the n -th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then

$$\int_1^n f(x) dx \leq A_n \leq f(1) + \int_1^n f(x) dx.$$

Problem 2. Use a variation of the argument given for $n = 6$ to prove this. \square

Theorem 15 (The Integral Test). *Let $f: [1, \infty) \rightarrow [0, \infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let*

$$A_n = \sum_{k=1}^n a_k$$

be the n -th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then

$$\sum_{k=1}^{\infty} a_k < \infty \quad \iff \quad \lim_{n \rightarrow \infty} \int_1^n f(x) dx \quad \text{exists and is finite.}$$

(Note that $\langle \int_1^n f(x) dx \rangle_{n=1}^{\infty}$ is a monotone increasing sequence, thus the limit exists, but might be $+\infty$.)

Problem 3. Prove this. \square

Problem 4. Use the Integral Test to give another proof of Theorem 13. \square

Problem 5. Use the Integral Test to show

$$\sum_{k=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

converges if and only if $p > 1$. \square

4. COMPARISON TESTS.

Proposition 16. *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series of non-negative terms. Assume there is a constant $C > 0$ such that*

$$a_k \leq Cb_k$$

for all k . Then

- (a) If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.
- (b) If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Problem 6. Prove this. *Hint:* Consider partial sums. \square

Theorem 17 (Limit Comparison Test). Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series of positive terms. Assume that

$$L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

exists. Then

- (a) $\sum_{k=1}^{\infty} b_k < \infty$ implies $\sum_{k=1}^{\infty} a_k < \infty$
- (b) If $L \neq 0$ and $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

Often the following special case is enough.

Corollary 18. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series of positive terms. Assume that

$$L = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

exists and $L \neq 0$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. \square

Problem 7. Prove Theorem 17. *Hint:* Recall that a convergent sequence is bounded. Thus $\langle a_k/b_k \rangle_{k=1}^{\infty}$ is bounded and therefore there is a constant C such that $a_k/b_k \leq C$. Thus Proposition 16 applies.

Here some applications of these results.

Example 19. Does the series $\sum_{k=1}^{\infty} \frac{k^3+2k^2+7}{3k^5+2}$ converge? Let this series be $\sum_{k=1}^{\infty} a_k$ and let $\sum_{k=1}^{\infty} b_k$ be the p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Then it is not hard to check that

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{1}{3}.$$

Therefore, by Corollary 18, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. But $\sum_{k=1}^{\infty} b_k$ is a p series with $p = 2 > 1$ and so both series converge. \square

Example 20. Does the series $\sum_{k=1}^{\infty} (\sqrt[3]{k+5} - \sqrt[3]{k-2})$ converge? Let $f(x) = \sqrt[3]{x} = x^{1/3}$. Then for $n > 2$ by the mean value theorem there is a ξ_n between -2 and 5 such that

$$a_n = f(n+5) - f(n-2) = f'(n+\xi_n)((n+5) - (n-2)) = \frac{1}{3}(n+\xi_n)^{-2/3}7.$$

Therefore if $\sum_{k=1}^{\infty} b_k$ is the divergent p -series $\sum_{k=1}^{\infty} 1/n^{2/3}$ we have

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \frac{7}{3}.$$

So $\sum_{k=1}^{\infty} a_k$ diverges by limit comparison to $\sum_{k=1}^{\infty} b_k$.

Problem 8. For practice in these ideas do Problems 10 and 11 on Page 161 of the text. *Hint:* For Problem 11 it might help to notice that

$$\frac{1}{n} - \frac{1}{n+x} = \frac{x}{n(n+x)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1/n^2}{1/(n(n+x))} = 1. \quad \square$$

5. THE ROOT AND RATIO TESTS

These are basically just limit comparisons with a geometric series. To get started here is a version of the comparison where we only worry about the comparison for large values.

Lemma 21. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series of positive terms. Assume there is an N such that

$$a_k \leq b_k \quad \text{for all} \quad k > N$$

and that $\sum_{k=1}^{\infty} b_k < \infty$. Then $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. Let A_n and B_n be the partial sums of these series. Let

$$C_1 = \max\{A_n : 1 \leq n \leq N\}.$$

If $n > N$ then

$$\begin{aligned} A_n &= (a_1 + \cdots + a_N) + (a_{N+1} + \cdots + a_n) \\ &\leq (a_1 + \cdots + a_N) + (b_{N+1} + \cdots + b_n) \\ &= (a_1 + \cdots + a_N) - (b_1 + \cdots + b_N) + (b_1 + \cdots + b_N + b_{N+1} + \cdots + b_n) \\ &= A_N - B_N + B_n \\ &\leq A_N - B_N + \sum_{k=1}^{\infty} b_k < \infty. \end{aligned}$$

Therefore if

$$C = \max \left\{ C_1, A_N - B_N + \sum_{k=1}^{\infty} b_k \right\}$$

we have

$$A_n \leq C$$

for all n . Thus the partial sums of $\sum_{k=1}^{\infty} a_k$ are bounded which implies that it is convergent. \square

The following is a dressed up version of doing a comparison with a geometric series.

