The Elements of the Calculus of Finite Difference

1. The fundamental theorem of summation theory.

Let $f\colon \mathbf{Z} \to \mathbf{R}$ be a function from the integers, $\mathbf{Z}$, to the real numbers, $\mathbf{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

Definition 1. Let $f\colon \mathbf{Z} \to \mathbf{R}$. Then the ***diffenence***, Δf, of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. $\square$

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1)+2) - (3x+2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax + b$	a
cr^x	$c(r-1)r^x$

Problem 1. Verify these. $\square$

Theorem 1 (Fundamental Theorem of Summation Theory). *Let $f\colon \mathbf{Z} \to \mathbf{R}$ and let F be an* ***anti-difference*** *of f. That is $\Delta F = f$. Then for $a, b \in \mathbf{Z}$ with $a < b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b}(F(k+1)-F(k)) \\ &= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k) \\ &= \big(F(a+1)+F(a+2)+\cdots+F(b)+F(b+1)\big) \\ &\qquad - \big(F(a)+F(a+1)+\cdots+F(b-1)+F(b)\big) \\ &= F(b+1)-F(a)\end{aligned}$$

as required. □

Theorem 1 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a, r and b are constants).

$f(x)$	$F(x)$
$ax+b$	$a\dfrac{x(x-1)}{2}+bx$
ar^x	$\dfrac{ar^x}{1-r}$

Problem 2. Verify these. (You just need to check $F(x+1)-F(x) = f(x)$). □

Problem 3 (Sum of finite geometric series)**.** Use that $\dfrac{ar^x}{1-r}$ as the anti-difference of ar^x and Theorem 1 to show

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

□

2. Falling factorial powers and sums of powers.

For Theorem 1 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 2. For natural number p define the ***falling factorial power*** of $x \in \mathbf{R}$ as $x^{\underline{0}} = 1$ and for $p \geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has p terms.) □

For small values of p this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1\\
x^{\underline{1}} &= x\\
x^{\underline{2}} &= x(x-1)\\
x^{\underline{3}} &= x(x-1)(x-2)\\
x^{\underline{4}} &= x(x-1)(x-2)(x-3)\\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

Proposition 1. *If $f(x) = x^{\underline{p}}$ where p is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

Problem 4. Prove this. □

Remark 1. The formula should remind you of the c formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. □

Proposition 2. *If $f(x) = x^{\underline{p}}$ where p is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of f.*

Problem 5. Prove this as a corollary of Proposition 1 by noting (by replacing p by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. □

Problem 6. Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) □

Proposition 3. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Remark 2. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

Problem 7. Prove this. Hint: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 1

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 6 to see that $F(1) = 0$. □

Proposition 4. *The equalities*

$$\begin{aligned} x &= x^{\underline{1}} \\ x^2 &= x^{\underline{2}} + x^{\underline{1}} \\ x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\ x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\ x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}} \end{aligned}$$

hold.

Problem 8. Verify the first three of these. □

Problem 9. Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

Hint: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 3

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\ &= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}. \end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^{n} k^3$. □

3. A couple of trigonometric sums.

For your convenience we recall some trig identities:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \end{aligned}$$

Problem 10. Let θ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show

$$\sin\left(\theta\left(x+\frac{1}{2}\right)\right) - \sin\left(\theta\left(x-\frac{1}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\theta x\right)$$

and therefore

$$F(x) = \frac{\sin\left(\theta\left(x-\frac{1}{2}\right)\right)}{2\sin(\frac{\theta}{2})}.$$

is an anti-difference of

$$f(x) = \cos(\theta x).$$

□

Proposition 5. *If* $\sin\left(\frac{\theta}{2}\right) \neq 0$, *then*

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

Problem 11. Use Problem 10 and Theorem 1 to prove this. □

There is a similar formula for sums for the sine function.

Proposition 6. *If* $\sin(\frac{\theta}{2}) \neq 0$, *then*

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

Problem 12. Prove this. □