Extremal Problems on Posets and Hypergraphs

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Travis Johnston

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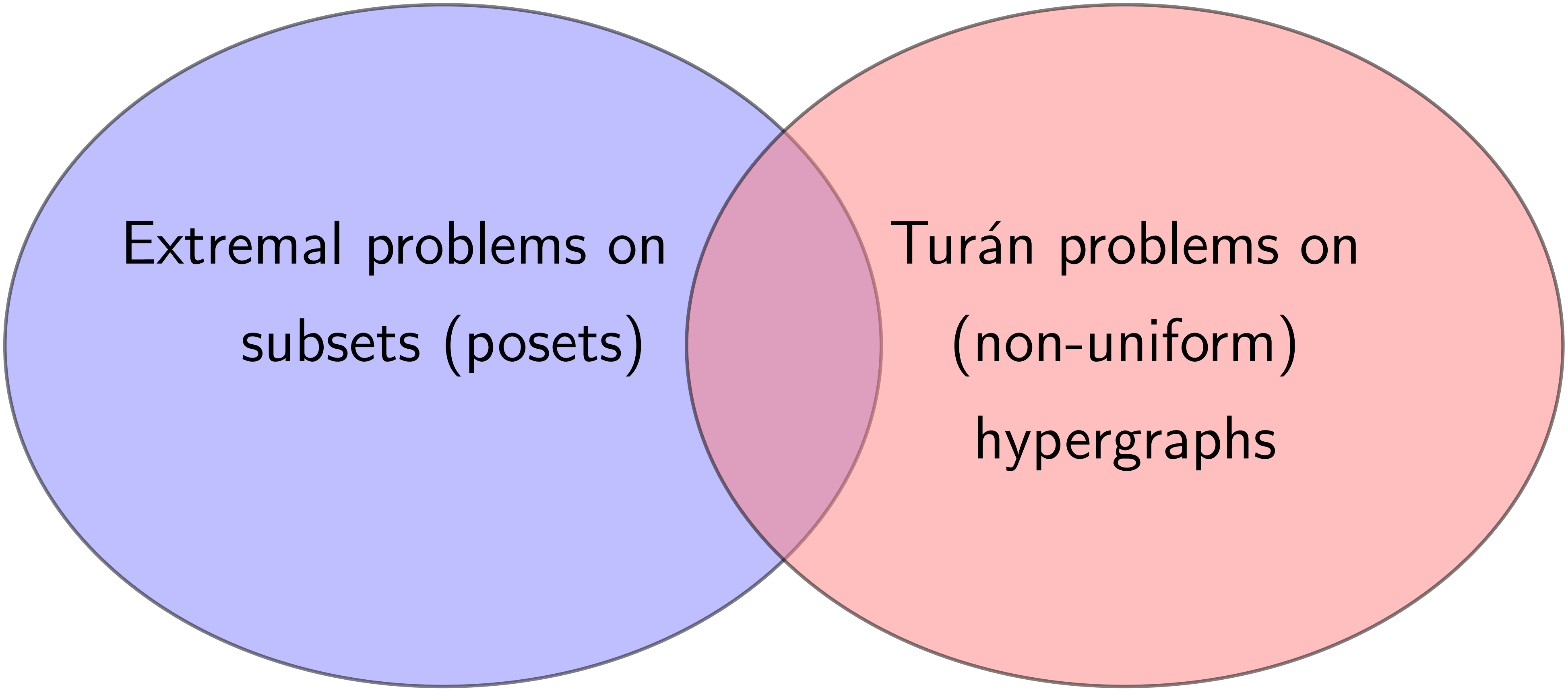
Overview


Extremal problems on subsets (posets)
Turán problems on (non-uniform) hypergraphs

Part I: Posets

Notation:

- $[n]$: the set of first n positive integers.
- $\mathcal{F}$: a family of subsets of $[n]$.

