Forbidden Structures in the Boolean Lattice

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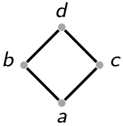

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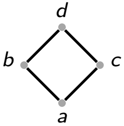

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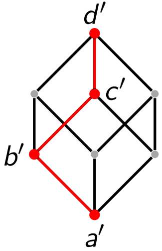

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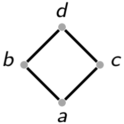

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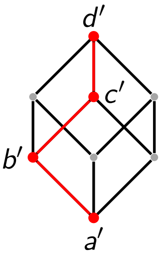

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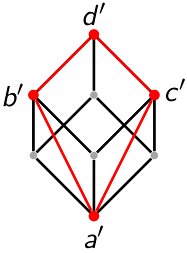

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Theorem (Sperner (1928); Erdős (1945))

$\mathrm{La}^*(n, K_r)$ *equals the sum of the* $r-1$ *largest binomial coefficients in* $\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\}$. *For fixed* r *and* $n \to \infty$,

$$\mathrm{La}^*(n, K_r) = (r - 1 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

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Main Conjecture

For fixed P, we have $\mathrm{La}^*(n, P) = O(2^n/\sqrt{n})$.

The Turán Threshold

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- Cor: $\pi^*(2^{[3]}) \le 24$.

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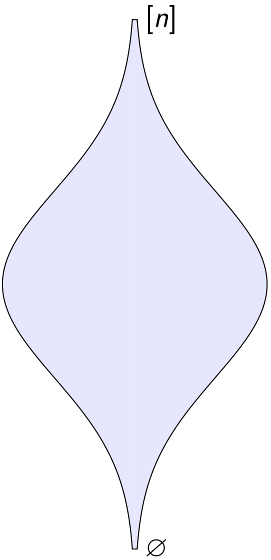

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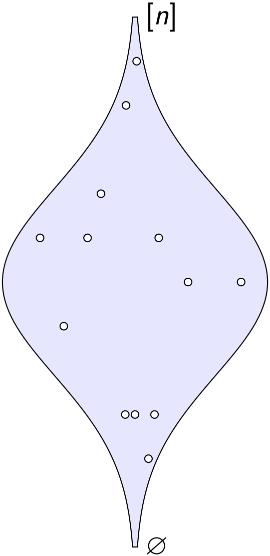

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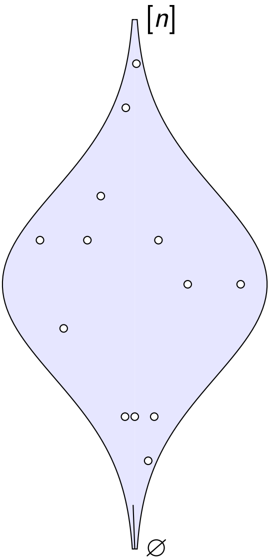

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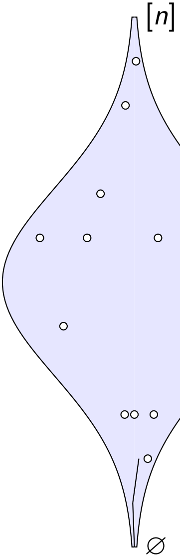

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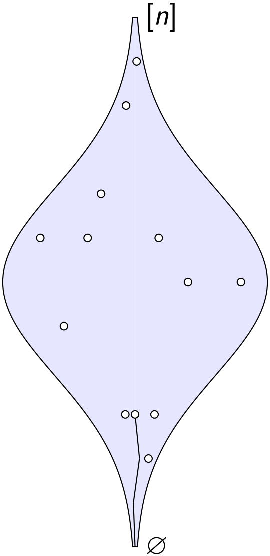

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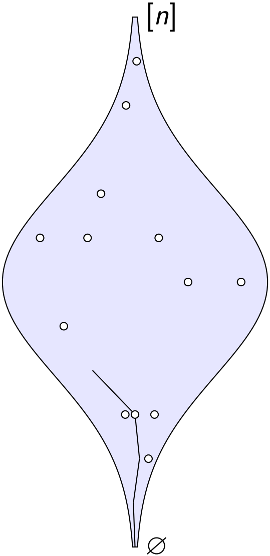

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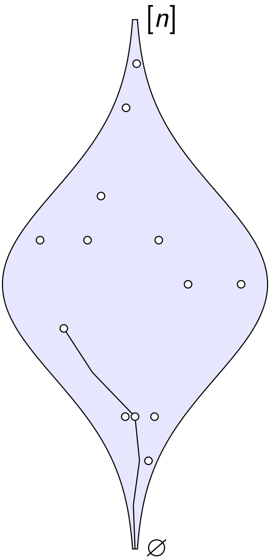

