Packing Posets in the Boolean Lattice

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Abstract

We are interested in maximizing the number of pairwise unrelated copies of a poset P in the family of all subsets of $[n]$. For instance, Sperner showed that when P is one element, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the maximum number of copies of P. Griggs, Stahl, and Trotter have shown that when P is a chain on k elements, $\frac{1}{2^{k-1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is asymptotically the maximum number of copies of P. We prove that for any P the maximum number of unrelated copies of P is asymptotic to a constant times $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover, the constant has the form $\frac{1}{c(P)}$, where $c(P)$ is the size of the smallest convex closure over all embeddings of P into the Boolean lattice.

1 Introduction

Using standard notation, let $\mathcal{B}_n$ be the inclusion poset of all subsets of $[n] = \{1, 2, \ldots, n\}$. Let P be any poset. Let $f : P \to \mathcal{B}_n$ be a weak embedding of the poset P into $\mathcal{B}_n$, i.e., if $a < b \in P$, then $f(a) \subset f(b)$. We call $f(P)$ a copy of P in $\mathcal{B}_n$. Let $\{\mathcal{F}_i\}_{i\geq 1}$ be pairwise

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unrelated copies of P, i.e., if $A_i \in \mathcal{F}_i$, $A_j \in \mathcal{F}_j$, and $i \neq j$, then A_i and A_j are unrelated. We say the family $\mathcal{F} = \cup_i \mathcal{F}_i$ is a family constructed from pairwise unrelated copies of P. Let $\mathrm{Pa}(n, P)$ denote the maximum size of a family constructed from pairwise unrelated copies of P in $\mathcal{B}_n$. This quantity can be generalized to apply to a collection of posets; let $\mathrm{Pa}(n, \{P_i\}_{i\geq 1})$ denote the maximum size of a family in $\mathcal{B}_n$ constructed from pairwise unrelated copies of posets chosen from the collection of posets $\{P_i\}_{i\geq 1}$.

We may also ask the similar question, what is the maximum number of pairwise unrelated induced copies of P in $\mathcal{B}_n$, where each copy is a strong embedding of P? A strong embedding f of P is such that for $a, b \in P$, $a < b$ *if and only if* $f(a) \subset f(b)$. We will denote the maximum size of a family in $\mathcal{B}_n$ constructed from induced copies of P as $\mathrm{Pa}^*(n, P)$. We can also define the more general quantity $\mathrm{Pa}^*(n, \{P_i\}_{i\geq 1})$.

The motivation for finding $\mathrm{Pa}(n, \{P_i\}_{i\geq 1})$ comes from a question under intensive study in recent years, that of finding the maximum size $\mathrm{La}(n, Q)$, which is the maximum size of a family $\mathcal{F} \subseteq \mathcal{B}_n$ that contains no copy of poset Q. This seems to be a challenging problem in extremal set theory, even determining the asymptotic growth of $\mathrm{La}(n, Q)$, as $n \to \infty$, for posets as simple as the four element diamond (which is $\mathcal{B}_2$). For a survey on the topic, see [3]. For the most recent progress on the diamond, see [8].

It is natural to extend this notion to collections of posets $\{Q_j\}_{j\geq 1}$, seeking to find the maximum size $\mathrm{La}(n, \{Q_j\}_{j\geq 1})$ of a family $\mathcal{F} \in \mathcal{B}_n$ that contains no copy of any poset Q_j in the collection. We noticed that for the collection $\{\mathcal{V}, \Lambda\}$, where $\mathcal{V} = \mathcal{V}_2$ is the poset on $\{a, b, c\}$ with $a < b$ and $a < c$, and Λ is the poset on $\{a, b, c\}$ with $a > b$ and $a > c$, $\mathrm{La}(n, \{\mathcal{V}, \Lambda\})$ is the same as $\mathrm{Pa}(n, \{\mathcal{B}_0, \mathcal{B}_1\})$, since any collection of subsets that contains no copy of $\mathcal{V}$ or Λ has components consisting only of single sets and/or two-element chains, all unrelated to each other. We recently learned that Katona and Tarján [7] solved this problem years ago, showing that $\mathrm{La}(n, \{\mathcal{V}, \Lambda\}) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. We were able to derive the same result, applying a 1984 result of Griggs, Stahl, and Trotter [4] that gives $\mathrm{Pa}(n, \mathcal{B}_1)$;

we present their more general result for the path $\mathcal{P}_k$ below.