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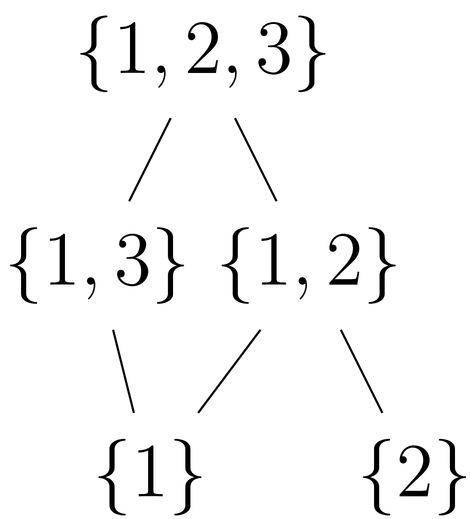
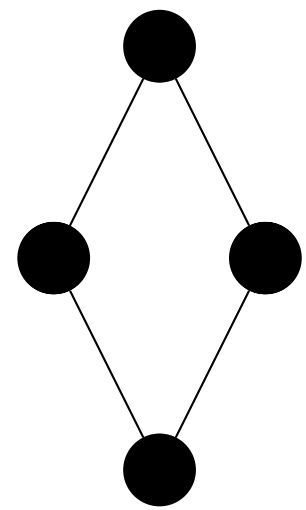
Example: $\mathcal{F} = \{\{\{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}\}$



The inclusion relations of $\mathcal{F}$.



A diamond pattern

Sperner theorem

Sperner theorem [1928]: Let $\mathcal{F}$ be an inclusion-free family of subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

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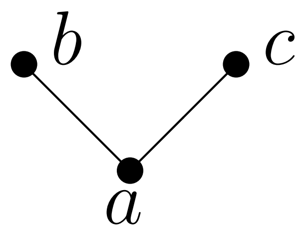
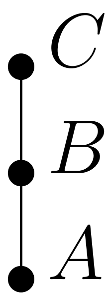
LYM $\Rightarrow$ Sperner theorem

Posets

A **poset** is a set P equipped with a partial-order $\leq$. A poset $(P, \leq)$ can be represented by Hasse Diagram.



Poset V



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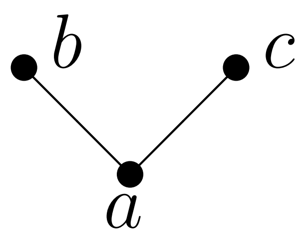

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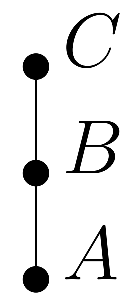

Chain P_3

A poset $P_1 = (P_1, \leq_1)$ is a *(weak) subposet* of a poset $P_2 = (P_2, \leq 2)$ if there exists an injection f from P_1 to P_2 such that $f(a) \leq_2 f(b)$ whenever $a \leq_1 b$.

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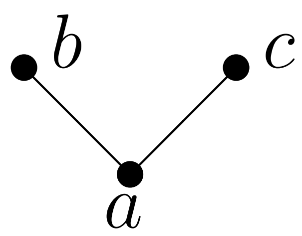

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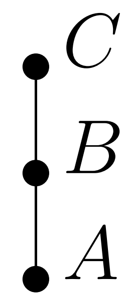

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The poset V is a (weak) sub-poset of the Chain P_3.

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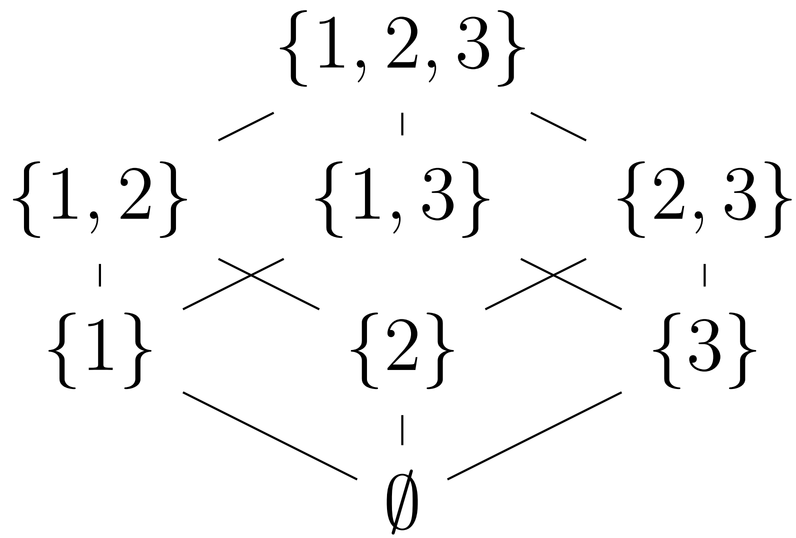