Theorem 22 (Root Test). Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms and assume the limit

$$\rho := \lim_{k \rightarrow \infty} (a_k)^{1/k}.$$

exists.

- (a) If $\rho < 1$ then the series converges.
- (b) If $\rho > 1$ then the series diverges.

Problem 9. Prove this. *Hint:* For (a) let r be any number such that $\rho < r < 1$. Then $\rho = \lim_{k \rightarrow \infty} (a_k)^{1/k} < r$ implies there is a N such that

$$k > N \quad \implies \quad (a_k)^{1/k} < r.$$

Then

$$a_k < r^k \quad \text{for all} \quad k > N.$$

Now consider Lemma 21 and Theorem 9.

For (b) show that if $\rho > 1$ then $\lim_{k \rightarrow \infty} a_k \neq 0$. □

Here is another dressed up version of comparison with a geometric series.

Theorem 23 (Ratio Test). Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms assume the limit

$$\rho := \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k}$$

exists.

- (a) If $\rho < 1$, then the series converges.
- (b) If $\rho > 1$, then the series diverges.

Problem 10. Prove this. *Hint:* For (a) let r be a number such that $\rho < r < 1$. Then, by the definition of \lim , there is a N such that

$$k > N \quad \implies \quad \frac{a_{k+1}}{a_k} < r.$$

Thus for $k > N$ we have

$$a_k = a_{N+1} \frac{a_{N+2}}{a_{N+1}} \frac{a_{N+3}}{a_{N+2}} \cdots \frac{a_{k-1}}{a_{k-2}} \frac{a_k}{a_{k-1}} = (a_{N+1}) \prod_{j=N+1}^{k-1} \frac{a_{j+1}}{a_j} < a_{N+1} r^{k-N-1}.$$

The series

$$\sum_{k=1}^{\infty} (a_{N+1}) r^{k-N-1} = \sum_{k=1}^{\infty} (a_{N+1} r^{-N-1}) r^k = \sum_{k=1}^{\infty} C r^k$$

(where $C = (a_{N+1} r^{-N-1})$) is a convergent geometric series. You should now be able to do a comparison by use of Lemma 21.

For (b) show $\rho > 1$ implies $\lim_{k \rightarrow \infty} a_k \neq 0$. □

6. ABSOLUTELY AND CONDITIONAL CONVERGENT SERIES.

Definition 24. The series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** iff the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is convergent. \square

Theorem 25. If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then it is convergent and

$$\left| \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=1}^{\infty} |a_k|.$$

Problem 11. Prove this. *Hint:* Proposition 5 and the triangle inequality applied to partial sums. \square

This, together with Proposition 16 implies

Proposition 26. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with $|a_k| \leq Cb_k$ for some positive constant C . Assume $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges absolutely. \square

Example 27. The last proposition implies all the following

$$\sum_{k=1}^{\infty} \frac{\cos(k)}{k^2}, \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{n2^n}, \quad \sum_{k=1}^{\infty} \frac{3 + (-1)^k}{(k+1) \ln^2(k+1)}.$$

converge absolutely. \square

Definition 28. The series $\sum_{k=1}^{\infty} a_k$ is **conditional convergent** iff $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k| = \infty$. \square

The following gives one of the main methods of producing conditional convergent series.

Theorem 29. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers with

- (a) $a_k \geq a_{k+1}$ (that is it is monotone decreasing),
- (b) $\lim_{k \rightarrow \infty} a_k = 0$.

Then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges. If $A = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is the sum and $A_n = \sum_{k=1}^n a_k$ is the n -th partial sum then

$$|A - A_n| \leq a_{n+1}.$$

That is the error at stopping at the n -th term is at most the $(n+1)$ -st term.