More generally, for any collection $\{Q_j\}_{j\geq 1}$, $\mathrm{La}(n, \{Q_j\}_{j\geq 1})$ is equivalent to $\mathrm{Pa}^*(n, \{P_i\}_{i\geq 1})$, where $\{P_i\}_{i\geq 1}$ is the collection of all possible connected posets that do not contain any of the posets in $\{Q_j\}_{j\geq 1}$ as a subposet. Note that the collection $\{P_i\}_{i\geq 1}$ may be infinite. For instance, $\mathrm{La}(n, \mathcal{V})$ is the same as $\mathrm{Pa}^*(n, \{P_i\}_{i\geq 1})$ where P_i is the i-fork consisting of one set that contains i (unrelated) sets, $i \geq 0$. So the problem of determining $\mathrm{Pa}^*(n, \{P_i\}_{i\geq 1})$ can be viewed as more general than the $\mathrm{La}(n, \{Q_j\}_{j\geq 1})$ problem.

In this paper we now concentrate on finding the asymptotic behavior of $\mathrm{Pa}(n, P)$ for single posets P. We hope that these ideas might help with solving the more difficult problem of finding $\mathrm{La}(n, \{Q_j\})$. To start us off, here are some examples in the literature of finding $\mathrm{Pa}(n, P)$ for specific P. A natural technique in proving results on posets is to count full chains. We define a *full chain* in $\mathcal{B}_n$ to be a chain of $\mathcal{B}_n$ that includes a subset of $[n]$ of every size. In Sperner's classic theorem (1928), he finds $\mathrm{Pa}(n, P)$ for $P = \mathcal{B}_0$.

Theorem 1.1 ([10]) $\mathrm{Pa}(n, \mathcal{B}_0)$ *is* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We include a proof of Sperner's Theorem to demonstrate a basic outline for our later proofs. Here is a proof introduced by Lubell [9]:

Proof. Each copy of $\mathcal{B}_0$ is just a subset of $[n]$. If a copy of $\mathcal{B}_0$ is a subset of size a, then it meets $a!(n-a)!$ full chains in $\mathcal{B}_n$; this is at least $\lfloor n/2 \rfloor! \lceil n/2 \rceil!$ full chains. There are only $n!$ full chains in $\mathcal{B}_n$. No chain may hit more than one copy of $\mathcal{B}_0$. The number of copies of $\mathcal{B}_0$ is $\frac{\mathrm{Pa}(n, \mathcal{B}_0)}{|\mathcal{B}_0|}$. Counting the full chains gives the inequality

$$\frac{\mathrm{Pa}(n, \mathcal{B}_0)}{|\mathcal{B}_0|}\Big(\lfloor n/2 \rfloor! \lceil n/2 \rceil!\Big) \leq n!, \text{ and hence}$$

$$\mathrm{Pa}(n, \mathcal{B}_0) \leq |\mathcal{B}_0| \frac{n!}{\lfloor n/2 \rfloor! \lceil n/2 \rceil!} = |\mathcal{B}_0| \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This bound is tight, as the copies of $\mathcal{B}_0$ may be chosen to be the middle level of $\mathcal{B}_n$. $\square$

This proof was generalized by Griggs, Stahl, and Trotter (1984) for $P = \mathcal{P}_k$, the chain (or path) on k elements. The earlier paper of Bollobás [2] also implies this result.

Theorem 1.2 ([4]) *The value of* $\mathrm{Pa}(n, \mathcal{P}_k)$ *is* $(k)\binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor}$. *For fixed* k*, as* n *goes to infinity,* $\mathrm{Pa}(n, \mathcal{P}_k)$ *is asymptotically* $\frac{k}{2^{k-1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. For a chain $\mathcal{P}$, with a minimum set $A \subseteq [n]$ and maximum $B \subseteq [n]$, define $\mathcal{I}_{\mathcal{P}}$ as $[A, B]$, the interval from A to B. For two chains $\mathcal{P}$ and $\mathcal{P}'$ to be pairwise unrelated copies of $\mathcal{P}_k$, a full chain in $\mathcal{B}_n$ hits at most one of $\mathcal{I}_{\mathcal{P}}$ or $\mathcal{I}_{\mathcal{P}'}$. A full chain that hits $\mathcal{I}_{\mathcal{P}}$ is a chain constructed from all the elements of A before any of the elements of $[n] \setminus B$. There are $n - |B \setminus A|$ elements in A or not in B. Therefore, the number of chains that hit $\mathcal{I}_{\mathcal{P}}$ is

$$\frac{n!}{\binom{n-|B \setminus A|}{|A|}} \geq \frac{n!}{\binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor}}$$

so each of the $\frac{\mathrm{Pa}(n, \mathcal{P}_k)}{|\mathcal{P}_k|}$ intervals from the copies of $\mathcal{P}_k$ meets at least $\frac{n!}{\binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor}}$ full chains. This gives that

