Planar Posets and Minimal Elements

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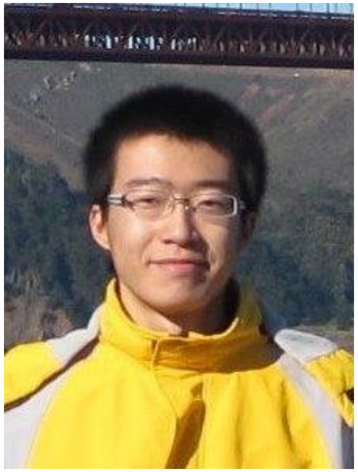
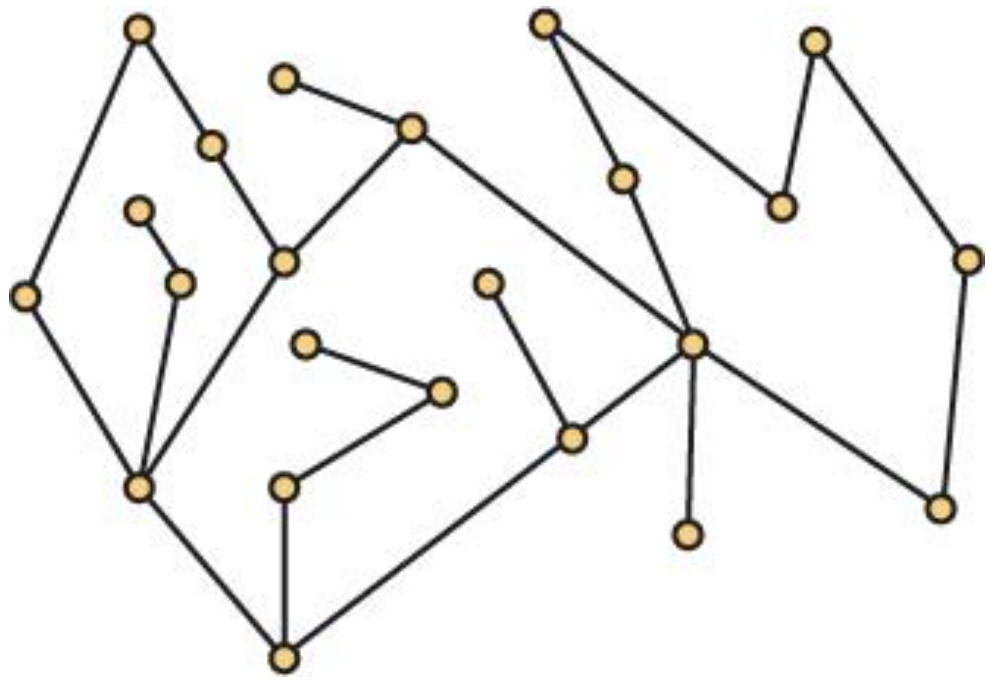
Joint Research with Ruidong Wang



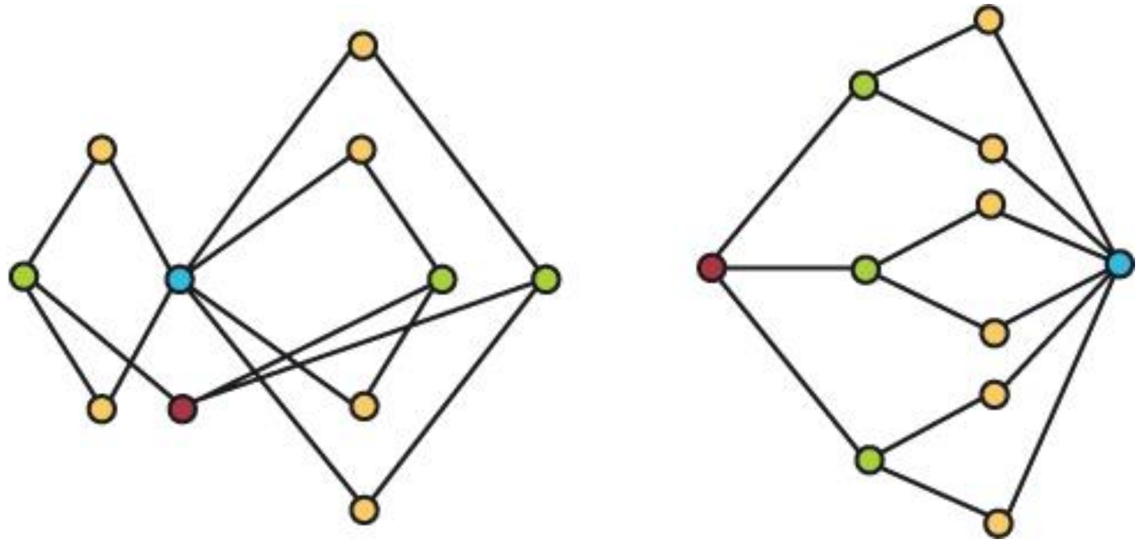
Ph.D. Georgia Tech, 2015 (anticipated)

Planar Posets



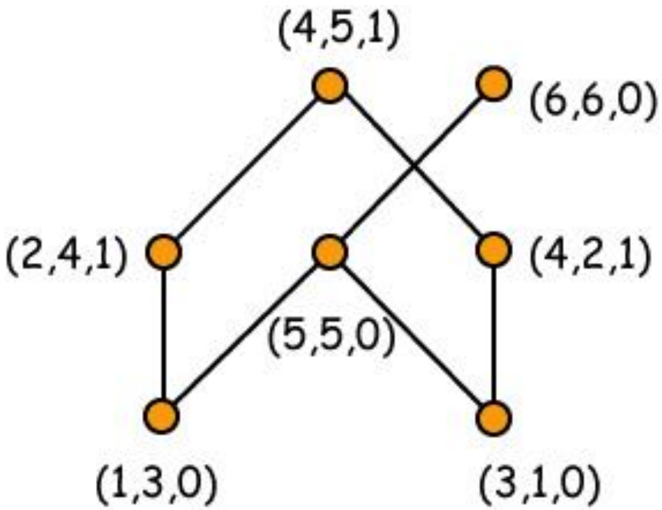
Definition A poset P is **planar** when it has an order diagram with no edge crossings.