Chain: $\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}$.
Anti-chain: $\{1,2\}, \{1,3\}, \{2,3\}$.

$\mathrm{La}(n, P)$

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$$\mathrm{La}(n, P) = \max\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \text{ contains no subposet } P\}.$$

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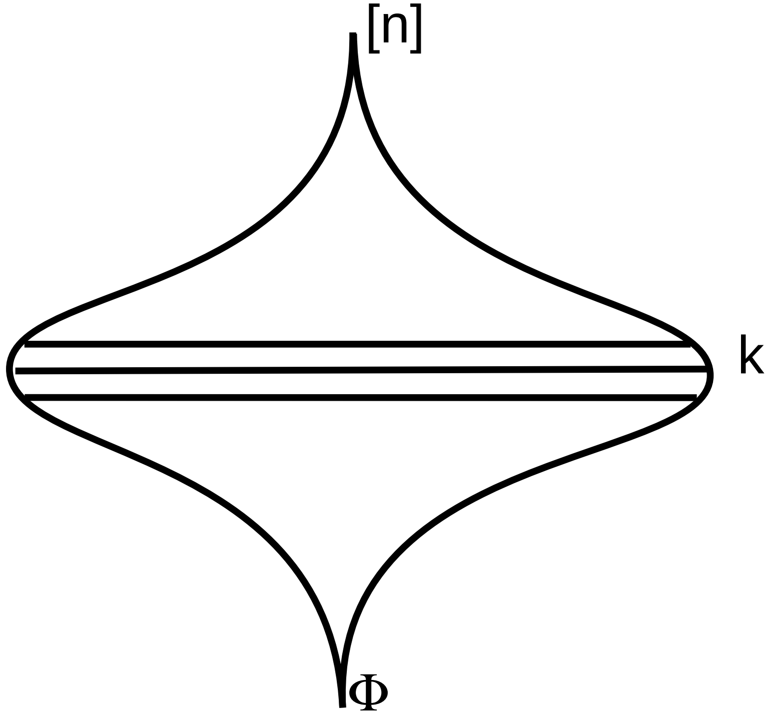