$$\frac{\mathrm{Pa}(n, \mathcal{P}_k)}{|\mathcal{P}_k|} \frac{n!}{\binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor}} \leq n!, \text{ or}$$

$$\mathrm{Pa}(n, \mathcal{P}_k) \leq |\mathcal{P}_k| \binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor} = (k)\binom{n-(k-1)}{\lfloor \frac{n-(k-1)}{2} \rfloor} \sim \frac{k}{2^{k-1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The bound is tight, as the following construction demonstrates. Fix a set $S \subseteq [n]$ such that $|S| = k - 1$. The A's corresponding to the chains are all the $\lfloor \frac{n-(k-1)}{2} \rfloor$-sets of $[n] \setminus S$, and the $\mathcal{P}_k$'s are chosen as any full chain in the interval $[A, A \cup S]$ for each A. $\square$

Notice the steps in the proof above. For the upper bound, first, each copy of $\mathcal{P}_k$ is contained in a larger set system $\mathcal{I}_{\mathcal{P}}$, where a full chain hits at most one $\mathcal{I}_{\mathcal{P}}$. Second, a lower bound on the number of full chains that hit an $\mathcal{I}_{\mathcal{P}}$ is found. Now $n!$ divided by this lower bound is an upper bound on $\dfrac{\mathrm{Pa}(n,P)}{|P|}$, the number of copies of P. As for the lower bound, a construction is found. In this particular example, the construction is multiple copies of $\mathcal{P}_k$, where each copy's minimum is on a base rank $\lfloor\frac{n-(k-1)}{2}\rfloor$, and the minimum is constructed from just the elements of $[n] \setminus S$. The set S restricts the choices of which sets in the base level to include in the packing. Each copy of P can then easily be built on top of its minimum using the elements of S.

A similar method may be used to find $\mathrm{Pa}(n,P)$ for general P. One important concept is how each copy of $\mathcal{P}_k$ is contained in a larger set system $\mathcal{I}_{\mathcal{P}}$, where a full chain meets at most one $\mathcal{I}_{\mathcal{P}}$. Similarly, we contain a copy of P inside the convex closure of that copy. The convex closure of a set system is defined as follows: Let $\mathcal{F} \subseteq \mathcal{B}_n$. In $\mathcal{B}_n$, $\mathcal{F}$ generates an ideal (or down-set) and a filter (or up-set) denoted as follows:

$$D(\mathcal{F}) = \{S \in \mathcal{B}_n | S \subseteq A \text{ for some } A \in \mathcal{F}\}, \text{ and}$$

$$U(\mathcal{F}) = \{S \in \mathcal{B}_n | A \subseteq S \text{ for some } A \in \mathcal{F}\}.$$

We define a closure operator on $\mathcal{F}$ as $\overline{\mathcal{F}} := D(\mathcal{F}) \cap U(\mathcal{F})$. Another definition would be

$$\overline{\mathcal{F}} := \{S \in \mathcal{B}_n | A \subseteq S \subseteq B \text{ for some } A, B \in \mathcal{F}\}.$$

Here S, A, and B could be equal, so clearly $\mathcal{F} \subseteq \overline{\mathcal{F}}$. A family $\mathcal{F}$ such that $\mathcal{F} = \overline{\mathcal{F}}$ is called convex. Note that convex families appear in the literature, including the conjecture by P. Frankl and J. Akiyama:

Conjecture 1.1 ([1]) *For every convex family $\mathcal{F} \subseteq \mathcal{B}_n$, there exists an antichain $\mathcal{A} \subseteq \mathcal{F}$*

such that $|\mathcal{A}| / |\mathcal{F}| \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}/2^n$.

If two copies of P are unrelated, then their closures must be unrelated as well. Therefore, we are more interested in the size and structure of the closure of a copy of P than of the copy of P itself. For a weak embedding f of P into $\mathcal{B}_k$, there exists a minimum value of $\left|\overline{f(P)}\right|$ over all choices of f and k. Denote this minimum as $c(P)$.

Here are some examples. If P is the three-element poset $\mathcal{V}$, we have that f may embed $\mathcal{V}$ into $\mathcal{B}_2$ such that $f(\mathcal{V}) = \{\emptyset, \{1\}, \{2\}\}$. Now $\overline{f(\mathcal{V})} = f(\mathcal{V})$ so $c(\mathcal{V}) = |\mathcal{V}| = 3$. In the proof of Theorem 1.2, the closure of an embedding of a chain $\mathcal{P}_k$ is the smallest interval in which it is enclosed, $\mathcal{I}_{\mathcal{P}}$ in the proof. The smallest size of this interval is 2^k so $c(\mathcal{P}_k) = 2^k$.

Here is one of the two main theorems, finding $\mathrm{Pa}(n, P)$ asymptotically for any P in terms of $c(P)$ and $|P|$.