A Non-planar Poset



Observation The height 3 non-planar poset shown on the left has a planar cover graph as evidenced by the drawing on the right.

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

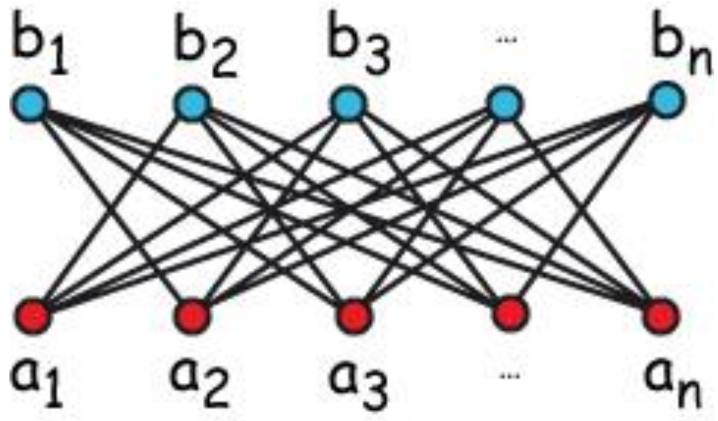
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

The **dimension** of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

Standard Examples



S_n

Fact For $n \geq 2$, the **standard example** S_n is a poset of dimension n.

Excluding the Standard Example S_2

Theorem (Füredi, Rödl, Hajnal and Trotter, '91) The maximum dimension of posets excluding S_2 and having height h is

$$\lg \lg h + 1/2 \lg \lg \lg h + O(1)$$

Theorem (Fishburn, '84) The posets which exclude S_2 are interval orders.

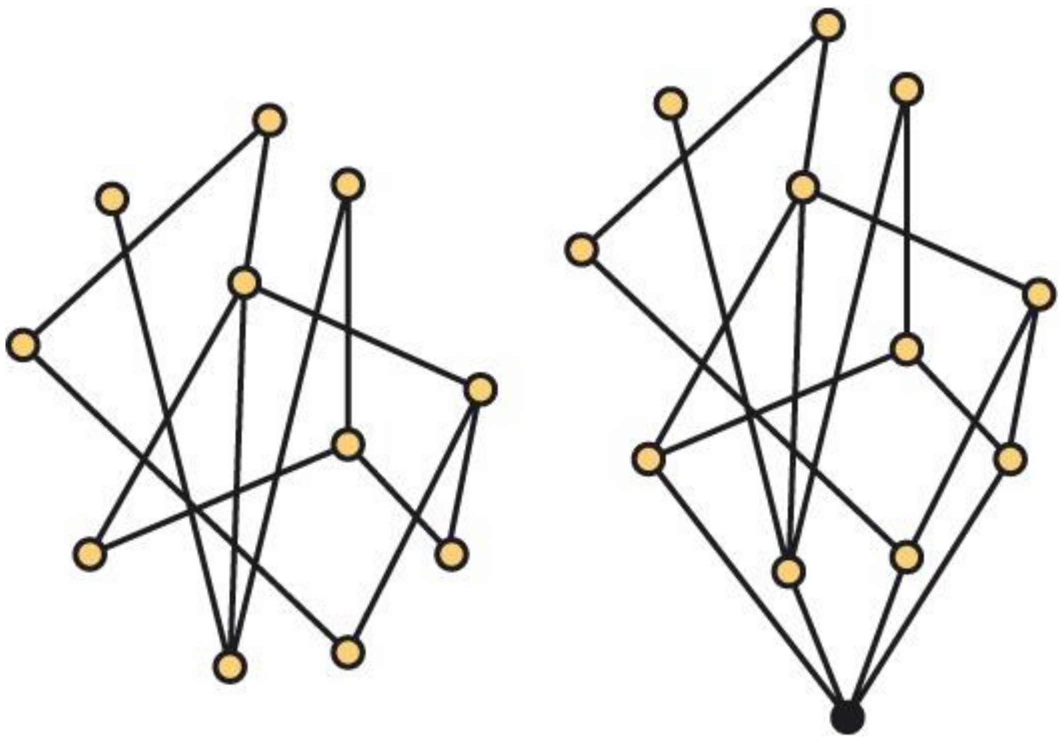
Dimension and Girth

Theorem (Felsner and Trotter, '00) For every pair (g, d) of positive integers, there is a poset P of height 2 so that $\dim(P) \geq d$, and the girth of P, considered as a bipartite graph, is at least g.

Note When $d \geq 3$, these posets contain the standard example S_2, i.e., they are not interval orders. On the other hand, when $g \geq 7$, they do not contain S_3.

Zeroes and Dimension

Fact If P is a poset, and Q is obtained from P by attaching a "zero", then $\dim(P) = \dim(Q)$.



Comparability Invariants

Remark Height, width, dimension and number of linear extensions are comparability invariants.