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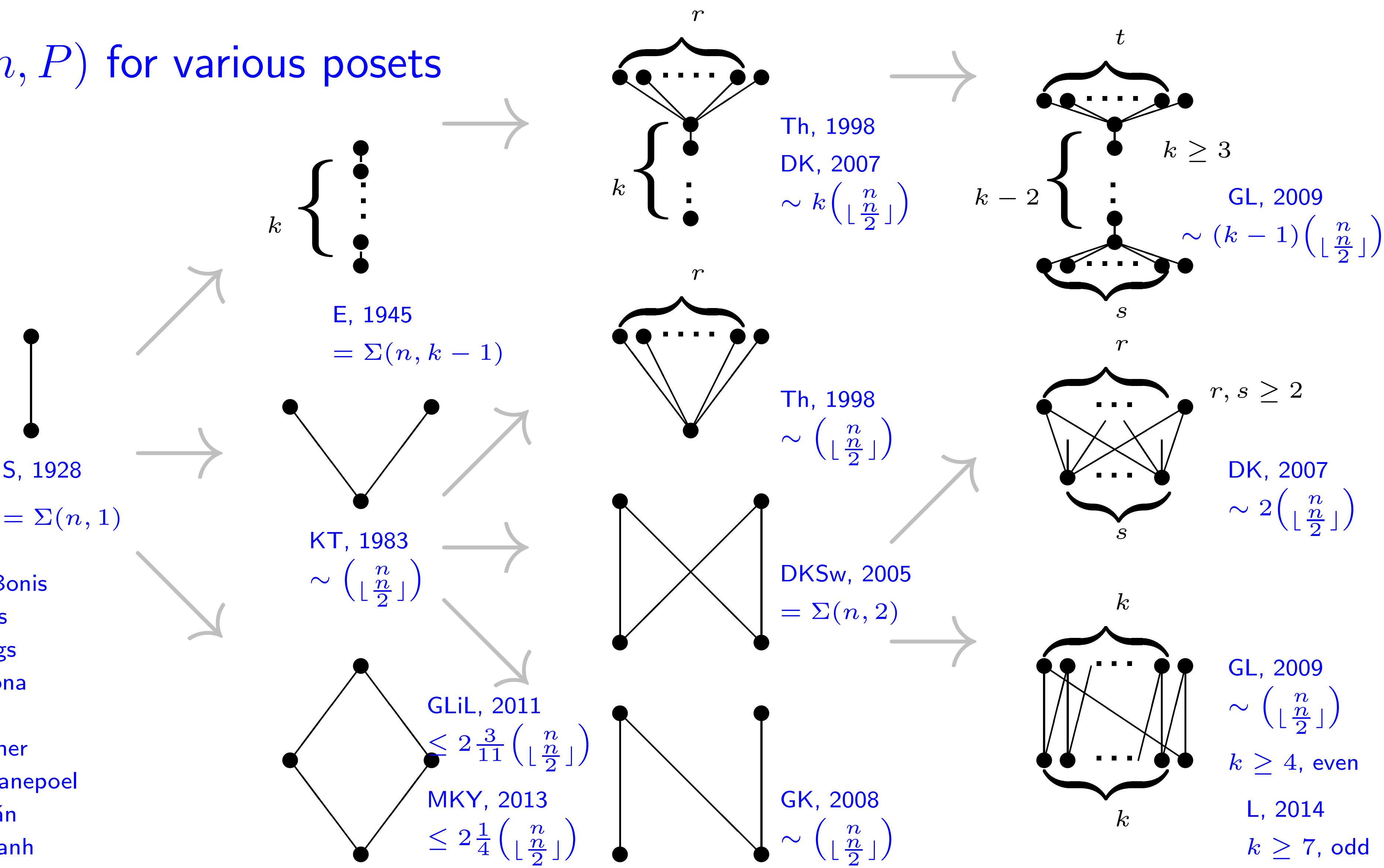
Erdős [1945]:
$\mathrm{La}(n, P_k) = \Sigma(n, k-1)$
$= (k - 1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$

Summary of Results

$\mathrm{La}(n, P)$ for various posets



D: De Bonis
E: Erdős
G: Griggs
K: Katona
L: Lu
S: Sperner
Sw: Swanepoel
T: Tarján
Th: Thanh
MKY: Martin-Kramer-Young

Courtesy of Wei-Tian Li

A conjecture

Conjecture [Griggs-Lu, 2009]:

The limit $\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and is an integer.

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Given a poset P, let $e(P)$ be the maximum integer m such that for all n, $\mathcal{B}(n,m)$ does not contain P as a subposet.
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What about we restrict $\mathcal{F}$ to $e(P) + 1$ consecutive levels?