Theorem 1.3 *For any poset P, as $n \to \infty$,* $\mathrm{Pa}(n, P) \sim \frac{|P|}{c(P)} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We can similarly define $c^*(P)$ as the minimum size of the closure of a strong embedding of P in $\mathcal{B}_n$ over all possible n. In general, $c^*(P) \neq c(P)$. Take for instance the poset $J = \{a, b, c, d\}$, $a < b < c$, and $a < d$; J may be weakly embedded into $\mathcal{B}_2$ so $c(J) = 4$. As for f, a strong embedding of J into $\mathcal{B}_k$, there exists a set $B' \in \mathcal{B}_k$, $f(b) \neq B'$, such that $f(a) \subset B' \subset f(c)$ so $B' \in \overline{f(J)}$, but $f(d) \neq B'$ because $f(d) \not\subseteq f(c)$. Therefore, $c^*(J) \geq 5$. Also, a strong embedding of J into $\mathcal{B}_3$ is easy to find such that $c^*(J) = 5$.

The second main theorem finds $\mathrm{Pa}^*(n, P)$ asymptotically for any P in terms of $c^*(P)$ and $|P|$.

Theorem 1.4 *For any poset P, as $n \to \infty$,* $\mathrm{Pa}^*(n, P) \sim \frac{|P|}{c^*(P)} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

While preparing this manuscript, we learned that this problem of determining asymptotically the maximum number of unrelated copies of a poset P in $\mathcal{B}_n$ was already proposed by Katona at a conference lecture in 2010 [5]. We also learned that Katona and Nagy [6] have recently (and independently) obtained results essentially equivalent to our two main results above. Our extension of the problem to a family of posets, $\mathrm{Pa}(n, \{P_i\}_{i\geq 1})$, appears to be new.

The following two sections are a proof of Theorem 1.3. The proof of Theorem 1.4 will require only a few alterations. This will be demonstrated after the main proof.

2 The Upper Bound

We obtain the upper bound on the number of unrelated copies of poset P from an asymptotic lower bound on the number of full chains that meet the *closure* of a copy of P. For a family $\mathcal{F}$ of subsets of $[n]$, let $a(\mathcal{F})$ be the number of full chains in $\mathcal{B}_n$ that intersect $\mathcal{F}$. While $a(\mathcal{F})$ will be as large as $n!$, if, say, $\mathcal{F}$ contains $\emptyset$, we are interested in how small it can get. If $\mathcal{F}$ consists of m subsets of size k, then $a(\mathcal{F})$ will be $mk!(n-k)! = m(n!/\binom{n}{k})$, which is at least $m\Big(n!/\binom{n}{\lfloor \frac{n}{2}\rfloor}\Big)$. For fixed m, as n grows we expect this last formula to be the minimum asymptotically. Let us denote by $\overline{a}(m,n)$ the minimum of $a(\mathcal{F})$, over all families $\mathcal{F} \subseteq \mathcal{B}_n$ with $|\mathcal{F}| = m$.

Proposition 2.1 *Let integer $m \geq 1$. Then as $n \to \infty$ the minimum number of full chains in $\mathcal{B}_n$ that meet a family of m subsets in $\mathcal{B}_n$, $\overline{a}(m,n) \sim m\Big(n!/\binom{n}{\lfloor \frac{n}{2}\rfloor}\Big)$.*

Proof. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of m subsets of $[n]$. For convenience let us assume that the subsets are labeled so that for all $i < j$, $|A_i| \leq |A_j|$. For any $1 \leq i_1 < \cdots < i_k \leq m$ let $b(i_1, \ldots, i_k)$ denote the number of full chains that pass through all of $A_{i_1}, \ldots, A_{i_k}$. Of course, $b(i_1, \ldots, i_k)$ is nonzero if and only if the sets $A_{i_1}, \ldots, A_{i_k}$ form a chain. Inclusion-exclusion gives us that $a(\mathcal{F})$ is the sum of the $b(i_1)$ minus the sum of the $b(i_1, i_2)$ plus the sum of the $b(i_1, i_2, i_3)$ minus and so on. Our difficulty now is that some terms $b(i_1, \ldots, i_k)$ with $k \geq 2$ can actually be large compared to some singleton terms $b(i_1)$, so we cannot immediately dismiss them. For instance, if $n = 100$ and $\mathcal{F}$

happens to be a chain with $|A_i| = i$ for all i, then $b(1,2) = 1!1!98!$ is much larger than $b(50) = 50!50!$. However, we can exploit the fact that terms $b(i_1, \ldots, i_k)$ with $k \geq 2$ are considerably smaller than some $b(i_1)$ terms. In the example, we could instead compare $b(1,2)$ to $b(1) = 1!99!$.