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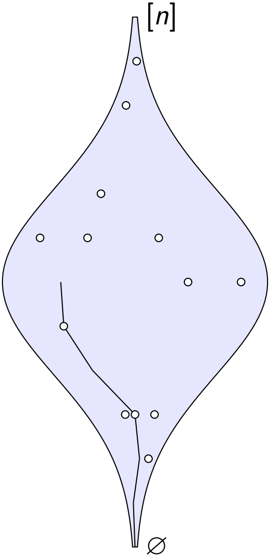

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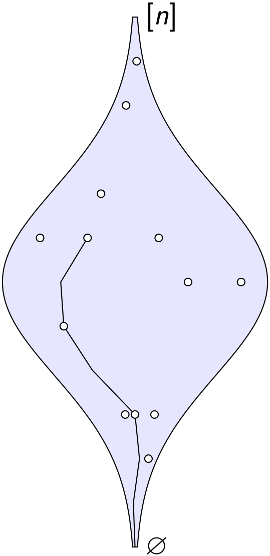

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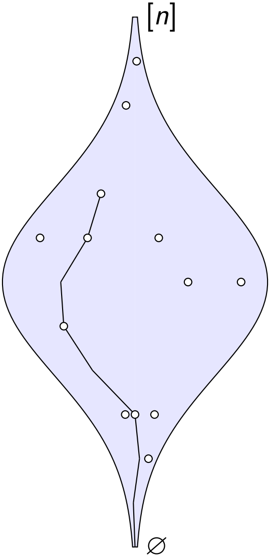

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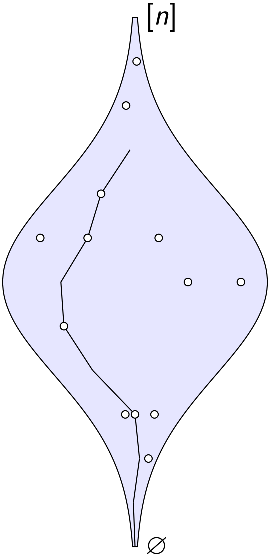

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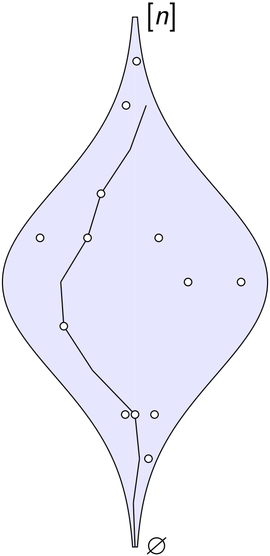

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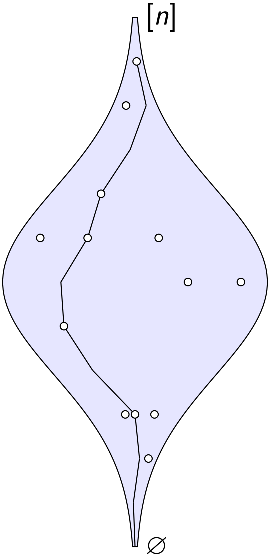

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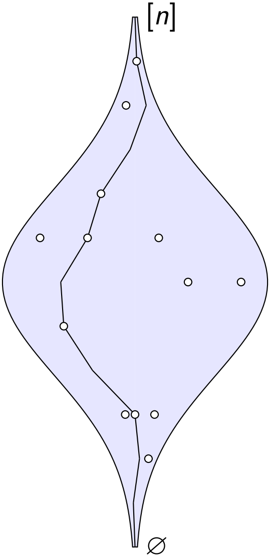

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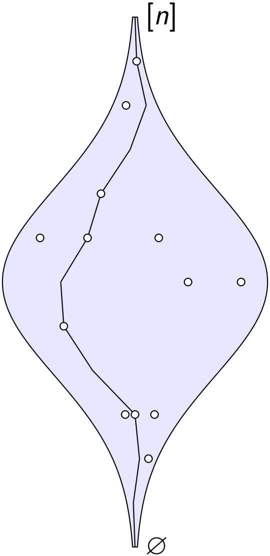