Lubell function

The Lubell function $h_n\colon 2^{2^{[n]}} \to \mathbb{R}$ defined as

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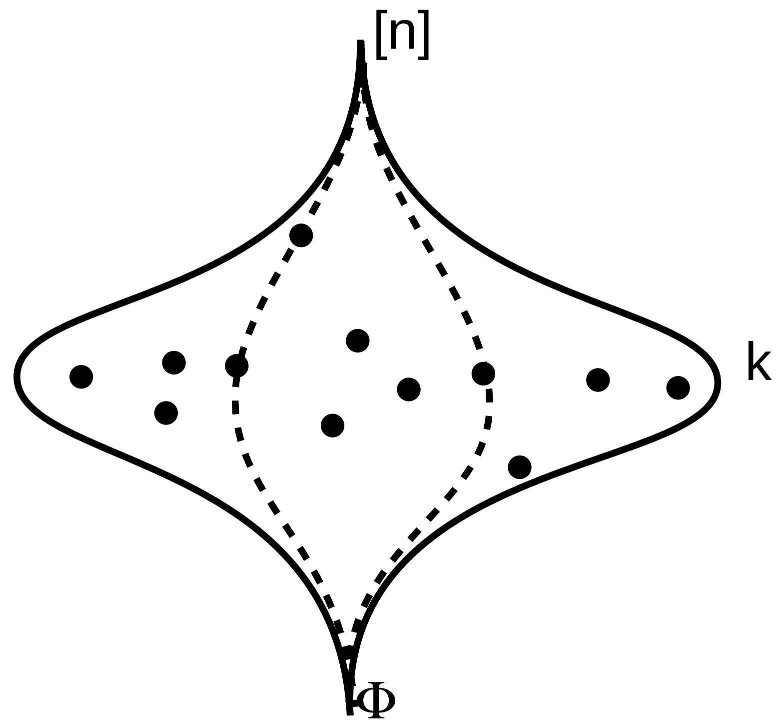
Let X be the number of elements in $\mathcal{F}$ hit by a random full chain. Then

$$h_n(\mathcal{F}) = \mathrm{E}(X).$$



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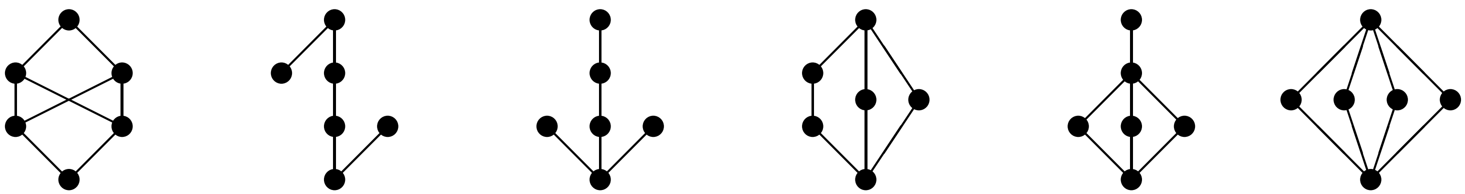

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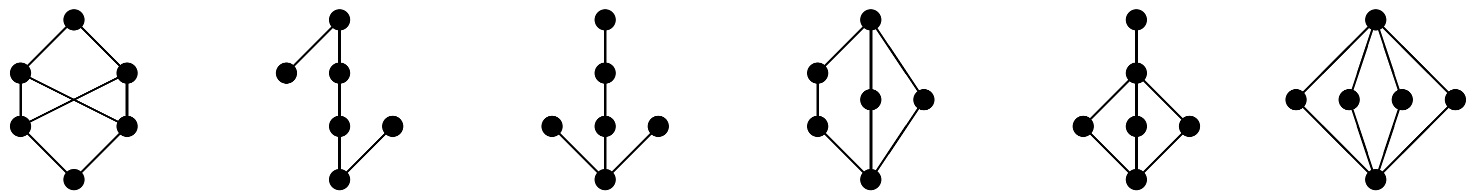

Lemma [Griggs, Li, and Lu, 2011]: If P is a uniformly L-bounded poset P, then the maximum P-free family must be $\mathcal{B}(n, e(P))$. In particular,

$$\mathrm{La}(n, P) = \Sigma(n, e(P)) \text{ for all } n.$$

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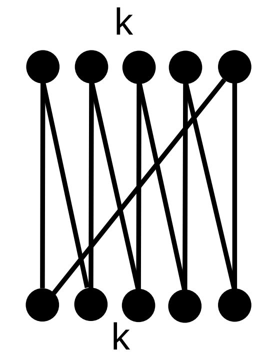

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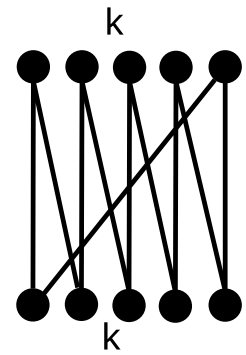