By making all signs for terms with $k \geq 2$ negative, our alternating sum lower bound above is at least the sum of the $b(i_1)$ minus the sum over all $k \geq 2$ of the terms $b(i_1, \ldots, i_k)$. For the $2^m - m - 1$ terms being subtracted, we assign each one to a particular positive singleton term $b(j)$ as follows: For a term $b(i_1, \ldots, i_k)$ with $k \geq 2$, by our labeling we have $|A_{i_1}| \leq \cdots \leq |A_{i_k}|$. Let $u := |A_{i_1}|$ and $v := |A_{i_k}|$. We assign this term to one of $b(i_1)$ or $b(i_k)$, resp., according to whether $|u - (n/2)|$ is at least (less than, resp.) $|v - (n/2)|$. For instance in the example above, the terms $b(20, 28)$ and $b(20, 30, 80)$ are assigned to $b(20)$, while $b(20, 30, 81)$ is assigned to $b(81)$.

We have then each singleton term $b(j) = |A_j|!(n - |A_j|)!$. There are less than 2^{m-1} terms $b(i_1, \ldots, i_k)$ with $k \geq 2$ assigned to $b(j)$. For those terms that are nonzero, it means that $A_{i_1} \subset \cdots \subset A_{i_k}$ and either i_1 or i_k is j, according to which is farther from $n/2$. Suppose $j = i_1$ (so $i_1 < n/2$). Then this term $b(i_1, \ldots, i_k)$ is a product of factorials that refines $b(i_1)$: While $i_1!$ is still a factor, $(n - i_1)!$ is replaced by a product of factorials no more than $1!(n - i_1 - 1)!$, so in total, we get at most $b(i_1)$ divided by $(n - i_1)$, which is at least $n/2$. In this case, and similarly when $j = i_k$, we see that the term $b(i_1, \ldots, i_k)$ is at most $b(j)$ divided by $n/2$. Therefore, the sum of all the terms assigned to $b(j)$ is at most $b(j)$ times $2^m/n$. Hence,

$$a(\mathcal{F}) \geq \sum_{j=1}^{m} b(j)(1 - (2^m/n)).$$

Since each term $b(j) = j!(n-j)! \geq n!/\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and this bound holds independent of $\mathcal{F}$, we see that as $n \to \infty$ for fixed m, $\overline{a}(m,n) \sim m\left(n!/\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$. □

Now we consider our poset packing problem. Assume that we have $\mathrm{Pa}(n,P)/|P|$ unrelated copies F_i of our poset P contained in the Boolean lattice $\mathcal{B}_n$. In fact, if a full chain passes through the closure $\overline{F_i}$ of one of these families F_i, it does not pass through the closure of any other F_j, since F_i and F_j are unrelated. That is, the closures $\overline{F_i}$ are also unrelated. Each closure $\overline{F_i}$ has at least $m = c(P)$ subsets in it so it meets at least $\overline{a}(m,n)$ full chains.

Altogether, the number of full chains that meet some closure $\overline{F_i}$ is then at least $\overline{a}(m,n)\,\mathrm{Pa}(n,P)/|P|$. This is in turn at most the total number of full chains, $n!$. Hence, $\mathrm{Pa}(n,P)/|P|$ is at most $n!/\overline{a}(m,n)$, which is asymptotic to $(1/m)\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for large n. This gives the desired asymptotic upper bound.

3 The Lower Bound Construction

Let m, k, and f be such that f embeds P into $\mathcal{B}_k$, and $\left|\overline{f(P)}\right| = m = c(P)$. We will construct an $\mathcal{F} \subseteq \mathcal{B}_n$ from pairwise unrelated copies of $f(P)$ so that the number of copies of P in $\mathcal{F}$ is $\dfrac{|\mathcal{F}|}{|P|} \sim \dfrac{1}{m}\dbinom{n}{\lfloor \frac{n}{2} \rfloor}$.

We will construct F through a finite number of iterations. Fix an $i \in \mathbb{N}$. This i is the number of iterations for which we construct asymptotically $\dfrac{(2^k - m)^j}{(2^k)^{j+1}}\dbinom{n}{\lfloor \frac{n}{2} \rfloor}$ unrelated copies of P for each $0 \le j \le i-1$. Because we may choose i to be arbitrarily large, we will have

$$
\begin{aligned}
\frac{|\mathcal{F}|}{|P|} &\sim \sum_{j=0}^{i-1} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \\
&\sim \sum_{j=0}^{\infty} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}}\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \\
&= \frac{1}{2^k}\left[\frac{1}{1 - \frac{2^k - m}{2^k}}\right]\binom{n}{\lfloor \frac{n}{2} \rfloor} = \frac{1}{m}\binom{n}{\lfloor \frac{n}{2} \rfloor}.
\end{aligned}
$$

Let's now create such an $\mathcal{F} \subseteq \mathcal{B}_n$ for each n. For the rest of the argument, let $(A + x)$

be the translation $\{a + x \mid a \in A\}$ for a set $A \subseteq [n]$ and an integer x. For the ease of notation, define $S_j := [k(j+1)] \setminus [kj] = \{kj+1, kj+2, \ldots, kj+k\} = ([k]+kj)$, the set $[k]$ translated by a multiple of k.