Problem 12. Prove this. *Hint:* Note

$$A_3 = A_1 - a_2 + a_3 = A_1 - (a_2 - a_3) \leq A_1$$

as $a_2 \geq a_3$. Likewise

$$A_5 = A_3 - a_4 + a_5 = A_3 - (a_4 - a_5) \leq A_3$$

as $a_4 \geq a_5$. In general

$$A_{2m+3} = A_{2m+1} - (a_{2m} - a_{2m+1}) \leq A_{2m+1}$$

Give an analogous argument to show

$$A_{2m+2} = A_{2m} + (a_{2m+1} - a_{2m+2}) \geq A_{2m}.$$

Now use this to show that if $\ell \geq n$ then for n odd

$$A_{n+1} \leq A_\ell \leq A_n$$

and for n even

$$A_n \leq A_\ell \leq A_{n+1}.$$

Therefore if $\ell \geq n$ the partial sum A_ℓ is between A_n and A_{n+1} . Also show $|A_{n+1} - A_n| = a_{n+1}$. It should not be hard to finish from here. \square

Problem 13. Show that if $0 < p \leq 1$ that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

is conditional convergent. \square

Therefore when $0 < p \leq 1$ (which implies $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges) the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ is conditionally convergent.

7. POWER SERIES.

Theorem 30. Let a_0, a_1, a_2, \dots be a sequence of numbers and let $f(x)$ be defined on \mathbf{R} by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all x where this converges. If the series converges for $x = x_0$, then it converges absolutely for all x with $|x| < |x_0|$.

Problem 14. Prove this. *Hint:* As

$$f(x_0) = \sum_{k=0}^{\infty} a_k (x_0)^k$$

converges we have $\lim_{k \rightarrow \infty} a_k (x_0)^k = 0$ by Theorem 3. This implies that $\langle a_k (x_0)^k \rangle_{k=0}^{\infty}$ is bounded. So there is a constant C with

$$|a_k (x_0)^k| = |a_k| |x_0|^k \leq C.$$

Then for $|x| < |x_0|$ we have

$$|a_k x^k| = |a_k| |x|^k = |a_k| |x_0|^k \left(\frac{|x|}{|x_0|} \right)^k \leq C \left(\frac{|x|}{|x_0|} \right)^k = C r^k$$

where

$$r = \frac{|x|}{|x_0|} < 1. \quad \square$$

Lemma 31. Let $f(x)$ be as in the last theorem. If the series for $f(x)$ converges at $x = x_0$, then the series

$$f^*(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

converges absolutely for all x with $|x| < |x_0|$. We call f^* the **formal derivative** of f as it is what the derivative would be if we knew that we could take it term at a time. (Shortly we will show that this is the actual derivative.)

Problem 15. Prove this. *Hint:* With notation as in Problem 14 show

$$|k a_k x^{k-1}| \leq k C r^{k-1}$$

and then show $\sum_{k=1}^{\infty} k C r^{k-1}$ converges by either the root or ratio test. \square

Corollary 32. With the same hypothesis as in the last lemma for $|x| < |x_0|$ the series

$$f^{**}(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

converges absolutely. (This is the **formal second derivative**.)

Proof. As $|x| < |x_0|$ there is a number r_0 such that $|x| < r_0 < |x_0|$. By the lemma the series $f^*(r_0)$ converges absolutely. But (with what I hope is not confusing notation) $(f^*)^*(x) = f^{**}(x)$ so this corollary follows by applying Lemma 31 to f^* (with r_0 replacing x_0). \square

Lemma 33. Let k be a positive integer and x, x_1, r_0 real numbers with $|x|, |x_0| < r_0$. Then

$$\left| \frac{x^k - x_1^k}{x - x_1} - k x_1^{k-1} \right| \leq \frac{k(k-1)}{2} r_0^{k-2} |x - x_0|.$$

Problem 16. Prove this. *Hint:* This is yet another opportunity to use Taylor's theorem. Let $p(x)$ be any two times differentiable function. By Taylor's theorem

$$p(x) = p(x_1) + p'(x_1)(x - x_1) + \frac{p''(\xi)}{2}(x - x_1)^2$$

where ξ is between x and x_1 . This can be rearranged as

$$\frac{p(x) - p(x_1)}{x - x_1} - p'(x_1) = \frac{p''(\xi)}{2}(x - x_1)$$

and so

$$\left| \frac{p(x) - p(x_1)}{x - x_1} - p'(x_1) \right| = \frac{|p''(\xi)|}{2} |x - x_1|.$$

Now consider the special case where $p(x) = x^k$. Then $|p''(\xi)| = k(k-1)|\xi|^{k-2} < k(k-1)r_0^{k-2}$ as ξ is between x and x_1 and $|x|, |x_1| < r_0$. \square

Theorem 34. Let a_0, a_1, a_2, \dots be a sequence of numbers and let $f(x)$ be defined on \mathbf{R} by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all x where this converges. If the series converges for $x = x_0$, then the function $f(x)$ exists and is differentiable for all x with $|x| < |x_0|$ and the derivative is given by the formal derivative

$$f'(x) = f^*(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Problem 17. Prove this. *Hint:* That $f(x)$ exists for $|x| < |x_0|$ follows from Theorem 30. We need so show that if $|x_1| < |x_0|$ that f is differentiable at x_1 and the derivative is $f^*(x_1)$. Choose a number r_0 such that $|x_1| < r_0 < |x_0|$. Let x be such that $|x| < r_0$. Explain why the following hold.