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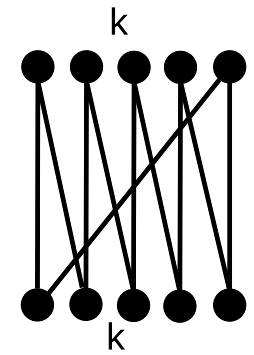

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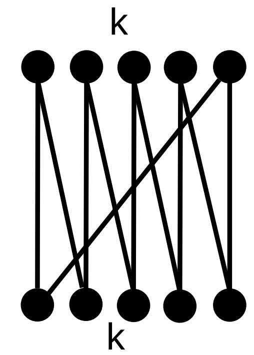

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- **Lu [2014]:** For odd $t \geq 7$, $\pi(\mathcal{O}_{2t}) = 1$.

$\pi(\mathcal{O}_6)$ and $\pi(\mathcal{O}_{10})$ are still open.

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If $\mathcal{F}$ is contained in three consecutive layers, then the upper bound can be further improved:

- **Axenovich-Manske-Martin [2009]:** $\pi^*(\mathcal{D}_2) \leq 2.2071$.
- **Manske-Shen [2012]:** $\pi^*(\mathcal{D}_2) \leq 2.1547$.
- **Balogh-Hu-Lidický-Liu [2014]:** $\pi^*(\mathcal{D}_2) \leq 2.15121$.

A weaker conjecture

A consective-level version: For any poset P, let

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- Is $\pi_c(P) = e(P)$?

Part II: hypergraphs

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Example: $H = (V, E)$ where

- $V = \{1, 2, 3\}$
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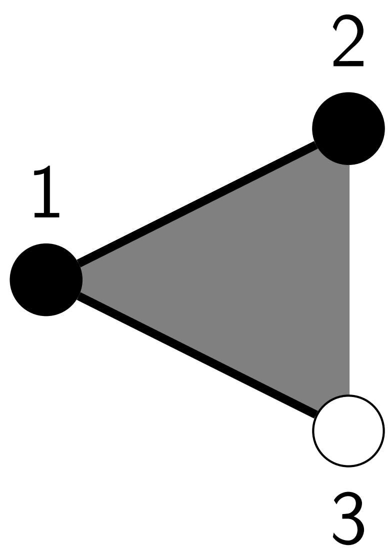

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- should be useful for problems in other areas.

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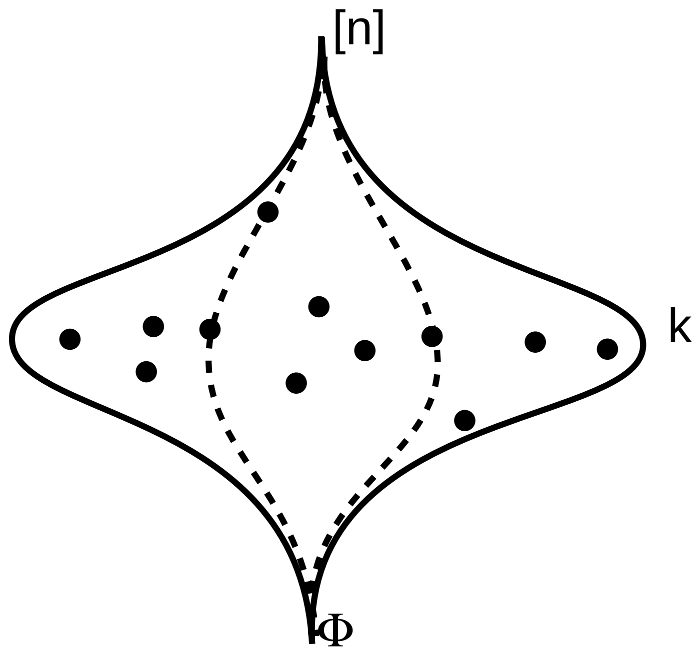

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It generalizes **Katona-Nemetz-Simonovits**' result for r-graphs.