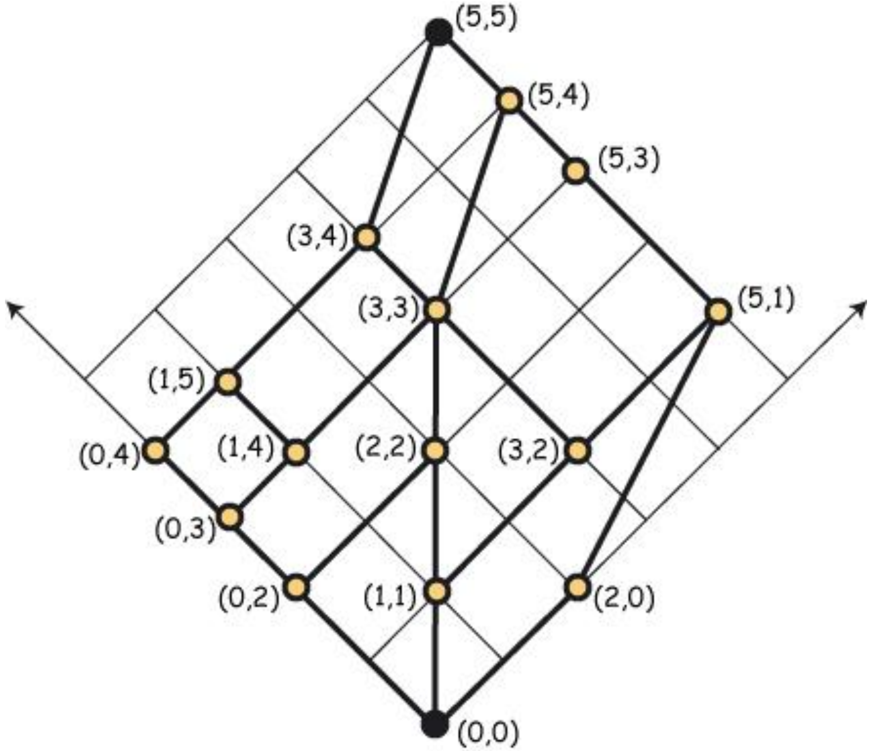
Remark All these parameters can vary wildly for posets with the same cover graph.

Meta Theorem The cover graph tells us almost nothing about the combinatorial properties of a poset!

Planar Posets with Zero and One

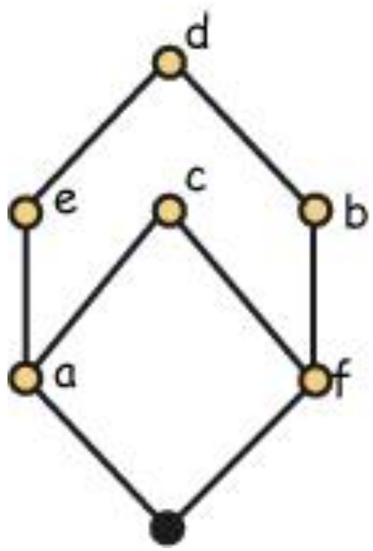
Theorem (Baker, Fishburn and Roberts, '71 + Folklore)

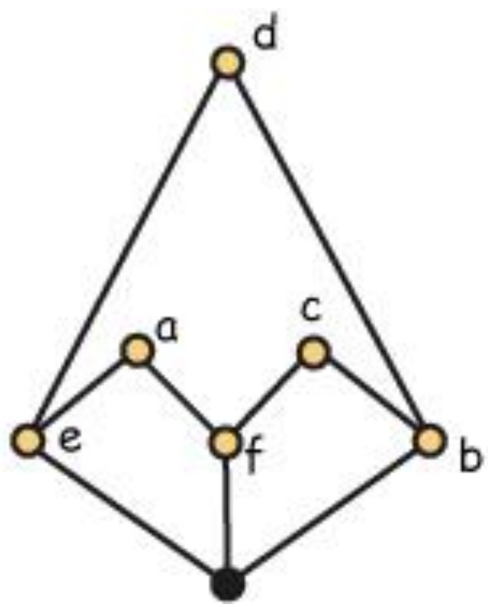
If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.

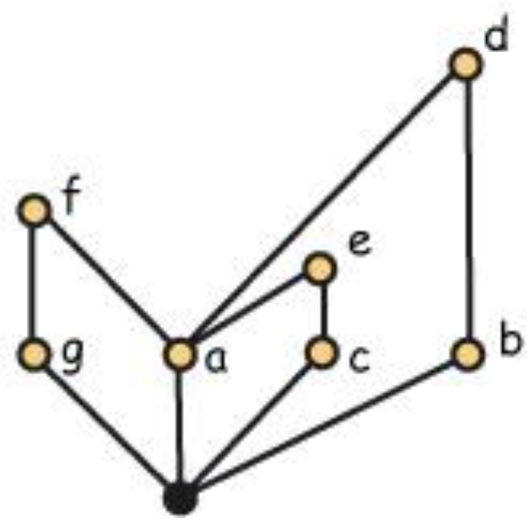


Dimension of Planar Poset with a Zero

Theorem (Trotter and Moore, '77) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.

