Mathematics 546 Final Name:______________________________

Show your work! Answers that do not have a justification will receive no credit.

(1) (25 Points) Define the following:

(a) H is a ***normal subgroup*** of the group G.

(b) b is a ***multiple*** of a where a and b are integers.

(c) The group G is ***Abelian***.

(d) The ***index*** of the subgroup H in the group G.

(e) The ***quotient group*** G/H where H is a normal subgroup of G.

(2) (15 Points) State the following:

(a) Lagrange's Theorem.

(b) The theorem about ideals in $\mathbf{Z}$.

(c) Gauss' Theorem about rational roots of polynomials with integer coefficients.

(3) (10 Points) Give the addition and multiplication tables for $\mathbf{Z}_6$, the integers modulo 6.

(4) (10 Points)

(a) Write the element $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 3 & 2 & 4 & 8 & 1 & 5 \end{pmatrix}$ of S_8 as in cycle notation.

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(b) Write the elemnt $(1543)(27)$ of S_8 in two line notation.

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(5) (10 Points) Find $\gcd(105, 186)$ and write it as a linear combiation of 105 and 186.

(6) (10 Points) Find a solution to $7x \equiv 3 \mod 22$. ________________________

(7) (10 Points) Show that for any integer n the gcd of n and $2n+5$ is either 1 or 5.

(8) (10 points) Let G be a group with $|G| = p$ where p is a prime. Let $a \in G$ with $a \neq e$. Show that $\langle a \rangle = G$.

(9) (15 Points) In the group $G = \mathbf{Z}_{15}^*$ let $H = \langle 2 \rangle$ be the cyclic subgroup generated by 2.

(a) List all the cosets of H in G.

(b) Give the operation table for the quotient group G/H.

(10) (10 Points)

(a) What is the order of the group $\mathbf{Z}_{24}^*$?

$|\mathbf{Z}_{24}^*| =$ ____________________

(b) Explain why if x is an integer with $\gcd(x, 24) = 1$ that $x^8 \equiv 1 \mod 24$.

(11) (10 Points) Recall the dihedral group D_6 is the group of elments $a^i b^j$ where

$$a^6 = e, \qquad b^2 = e, \qquad ba = a^{-1}b.$$

The subgroup $H = \langle a^2 \rangle = \{e, a^2, a^4\}$ is a normal subgroup of D_6. Find the operation table for the quotient group D_6/H.

(12) (10 Points) Use Lagrange's Theorem to show that if G is a finite group, $n = |G|$, and $a \in G$, that $a^n = e$.

(13) (10 Points) Show that if H, K, L are subgroups of a group G, then the intersection $H \cap K \cap L$ is also a subgroup.

Have a good time with the little bit of summer left.
You have been a fun group to teach.