Properties and examples

For any family of R-graphs $\mathcal{H}$, we have

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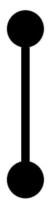
π () $= 1\frac{1}{4}.$

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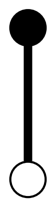
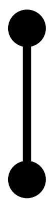
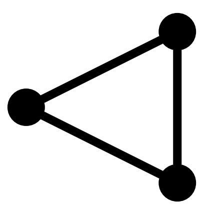
$\pi\Bigg($  $\Bigg) = 1.$

$\pi\Bigg($  $\Bigg) = 1\frac{1}{4}.$

$\pi\Bigg($  $\Bigg) = 1\frac{1}{2}.$

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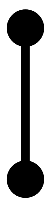
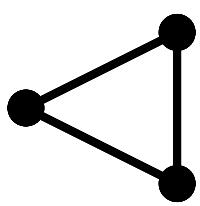
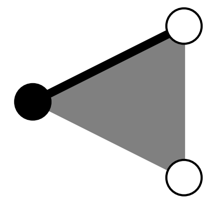
$\pi\Big($  $\Big) = 1\frac{1}{4}.$

$\pi\Big($  $\Big) = 1\frac{1}{2}.$

$\pi\Big($  $\Big) = 2.$

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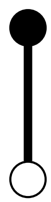
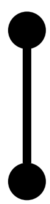
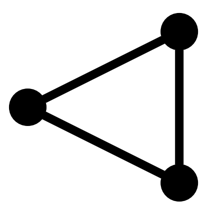
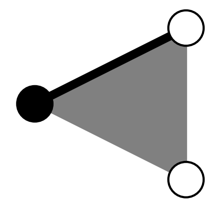
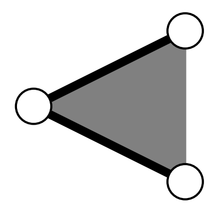
$\pi\Big($  $\Big) = 1.$

$\pi\Big($  $\Big) = 1\frac{1}{4}.$

$\pi\Big($  $\Big) = 1\frac{1}{2}.$

$\pi\Big($  $\Big) = 2.$

$\pi\Big($  $\Big) = 1\frac{1}{4}.$

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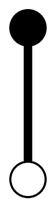
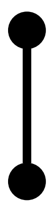
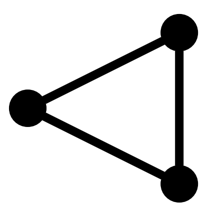
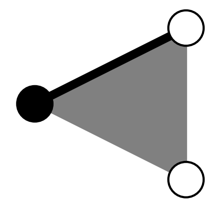
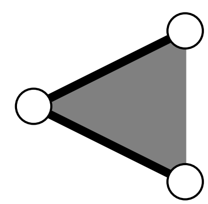
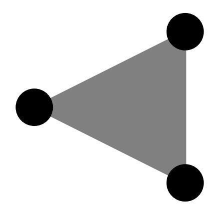
$\pi\Big($  $\Big) = 1.$

$\pi\Big($  $\Big) = 1\frac{1}{4}.$

$\pi\Big($  $\Big) = 1\frac{1}{2}.$

$\pi\Big($  $\Big) = 2.$

$\pi\Big($  $\Big) = 1\frac{1}{4}.$

$\pi\Big($  $\Big) = 1 + \frac{2\sqrt{3}}{9}.$

Supersaturation

Supersaturation Lemma for r-graphs: For any r-graph H and $a > 0$ there are b, $n_0 > 0$ so that if G is a r-graph on $n > n_0$ vertices with $|E(G)| > (\pi(H) + a)\binom{n}{r}$ then G contains at least $b\binom{n}{|V(H)|}$ copies of H.