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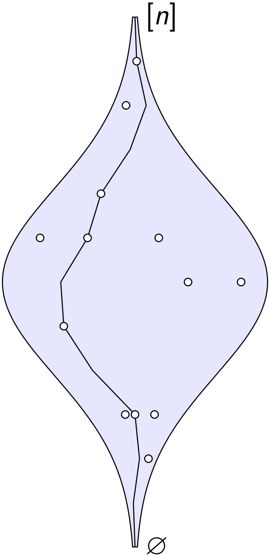

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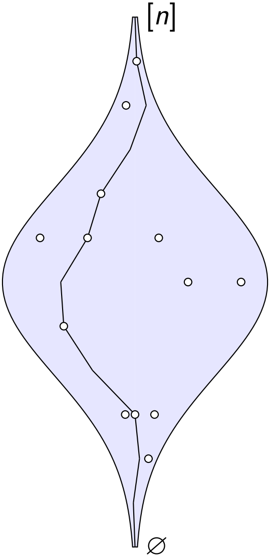

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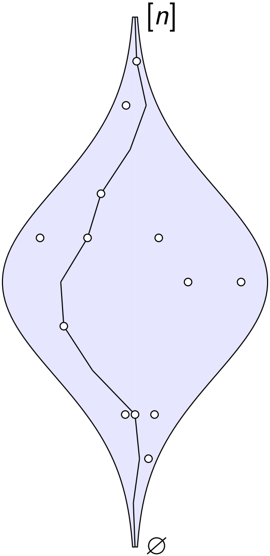

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- Think of $\ell_n(\mathcal{F})$ as a measure of the size of $\mathcal{F}$, with $0 \leq \ell_n(\mathcal{F}) \leq n+1$.
- For $A \subseteq B$, we define $\ell(\mathcal{F}; [A, B])$ to be the expected number of times a random, full chain from A to B meets $\mathcal{F}$.

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Conjecture

Always $\lambda^*(P)$ is finite.

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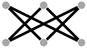

$$r - 2 \leq \lambda^*(\mathcal{S}_r) \leq 2r + O(\sqrt{r})$$

The Standard Example: Proof Sketch

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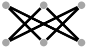

$$r - 2 \leq \lambda^*(\mathcal{S}_r) \leq 2r + O(\sqrt{r})$$

Proof (sketch).

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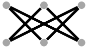

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Let $F_0 \subseteq 2^{[n]}$ with $\ell(\mathcal{F}_0) \geq 2r + \Theta(\sqrt{r})$.

1. Let $\mathcal{F}_1 = \{A \in \mathcal{F}_0 \colon\ |A| \leq n/2\}$. By self-duality, we may assume $\ell(\mathcal{F}_1) \geq \frac{1}{2}\ell(\mathcal{F}_0) \geq r + \Theta(\sqrt{r})$.

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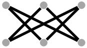

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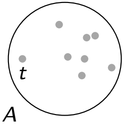

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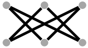

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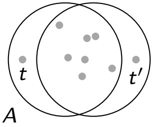

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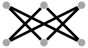

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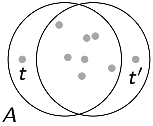

A set $A \in \mathcal{F}$ is γ-flexible if it has at least $\gamma|A|$ pivots. $\square$

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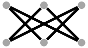

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3. Form $\mathcal{F}_2$ from $\mathcal{F}_1$ by throwing away all sets that are not $(1 - \frac{1}{\sqrt{r}})$-flexible. We have $\ell(\mathcal{F}_2) \geq \ell(\mathcal{F}_1) - O(\sqrt{r})$.

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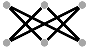

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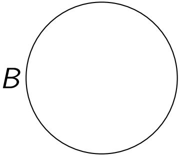

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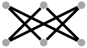

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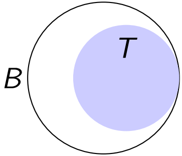

Let T be the set of pivots in B.

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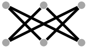

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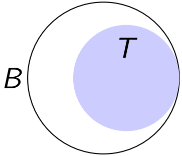

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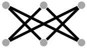

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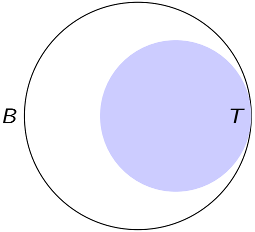
5. Since $\ell(\mathcal{F}_2; [\varnothing, B])$ is large and $|T| \geq \gamma |B|$, we find sets $A_1, \ldots, A_r \in \mathcal{F}_2$ and elements $t_1, \ldots, t_r \in T$ such that



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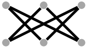

