Some Recent Results about Cross Intersecting Families

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This talk is based on joint work with

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1. Set up and Weighted Erdős–Ko–Rado

- $[n] := \{1, 2, \ldots, n\}, \quad p \in (0, 1)$.
- A family of subsets $\mathcal{A} \subset 2^{[n]}$.
- $\mathcal{A}$ is t-intersecting if $|A \cap A'| \geq t$ for all $A, A' \in \mathcal{A}$.

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- The p-weight (or product measure) of $\mathcal{A}$ is

$$\mu_p(\mathcal{A}) := \sum_{A \in \mathcal{A}} p^{|A|}(1-p)^{n-|A|}.$$

- Ex. $\mathcal{F}_0 := \{A \subset [n] : [t] \subset A\}$ is a t-intersecting family with $\mu_p(\mathcal{F}_0) = p^t$.

- Another example of t-intersecting family:

$$\mathcal{F}_1 := \{F \subset [n] : |F \cap [t+2]| \geq t+1\}.$$

It follows that

$$\mu_p(\mathcal{F}_1) = (t+2)p^{t+1}q + p^{t+2}$$

and

$$\mu_p(\mathcal{F}_0) \geq \mu_p(\mathcal{F}_1) \text{ iff } p \leq \frac{1}{t+1}.$$

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- $\mu_p(\mathcal{F}_i) \geq \mu_p(\mathcal{F}_{i+1})$ iff $p \leq \frac{i+1}{t+2i+1}$.

Theorem (Ahlswede–Khachatrian, Bey–Engel, Dinur–Safra, T)

Let $\mathcal{A} \subset 2^{[n]}$ be t-intersecting. Then

$$\mu_p(\mathcal{A}) \leq \max_i \mu_p(\mathcal{F}_i).$$

Corollary

Let $\mathcal{A} \subset 2^{[n]}$ be t-intersecting.

1. If $p \leq \frac{1}{t+1}$ then
$$\mu_p(\mathcal{A}) \leq \mu_p(\mathcal{F}_0) = p^t.$$
2. If $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$ then
$$\mu_p(\mathcal{A}) \leq \mu_p(\mathcal{F}_1).$$

2. Extension to cross intersecting families

- Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$.
- $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting if $|A \cap B| \geq t$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

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Theorem (Frankl–Lee–Siggers–T)

Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ be cross t-intersecting.

1. If $t \geq 14$ and $p \leq \frac{1}{t+1}$ then (arXiv 1303.0657)

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq (\mu_p(\mathcal{F}_0))^2 = p^{2t}.$$

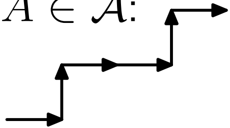
2. If $t \geq 52$ and $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$ then

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq (\mu_p(\mathcal{F}_1))^2.$$

Some ideas in the proof

- Let $\mathcal{A}$ and $\mathcal{B}$ be cross t-intersecting.
- Assign a walk in $\mathbb{Z}^2$ to each $A \in \mathcal{A}$:

$$n = 6,\ A = \{2, 5\} \Longleftrightarrow$$



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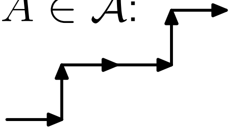

- We may assume that $\mathcal{A}, \mathcal{B}$ are shifted.
- Key fact: there are a, b such that
  1. all walks in $\mathcal{A}$ hit $y = x + a$,
  2. all walks in $\mathcal{B}$ hit $y = x + b$,
  3. $a + b \geq 2t$.

Some ideas in the proof (continued)

- Consider the infinite random walk on $\mathbb{Z}^2$ where i-th step is "$\uparrow$" with probability p, and "$\rightarrow$" with probability $1-p$.
- $\mu_p(\mathcal{A})$ is bounded as follows:

$$\begin{aligned}
&\mu_p(\mathcal{A})\\
&\quad \le \Pr\,(\text{the random walk hits } y = x + a)\\
&\quad = \left(\frac{p}{1-p}\right)^a.
\end{aligned}$$

3. Different measures and algebraic approach

- Let G be a bi-regular bipartite graph with $V(G) = V_1 \cup V_2$.
- $U_1 \subset V_1$ and $U_2 \subset V_2$ are cross independent if $uv \notin E(G)$ for all $u \in V_1$, $v \in V_2$.
- For $i = 1, 2$ let $\tilde{\mu}_i$ be a general measure:

$$\tilde{\mu}_i : V_i \to [0, 1] \text{ and } \sum_{v \in V_i} \tilde{\mu}_i(v) = 1.$$

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- (Key fact): Let $\sigma_1 \geq \sigma_2 \geq \cdots$ be singular values of a bip. adjacency matrix of G. Then

$$\sqrt{\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2)} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

Recall $\mu_p(\mathcal{A}) := \sum_{A\in\mathcal{A}} p^{|A|}(1-p)^{n-|A|}$.
Let $p_1, p_2 \in (0,1)$ and let $q_i := 1 - p_i$ $(i = 1, 2)$.