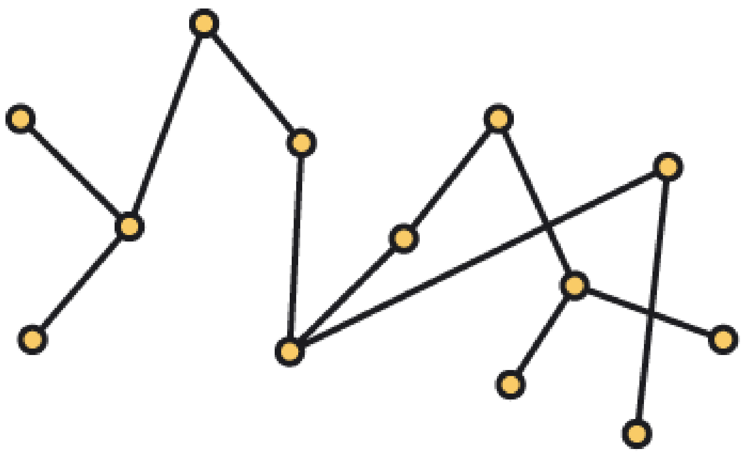






The Dimension of a Tree

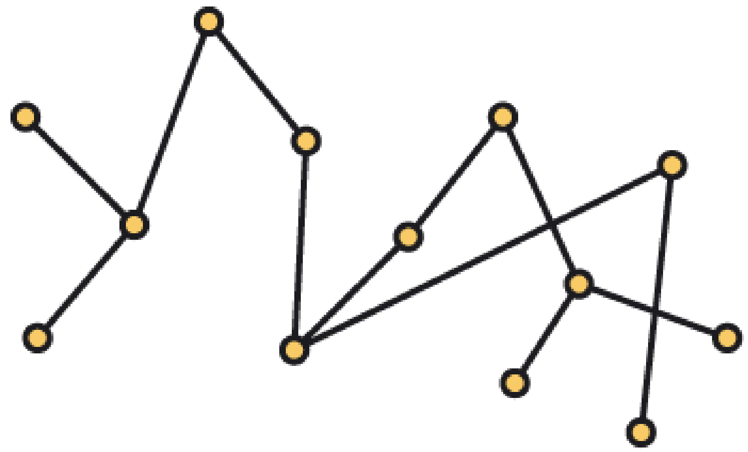
Corollary (Trotter and Moore, '77) If the cover graph of P is a tree, then $\dim(P) \leq 3$.



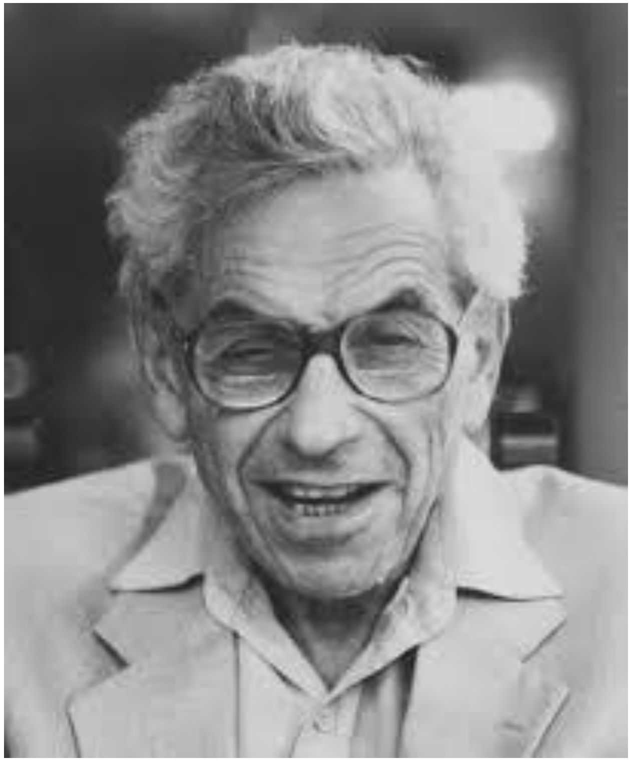
Remark Of course, the corollary follows by showing that the poset obtained by adding a zero to a tree is planar.

A Restatement - With Hindsight

Corollary (Trotter and Moore, '77) If the cover graph of P has tree-width 1, then $\dim(P) \leq 3$.

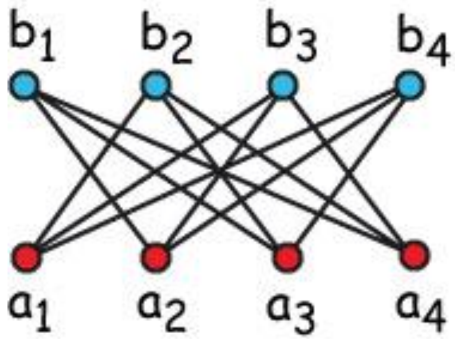


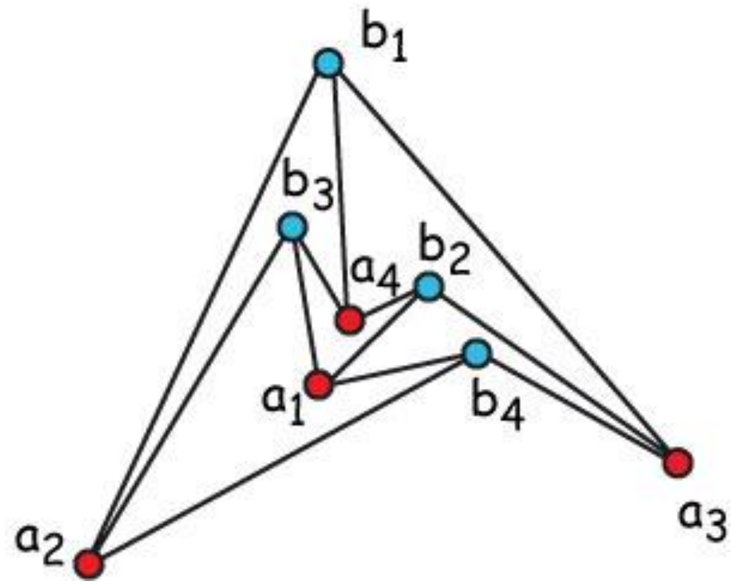
Paul Erdős: Is your Brain Open?



A 4-dimensional planar poset

Fact The standard example S_4 is planar!





Wishful Thinking: If Frogs Had Wings …

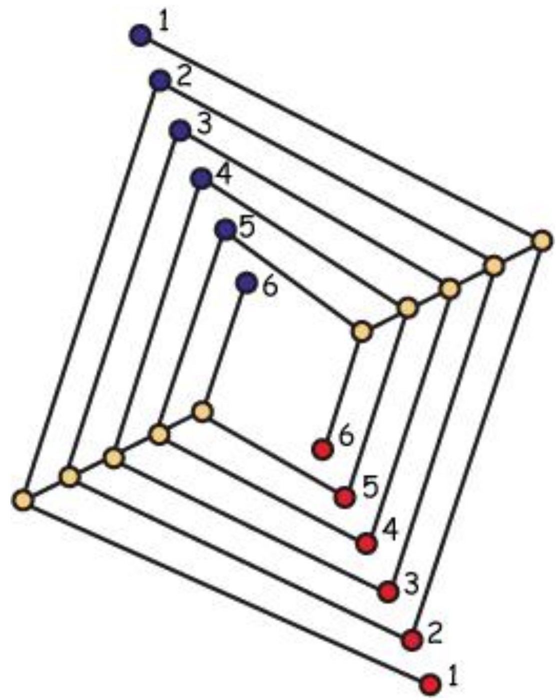
Question Could it possibly be true that $\dim(P) \le 4$ for every planar poset P?