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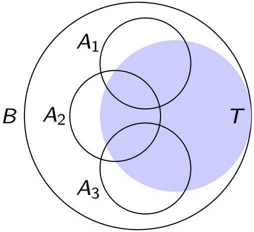

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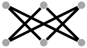

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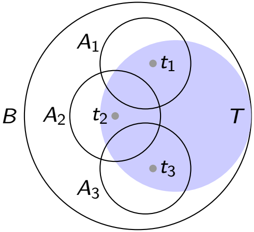

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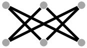

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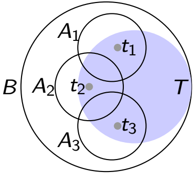

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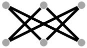

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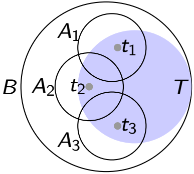

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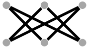

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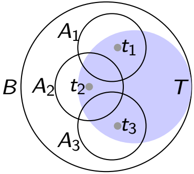

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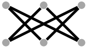

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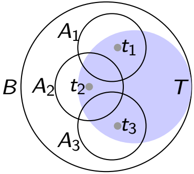

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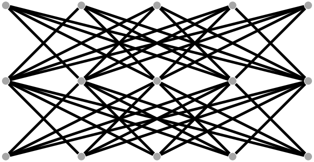
9. $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$ form a copy of $\mathcal{S}_r$. $\square$

Toward Height 3

Definition

The generalized standard example of width r and height h, denoted $\mathcal{S}_{r,h}$, has h disjoint r-antichains $A_1, \ldots, A_h$ where $A_i \cup A_{i+1}$ is a copy of $\mathcal{S}_r$.



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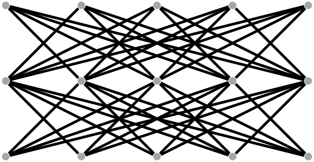

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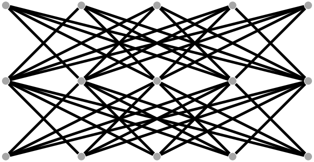

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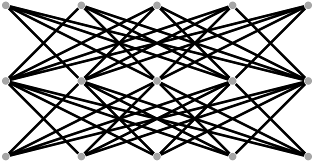

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- Proof involves about 10 cleaning steps.
- Not clear how to extend to other posets of height 3 or $\mathcal{S}_{r,h}$.

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- $\lambda^*(2^{[3]}) \leq 2 + \lambda^*(\mathcal{S}_3) \leq 24$.
- Open problem: show that $\lambda^*(2^{[4]})$ or $\pi^*(2^{[4]})$ is finite.

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Fact

There exists $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F}$ is $\mathcal{B}_2$-free and $|\mathcal{F}| \geq \Omega(n^{\frac{1}{4}} \cdot \frac{2^n}{\sqrt{n}})$.

The Turán Problem for Boolean Algebras

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- [Erdős–Kleitman 1971] For some constants c_1, c_2 and n sufficiently large

$$c_1 \cdot n^{-1/4} \cdot 2^n \leq b(n,2) \leq c_2 \cdot n^{-1/4} \cdot 2^n.$$

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- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}.$

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For each d, there exists c_d such that for n sufficiently large

$$n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \cdot 2^n \leq b(n,d) \leq c_d \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

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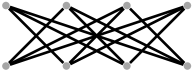
Theorem

$$b(n,d) \leq 50 \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

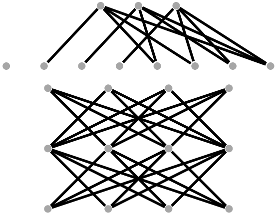
Summary & Open Problems

Theorems



$$r - 2 \leq \lambda^*(\mathcal{S}_r) \leq 2r + O(\sqrt{r})$$

If P has height 2, then $\lambda^*(P) \leq O(|P|)$.



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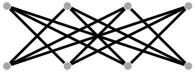

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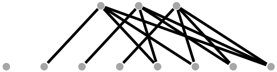

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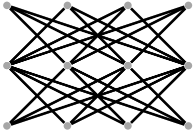

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Problems

- Conjecture: Always $\lambda^*(P)$ is finite.
- Show that $\lambda^*(P)$ is finite when P has height 3.
- Find a bound $\lambda^*(2^{[4]})$.

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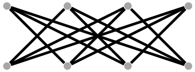

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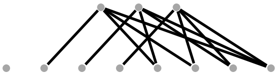

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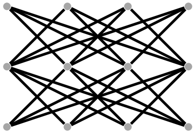

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Thank You.