A level (or row) of $\mathcal{B}_n$ is all subsets of $[n]$ of the same size, the rank of the level. The level of rank r is often denoted as $\binom{[n]}{r}$. Define a *layer* of $\mathcal{B}_n$ (denoted as ℓ) to be $k+1$ consecutive levels of $\mathcal{B}_n$. We call the smallest rank in layer ℓ its base rank, b_ℓ. Specifically, $\ell = \binom{[n]}{b_\ell} \cup \binom{[n]}{b_\ell+1} \cup \cdots \cup \binom{[n]}{b_\ell+k}$. We define our layers by taking the base ranks to be $\lfloor n/2 \rfloor + z(k+1)$ for all integers z; in this way, we partition the levels of $\mathcal{B}_n$ and any two layers are disjoint. We construct $\mathcal{F}$ by populating certain layers with many copies of $f(P)$. A layer ℓ that is populated corresponds to a triple (j_ℓ, R_ℓ, b_ℓ); ℓ has base rank b_ℓ, the iteration in which it is populated j_ℓ (ranges from 0 to $i-1$), and a restriction set $R_\ell \subseteq [kj_\ell]$, which defines which elements of ℓ are in $\mathcal{F}$. The following is exactly how $\mathcal{F}$ is constructed in a layer ℓ:

$$\begin{aligned} \ell \cap \mathcal{F} = \Big\{ R_\ell \cup A \cup B \;\Big|\; & A \subseteq S_{j_\ell}, (A - kj_\ell) \in f(P), \\ & B \subseteq [n] \setminus [k(j_\ell+1)], \text{ and} \\ & |R_\ell| + |B| = b_\ell \Big\}. \end{aligned}$$

Our choice for the R_ℓ and the order of the b_ℓ's, as we will show later, prevents any two copies of P in different layers from having any related sets. For a fixed B, the family of all the A's forms a copy of P that is $f(P)$ translated, from using the elements in $[k]$ to using the elements from S_{j_ℓ}. There is then one copy of P in ℓ for each choice of B. The purpose of B is to combine with R_ℓ to be in the base level of the layer, i.e., $B \cup R_\ell \in \binom{[n]}{b_\ell}$. There are $\dbinom{n - k(j_\ell+1)}{b_\ell - |R_\ell|}$ choices for B. Notice that copies of P within a layer are unrelated; every set in a copy of P has the same base set $R_\ell \cup B$, and the copies of P in a layer have unrelated base sets.

For each iteration j, we will be populating $(2^k - m)^j$ layers. This gives a total of only $L := i$ populated layers if $2^k - m = 1$, or $L := \sum_{j=0}^{i-1}(2^k - m)^j = \dfrac{(2^k - m)^i - 1}{(2^k - m) - 1}$ populated layers otherwise. The order of the b_ℓ's of the populated layers is important in preventing any two copies of P from being related, but as long as the order of the populated layers is maintained, the b_ℓ's for the populated layers may be chosen close to the middle level, i.e., $|b_\ell - \lfloor n/2 \rfloor| \leq (k+1)L$, where L is a constant defined above that does not depend on n and $k+1$ is the number of levels in each layer. Each layer has $\binom{n - k(j_\ell + 1)}{b_\ell - |R_\ell|}$ copies of P. This is now asymptotic to $\dfrac{1}{(2^k)^{j_\ell+1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies since $b_\ell - |R_\ell|$ is at most a fixed, finite distance from $n/2$. This results in our desired number of copies of P,

$$\frac{|\mathcal{F}|}{|P|} \sim \sum_{j=0}^{i-1} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We will now demonstrate how each R_ℓ is chosen and in what order are the populated layers to ensure that the copies of P are pairwise unrelated.