We observe that

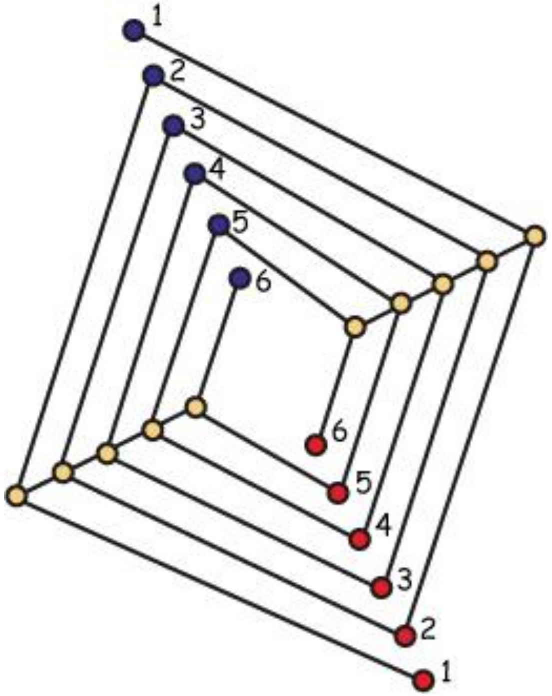
- $\dim(P) \le 2$ when P has a zero and a one.
- $\dim(P) \le 3$ when P has a zero or a one.
- So why not $\dim(P) \le 4$ in the general case?

No … by Kelly's Construction

Theorem (Kelly, '81) For every $n \geq 5$, the standard example S_n is non-planar but it is a subposet of a planar poset.



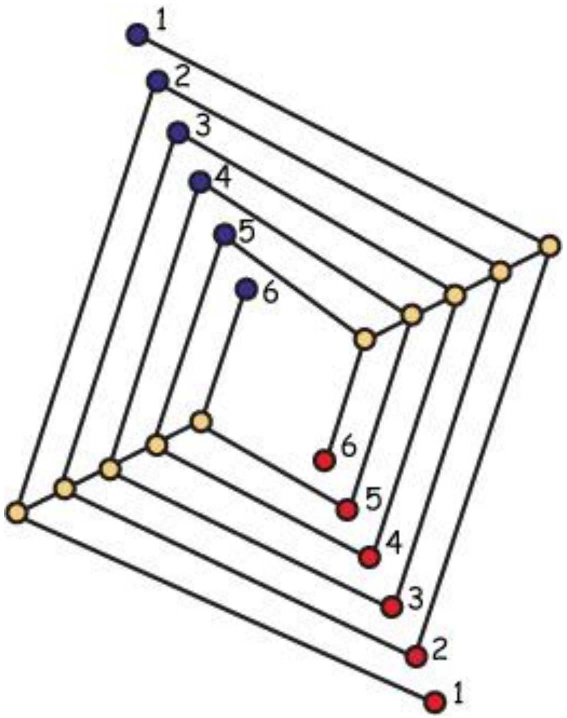
We Should Have Asked … But Didn't



Questions If P is planar and has large dimension, must P contain:

1. A long chain?
2. Many minimal elements?
3. A large standard example?

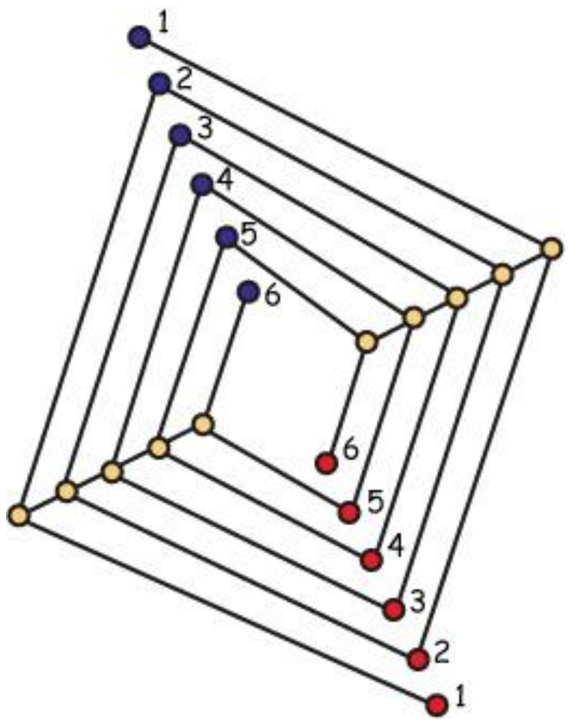
Question 4: Topological Graph Theory



Observations The cover graphs of the posets in Kelly's construction have tree-width at most 3. Is there a connection between the dimension of a poset and the tree-width of its cover graph?

Note In fact, the path-width of these cover graphs is at most 3.

A Brief Promotional Announcement



Observations The cover graphs of the posets in Kelly's construction have tree-width at most 3. Trotter and Moore showed that if the tree-width of the cover graph is 1, then the dimension is bounded. What is the situation when the tree-width is 2?

Remark Stay right here for the next talk by Gwenaël Joret!

Large Height is Necessary

Theorem (Streib and Trotter, '12) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation The proof uses Ramsey theory at several key places and the bound we obtain is **very** large in terms of h.

Tree-width and Dimension

Theorem (Joret, Micek, Milans, Trotter, Walczak, Wang, '14+) The dimension of a poset is bounded in terms of its height and the tree-width of its cover graph. Formally, for every pair (t, h), there is a constant $d = d(t, h)$ so that if P is a poset of height at most h and the tree-width of the cover graph of P is at most t, then $\dim(P) \leq d$.