Recall $\mu_p(\mathcal{A}) := \sum_{A\in\mathcal{A}} p^{|A|}(1-p)^{n-|A|}$.
Let $p_1, p_2 \in (0,1)$ and let $q_i := 1 - p_i$ $(i = 1,2)$.

Theorem

If $(p_1p_2)/(q_1q_2) < \left(\sqrt[t]{2}-1\right)^2$ $\cdots\cdots$ $(*)$,
and $\mathcal{A},\mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then

$$\sqrt{\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B})} \le \left(\frac{\sqrt{p_1p_2}}{\sqrt{p_1p_2}+\sqrt{q_1q_2}}\right)^t.$$

If $p_1 = p_2$, then the bound is sharp.
If $p_1, p_2 < \frac{\log 2}{t+1} < \frac{1}{t+1}$, then $(*)$ is satisfied.

For the case $t = 1$ we get the exact bound:

Theorem (Suda–Tanaka–T)

Let $p_1, p_2 \in (0, 1/2]$. $\qquad (1/2 = \frac{1}{t+1})$

If $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross 1-intersecting, then

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B}) \leq p_1 p_2.$$

The proof is done by solving a corresponding SDP problem. In fact we got a refined bipartite ratio bound based on SDP.

Our setup

Let G be a bi-regular bipartite graph with $V(G) = V_1 \cup V_2$ and $\tilde{\mu}_i : V_i \to [0,1]$ $(i = 1,2)$.
Let A be a bipartite adjacency matrix of G.
Suppose $U_1 \subset V_1$ and $U_2 \subset V_2$ are cross indep.

Easy ratio bound (reprise)

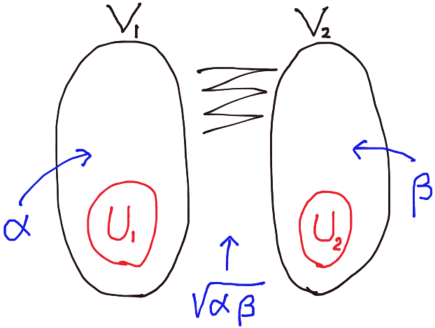
Let $\sigma_1 \geq \sigma_2 \geq \cdots$ be the singular values of A.
Then

$$\sqrt{\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2)} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

New ratio bound (idea)

If A has singular values $\sqrt{\alpha_1\beta_1} \geq \sqrt{\alpha_2\beta_2} \geq \cdots$ with some extra properties, then

$$\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2) \leq \frac{\alpha_1}{\alpha_1+\alpha_2}\,\frac{\beta_1}{\beta_1+\beta_2}.$$



New ratio bound (still oversimplified)

If there are nonsingular matrices P_1, P_2 and a nonnegative symmetric matrix A_1 such that

- $P_1^\mathsf{T} A P_2 = \bigoplus (-1)^s \sqrt{\alpha_s \beta_s}\, I_{m_s}$,
- $P_1^\mathsf{T} A_1 P_1 = \bigoplus (-1)^s \alpha_s\, I_{m_s}$,
- α_s and β_s satisfy some inequalities.

Then

$$\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2) \leq \frac{\alpha_1}{\alpha_1 + \alpha_2}\, \frac{\beta_1}{\beta_1 + \beta_2}.$$

This new ratio bound can be applied to the following type of cross 1-intersecting EKR:

- weighted subsets ($p_i \leq 1/2$)

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B}) \leq p_1 p_2,$$

- uniform subsets ($n \geq 2k_i$)

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1},$$

- subspaces ($n \geq 2k_i$) (Suda–Tanaka 2013)

$$|\mathcal{A}||\mathcal{B}| \leq \begin{bmatrix} n-1 \\ k_1-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k_2-1 \end{bmatrix}.$$

Conjectures

Conjecture 1

Let $p_1, p_2 \leq \frac{1}{t+1}$. If $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B}) \leq (p_1 p_2)^t.$$

True if

- $t = 1$,
- $t \geq 14$ and $p_1 = p_2$,
- $p_1 = p_2 \leq \frac{\log 2}{t+1}$.

$\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ are 3-cross intersecting if

$$A \cap B \cap C \neq \emptyset$$

for all $A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C}$.

Conjecture 2

Let $\mathcal{A} \subset \binom{[n]}{k_1}, \mathcal{B} \subset \binom{[n]}{k_2}, \mathcal{C} \subset \binom{[n]}{k_3}$ be 3-cross intersecting, and $2n \geq 3k_i$. Then

$$|\mathcal{A}||\mathcal{B}||\mathcal{C}| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}\binom{n-1}{k_3-1}.$$

True if $k_1 = k_2 = k_3$.

Conjecture 2 would imply

Conjecture 3

Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ be 3-cross intersecting, and $p_1, p_2, p_3 \leq 2/3$. Then

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B})\mu_{p_3}(\mathcal{C}) \leq p_1 p_2 p_3.$$

Not known even if $p_1 = p_2 = p_3$.