Let's start with $j = 0$. We start by populating one layer of F; let $\mathcal{F} \supseteq \{A \cup B \mid A \in f(P), B \in [n] \setminus [k], |B| = \lfloor n/2 \rfloor\}$. In other words, the layer ℓ with $b_\ell = \lfloor n/2 \rfloor$ is populated with $R_\ell = \emptyset$ and $j_\ell = 0$. Now $|\mathcal{F}| \geq |P| \binom{n-k}{\lfloor n/2 \rfloor}$, which is asymptotically $\dfrac{1}{2^k}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P. So if $m = 2^k$, (i.e., $\overline{f(P)} = \mathcal{B}_k$,) we are done. If not, we would like to add more copies of P then just those in our middle layer so we will need to know which of the elements of $\mathcal{B}_n$ are available to include in the family; we consider which elements of $\mathcal{B}_n$ are unrelated to any element of this middle, populated layer. Consider a set $B \in \mathcal{B}_n$ and $B_{[k]} := B \cap [k]$ and $b := \left|B \setminus B_{[k]}\right|$. This set B is unrelated to all sets in $\mathcal{F}$ if and only if one of the following is true:

1. $B_{[k]}$ is unrelated to all sets in $f(P)$;

2. $B_{[k]} \not\subseteq C$ for all $C \in f(P)$ and $b < \lfloor n/2 \rfloor$; or

3. $B_{[k]} \not\supseteq C$ for all $C \in f(P)$ and $b > \lfloor n/2 \rfloor$.

The choices for $B_{[k]}$ that can provide more sets to add to the family are exactly the sets $B_{[k]} \in \mathcal{B}_k \setminus \overline{f(P)}$. In fact, each one of the $B_{[k]} \in \mathcal{B}_k \setminus \overline{f(P)}$ can lead to a distinct *layer* of copies of P by choosing the base levels correctly; the new layers are the layers from the second iteration (so would have $j_\ell = 1$), and the layer's restriction set would be $B_{[k]}$. The next step is identifying appropriate base levels for each new layer and then demonstrating how this process iterates.

Let's order the elements of $U := \mathcal{B}_k \setminus \overline{f(P)}$. First, split U into two sets, U^+ and U^-:

$$\begin{aligned} U^+ :=& \mathcal{B}_k \setminus U(f(P)) \\ =& \{V \in U \mid V \not\supseteq C \text{ for all } C \in \overline{f(P)}\}, \text{ and} \\ U^- :=& U \setminus U^+ \\ =& U(f(P)) \setminus \overline{f(P)} \\ \subseteq& \{V \in U \mid V \not\subseteq C \text{ for all } C \in \overline{f(P)}\}. \end{aligned}$$

The set U^+ contains both the elements of U contained in some element of $f(P)$ and the subsets of $[k]$ that are unrelated to any element of $f(P)$. On the other hand, U^- contains the elements of U containing some element of $f(P)$. Let $\leq_U$ be any ordering of the elements in U such that if $V_1 \in U^-$ and $V_2 \in U^+$, then $V_1 \leq_U V_2$, else if $V_1 \supseteq V_2$, then $V_1 \leq_U V_2$. We will use this ordering $\leq_U$ to order the base ranks to guarantee all copies of P remain unrelated.

For $j = 0$, we have the populated layer corresponding to $(0, \emptyset, \lfloor n/2 \rfloor)$. For $j = 1$, we populate the layers corresponding to $(1, V, b_V)$ for each $V \in U$. We can choose the b_V's such that if $V \in U^-$, then $b_V < \lfloor n/2 \rfloor$, and if $V \in U^+$, then $b_V > \lfloor n/2 \rfloor$, and if $V_1 <_U V_2$, then $b_{V_1} < b_{V_2}$. For an iteration $j > 1$, for each layer corresponding to $(j-1, R, b)$ populated in iteration $j-1$, we can populate $2^k - m$ new layers, one for each set in U. These new layers correspond to $\Big(j, R \cup \big(V + k(j-1)\big), b_\ell\Big)$ for each $V \in U$. Inductively, there are then $(2^k - m)^j$ layers populated in iteration j, each with asymptotically $\frac{1}{(2^k)^{j+1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P, for a total of $\frac{(2^k-m)^j}{(2^k)^{j+1}}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P associated with iteration j. All that is left to prove is that we can put the layers in an appropriate order, i.e., the base ranks (b_ℓ) may be chosen in such a way as to prevent any two copies of P from being related.