Note This result was conjectured by Gwenaël Joret.

The Grand Theorem

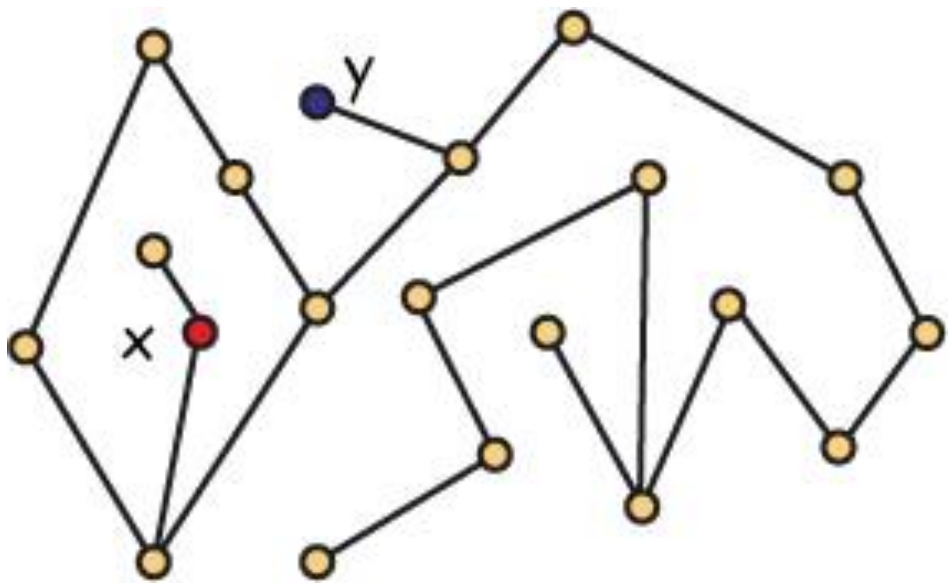
Theorem (Walczak, 14+) Let C be any proper minor closed family of graphs. Then there exists a function $f(C, h)$ so that if P is a poset whose cover graph belongs to C and the height of P is h, then $\dim(P) \leq f(C, h)$.

Must Have Many Minimal Elements

Answer a question posed by R. Stanley, we have been able to prove the following inequality.

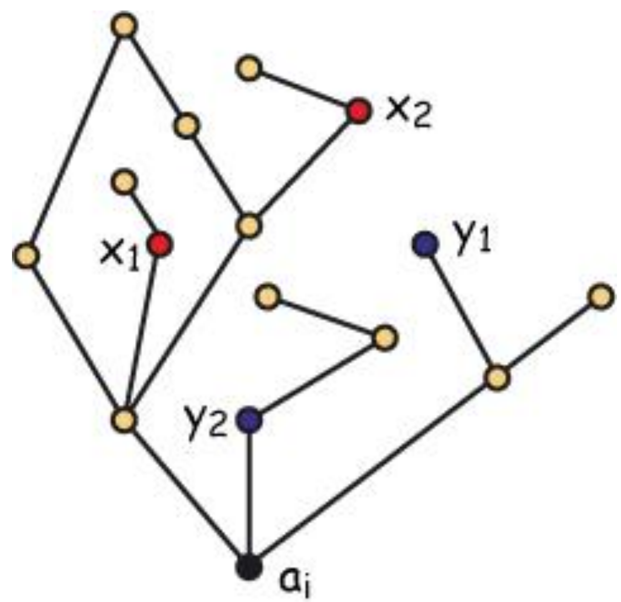
Theorem (Trotter and Wang, '13+) The maximum dimension $m(t)$ of a planar poset with t minimal elements is at most $2t + 1$.

Sketch of the Proof (1)



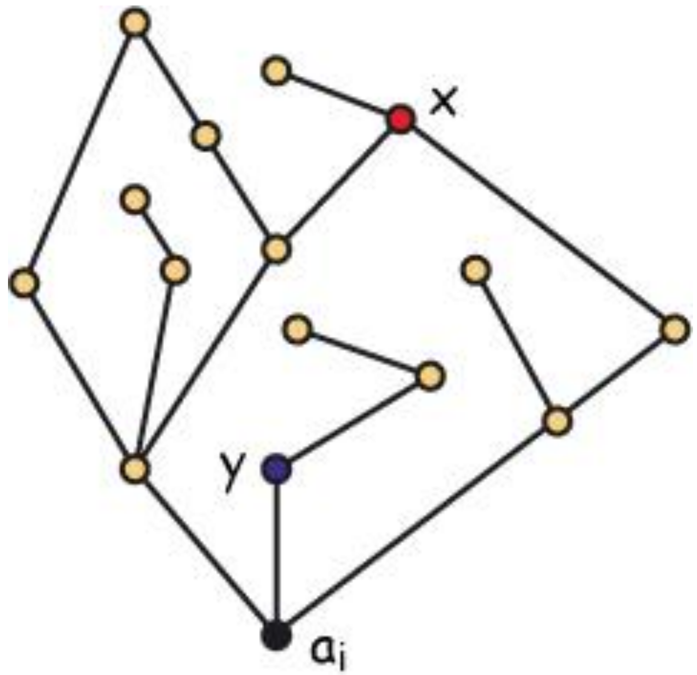
First Step Classify some incomparable pairs as **short**. Here x is short of y because all points z with $z \geq x$ in P are below y in the plane. One linear extension L_0 is used to put x over y whenever x is short of y.

Sketch of the Proof (2)



Second Step For each i, classify non-short incomparable pairs in $U(a_i)$ as **left**, **right** and **over**. Here, x_1 is left of y_1 while x_2 is left of y_2. Right is dual to left.

Sketch of the Proof (2 continued)



Second Step Continued Here x is over y.