(a) The series for the following all converge absolutely.

$$f(x), \quad f(x_1), \quad f^*(x_1), \quad f^{**}(r_0).$$

(b) We have

$$\frac{f(x) - f(x_1)}{x - x_1} - f^*(x_1) = \sum_{k=1}^{\infty} a_k \left(\frac{x^k - x_1^k}{x - x_1} - k x_1^{k-1} \right)$$

(c) The inequality

$$\left| \frac{f(x) - f(x_1)}{x - x_1} - f^*(x_1) \right| \leq C|x - x_1|$$

holds, where

$$C = \frac{1}{2} \sum_{k=2}^{\infty} k(k-1) |a_k| r_0^{k-1} < \infty$$

holds. (Part of the problem is explaining why $C < \infty$. The hint here is that the series for $f^{**}(r_0)$ converges absolutely.)

(d) To finish show

$$f'(x_1) = \lim_{x \rightarrow x_1} \frac{f(x) - f(x_1)}{x - x_1} = f^*(x_1). \quad \square$$

Now that we have differentiated we wish to integrate. Note that by Theorem 34 if the series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for $x = x_0$, then it is differentiable on the interval $(-|x_0|, |x_0|)$ and therefore also continuous on this interval. Thus if $|x| < |x_0|$ this implies $\int_0^x f(t) dt$ is the integral of a continuous function and thus it exists.

Theorem 35. Let a_0, a_1, a_2, \dots be a sequence of numbers and let $f(x)$ be defined on \mathbf{R} by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all x where this converges. If the series converges for $x = x_0$, then for any x with $|x| < |x_0|$

$$\int_0^x f(t) dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k} x^k.$$

That is we can integrate the $f(x)$ term at a time.

Problem 18. Prove this. *Hint:* Let $F(x)$ be defined to be the **formal integral** of $f(x)$. That is

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

Choose r_0 with $|x| < r_0 < |x_0|$. Then as the series for $f(x)$ is convergent, its terms are bounded. That is there is a constant C such that

$$|a_k x_0^k| \leq C.$$

Then

$$\left| \frac{a_k}{k+1} r_0^{k+1} \right| = \frac{r_0 |a_k x_0^k|}{k+1} \left| \frac{r_0}{x_0} \right|^k \leq \frac{r_0 C}{k+1} \left| \frac{r_0}{x_0} \right|^k = \frac{C_1}{k+1} r^k \leq C_1 r^k$$

where

$$C_1 = r_0 C \quad \text{and} \quad r = \left| \frac{r_0}{x_0} \right| < 1.$$

Now

- Explain why the series for $F(r_0)$ converges absolutely. *Hint:* Compare the the geometric series $\sum_{k=0}^{\infty} C_1 r^k$.
- Explain why $F(x)$ is differentiable on the interval $(-r_0, r_0)$. *Hint:* Theorem 34 with x_0 replaced by r_0 .
- The derivative of $F(x)$ on $(-r_0, r_0)$ is $f(x)$ *Hint:* Theorem 34 again.
- Finish the proof. *Hint:* Fundamental Theorem of Calculus. \square

Now that we know that we can integrate and differentiate power series we can find new series form old ones.

Example 36. Find the series for $(1+x)^{-2}$ on the interval $(-1, 1)$. We know

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \dots$$

This can be differentiated term at a time to get

$$-(1+x)^{-2} = 0 - 1 + 2x - 3x^2 + 4x^3 - 5x^4 + 6x^5 - \dots$$

so that

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 - \dots = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k. \quad \square$$

Similar examples can be done by integrating term at a time. Here are some for you to try.

Problem 19. (a) Find a series for $\ln(1+x)$ valid on $(-1, 1)$. *Hint:*

$$\ln(1+x) = \int_0^x \frac{dt}{1+t}$$

and you know how to expand $1/(1+t)$ in a series.

(b) For any positive integer n find the series for $(1+x)^{-n}$ valid on $(-1, 1)$.

(c) On $(-1, 1)$ we have the convergent geometric series:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots$$

Use this to find a power series for $\arctan(x)$ valid on $(-1, 1)$. \square