Let $(\ell_s)_{1 \leq s \leq L}$ be the sequence of populated layers, ℓ_s corresponding to (j_s, R_s, b_s), in the order of the rank of the base levels, i.e., for all $s_1 < s_2$, $b_{s_1} < b_{s_2}$. Let's consider our ordered set U again. Let's add to U the character E to indicate the 'end' of a word. Let E be between U^- and U^+ in $\leq_U$. Consider words $V_0V_1 \ldots V_{j-1}E$, where the letters come from U, the words always end in E, and E is only at the end of a word. We only consider words of length $j+1$, where $0 \leq j \leq i-1$. There is a bijection between the layers (ℓ_s) and the possible words of length at most i. Specifically, given a word $V_0V_1 \ldots V_{j-1}E$, its corresponding j_s is j and $R_s = \cup_{p=0}^{j-1}(V_p + kp)$. Let W be the set of all words of length at most i. Order these words lexicographically using $\leq_U$. Specifically, for any two words in W, $w_p = U_0 \ldots U_s$ and $w_q = V_0 \ldots V_t$, we say $w_1 < w_2$ if and only if $U_i = V_i$ for $0 \leq i \leq j-1$ and $U_j <_U V_j$ for some $j \geq 0$. Use this ordering of W and the bijection between the words and the layers to directly define the corresponding ordering of the (b_s). Specifically, for two layers ℓ_1 and ℓ_2 with base level ranks b_1 and b_2 respectively, $b_1 < b_2$ if and only if the word corresponding to ℓ_1 is less than the word corresponding to ℓ_2.

Now we show that no two copies of P are related. We have already seen that no two copies of P in the same layer can be related. For two copies of P, P_p in layer ℓ_p (with base rank b_p) and P_q in layer ℓ_q (with base rank b_q), consider their corresponding words, $w_p = U_0 \ldots U_s$ and $w_q = V_0 \ldots V_t$. Without loss of generality, let $w_p <_U w_q$ so $b_p < b_q$. Consider the subscript c for the first character where w_p and w_q differ, i.e., $U_0 \ldots U_{c-1} = V_0 \ldots V_{c-1}$ and $U_c \neq V_c$, $U_c <_U V_c$. Choose any representatives of the copies of P, $A_p \in P_p$ and $A_q \in P_q$, and define $B_p := A_p \cap S_c$ and $B_q := A_q \cap S_c$. Since $b_p < b_q$, we have that $A_q \nsubseteq A_p$; next we show that $A_p \nsubseteq A_q$.

The order of the words, and hence the order of the b_ℓ's, was chosen specifically to prevent any copies of P from being pairwise related. If $U_c = E$, then $(B_p - kc) \in f(P)$ and $V_c \in U^+$ so $V_c \nsupseteq C$ for all $C \in f(P)$, i.e., $(V_c + kc) = B_q \nsupseteq B_p$ for any B_p such that $(B_p - kc) \in f(P)$. But $B_q \nsupseteq B_p$ implies $A_q \nsupseteq A_p$. For similar reasoning, if $V_c = E$, then $A_p \nsubseteq A_q$. If neither $U_c = E$ nor $V_c = E$, then $U_c < V_c$ implies $U_c \nsubseteq V_c$, but $U_c = (B_p + kc)$ and $V_c = (B_q + kc)$ so $B_p \nsubseteq B_q$ so $A_p \nsubseteq A_q$. Either way, no set from P_p is related to any set from P_q. This completes the proof of Theorem 1.3.

4 Concluding Remarks

We now explain how we may modify the proof above to prove Theorem 1.4. In proving the upper bound of Theorem 1.3, we use the fact that the closure of a copy of P meets at least $\overline{a}(c(P), n)$ full chains in $\mathcal{B}_n$. For Theorem 1.4, using only strong embeddings, we have that a copy of P meets at least $\overline{a}(c^*(P), n)$, which similarly gives us the upper bound. In the lower bound of Theorem 1.3, we created a family $\mathcal{F} \subseteq \mathcal{B}_n$ constructed from multiple copies of $f(P)$, a weak embedding of P into $\mathcal{B}_k$ such that $\overline{f(P)} = c(P)$. If we instead take f to be a strong embedding such that $\overline{f(P)} = c^*(P)$, then the same method of construction will achieve the asymptotic lower bound.

For a finite collection of posets, the quantities $\mathrm{Pa}(n, \{P_1, \ldots, P_k\})$ and $\mathrm{Pa}^*(n, \{P_1, \ldots, P_k\})$ may be found asymptotically as well. Specifically, as n goes to infinity,

$$\mathrm{Pa}(n, \{P_1, P_2, \ldots, P_k\}) \sim \max_{1 \le i \le k} \left(\frac{|P_i|}{c(P_i)} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}, \text{ and}$$

$$\mathrm{Pa}^*(n, \{P_1, P_2, \ldots, P_k\}) \sim \max_{1 \le i \le k} \left(\frac{|P_i|}{c^*(P_i)} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

As for future work, it would be nice to know how to find $c(P)$ and $c^*(P)$ quickly for any P, or at least to find the complexity of such an algorithm. Also, in the examples in the introduction, the exact values for $\mathrm{Pa}(n, P)$ are found, not just their asymptotic values. It would be nice to have exact values for $\mathrm{Pa}(n, P)$ and $\mathrm{Pa}^*(n, P)$.

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