Sketch of the Proof (3)

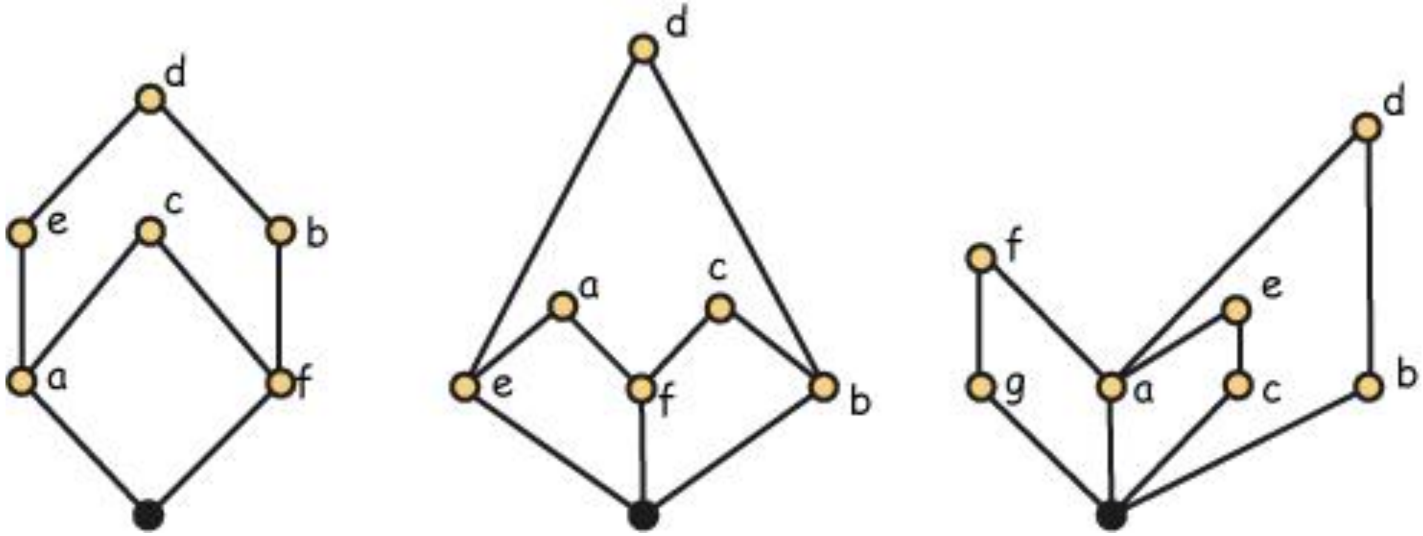
Third Step For each $i = 1, 2, \ldots, t$, there will be two linear extensions L_{2i-1} and L_{2i}.

In L_{2i-1}, we put all elements of $U[a_i]$ over the rest of P. Within $U[a_i]$, we put x over y when x is left of y.

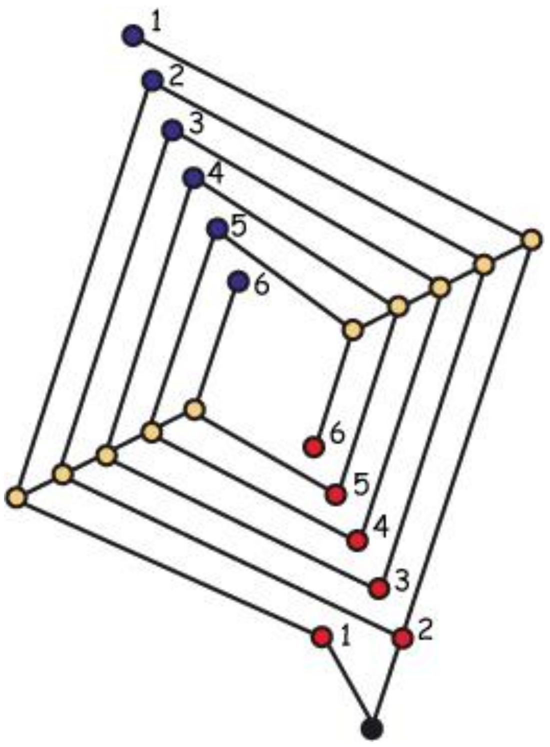
In L_{2i}, within $U[a_i]$, we put x over y when x is right of y or when x is over y.

m(1) = 3

Examples These 3-dimensional posets are planar and remain planar when a zero is added.



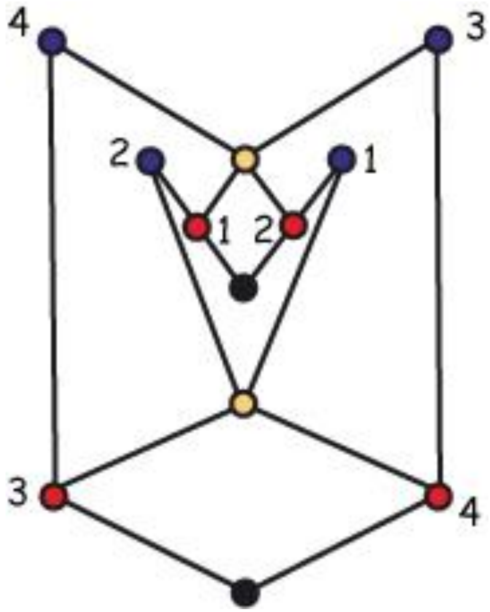
Lower Bound from Kelly's Construction



Note A trivial modification to Kelly's construction shows $m(t) \geq t + 1$, for all $t \geq 2$.

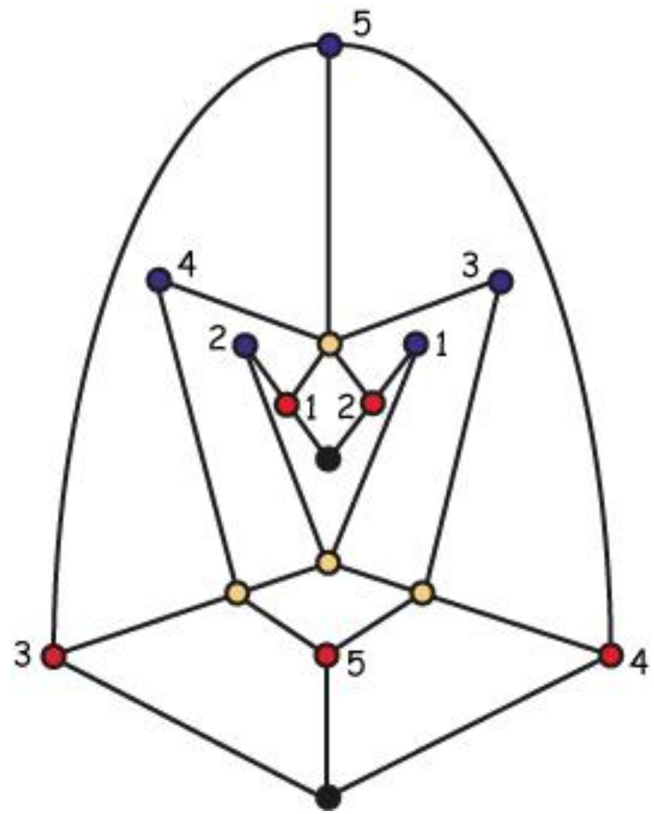
This only implies that $m(2) \geq 3$.

m(2) ≥ 4



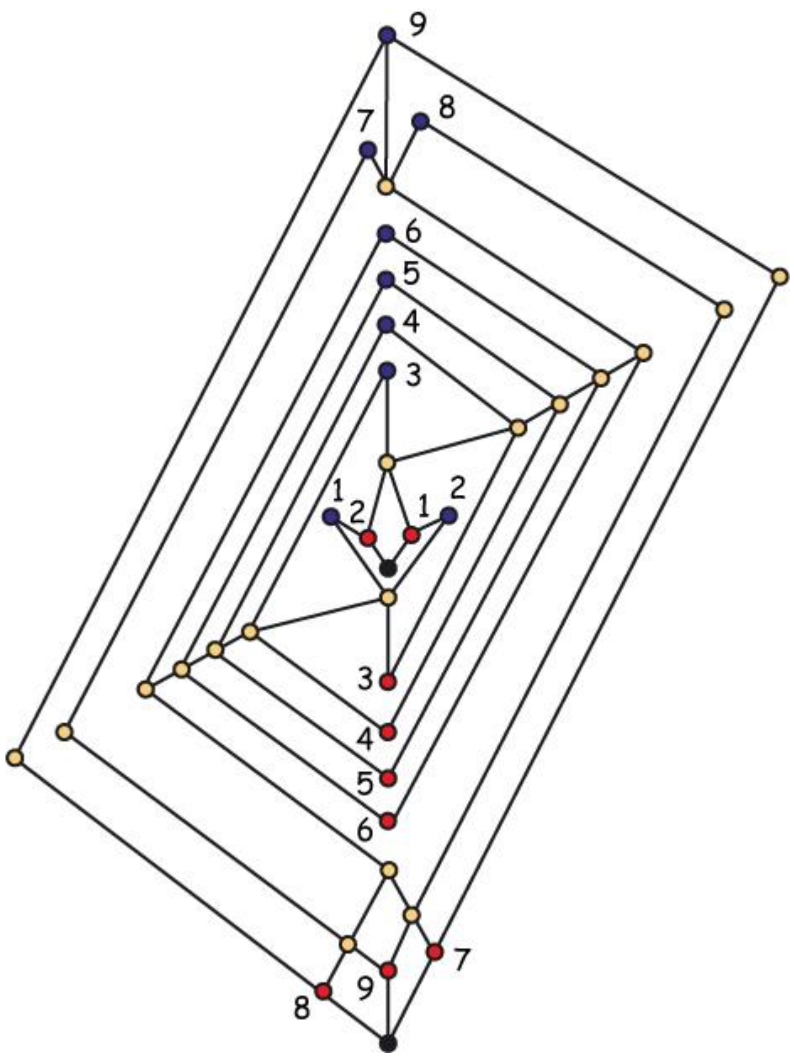
Example This construction shows $m(2) \geq 4$. We already know that $m(2) \leq 5$.

m(2) = 5



Example This construction shows m(2) = 5.

An Improved General Lower Bound



Example The construction shows

$$m(t) \geq t + 3,$$

for all $t \geq 2$.

Note This inequality is tight for $t = 2$. But for $t = 3$, we only know

$$6 \leq m(3) \leq 7.$$

Open Problems

Problem 1 For an integer $t \geq 3$, what is the maximum dimension $m(t)$ of a planar poset with t minimal elements. We know $t + 3 \leq m(t) \leq 2t + 1$.

Problem 2 We conjecture that for every $n \geq 2$, there is an integer d_n so that if P is a poset with a planar cover graph and $\dim(P) \geq d_n$, then P contains the standard example S_n.

Remarks on the Open Problems

Remark We can show that there is an integer d_2 so that if P is a poset with a planar cover graph and $\dim(P) \geq d_2$, then P is not an interval order, i.e., P contains the standard example S_2.

A Second Promotional Announcement

Theorem (Trotter and Wang, 14+) If $\dim(P) = d \geq 3$, there is a matching of size d in the comparability graph of P.

Theorem (Trotter and Wang, 14+) If $\dim(P) = d \geq 3$, there is a matching of size d in the incomparability graph of P.

Corollary (Hiraguchi, '51) If P is a poset on n points and $n \geq 4$, then $\dim(P) \leq n/2$.

Kleitman's Rule

General Counsel Never solve a difficult problem completely. If you do and write a 100+ page paper, it will be read by at most 10 people. Instead, make a substantive advancement on an interesting problem, one that opens up two new problems and present your work in a nifty paper of at most 10 pages. Then hundreds of researchers will read your paper and you will have major impact on the field.

Disclaimer Words spoken with tongue firmly in cheek. However, our paper is 9 pages in length.