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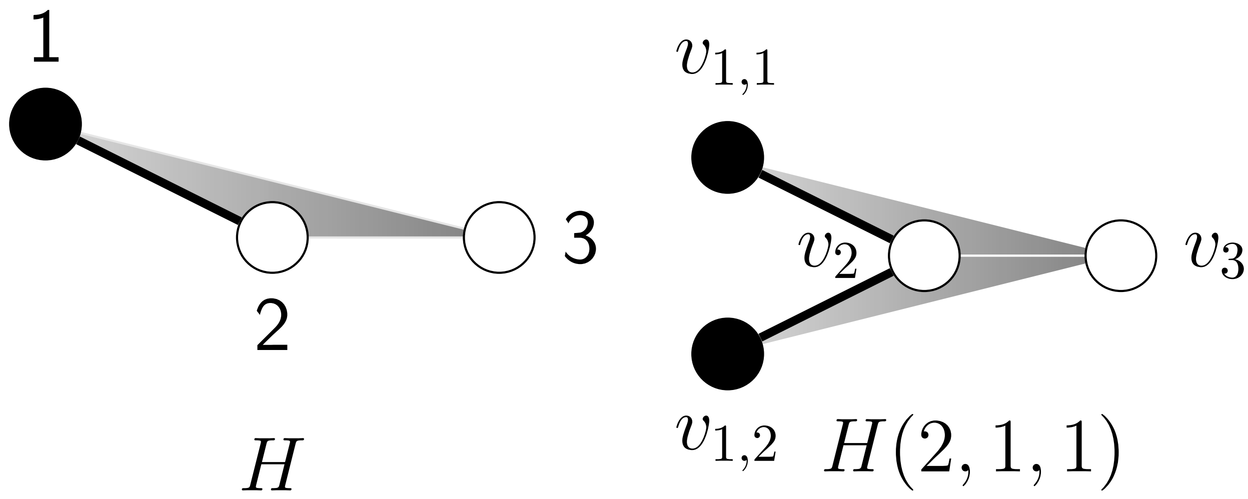
For any hypergraph H_n and positive integers $s_1, s_2, \ldots, s_n$, the blowup of H is a new hypergraph (V, E), denoted by $H_n(s_1, s_2, \ldots, s_n)$, satisfying

1. $V := \sqcup_{i=1}^{n} V_i$, where $|V_i| = s_i$.
2. $E = \cup_{F \in \mathrm{E}(H)} \prod_{i \in F} V_i$.

When $s_1 = s_2 = \cdots = s_n = s$, we simply write it as $H(s)$.



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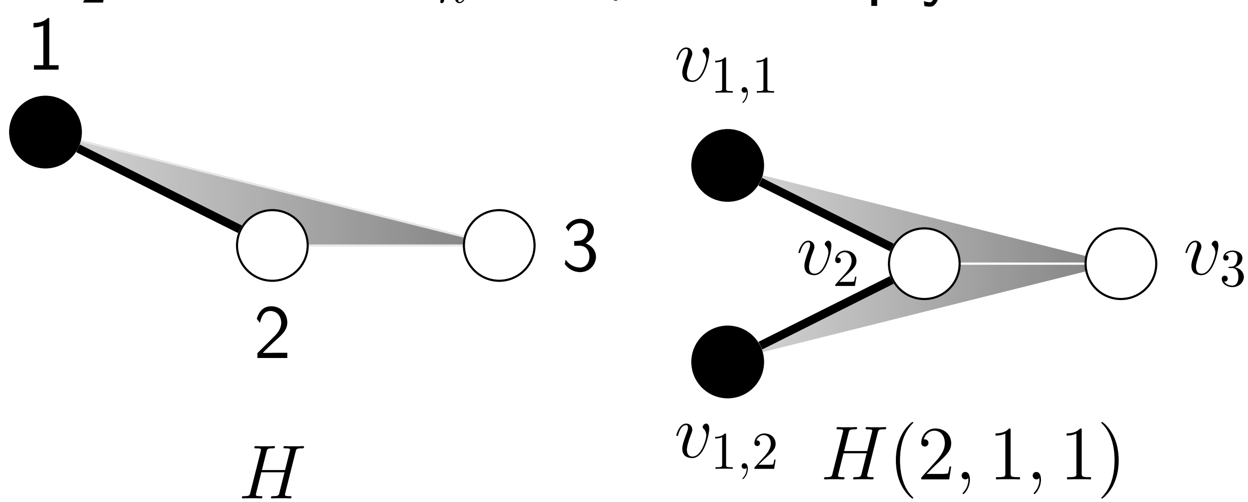

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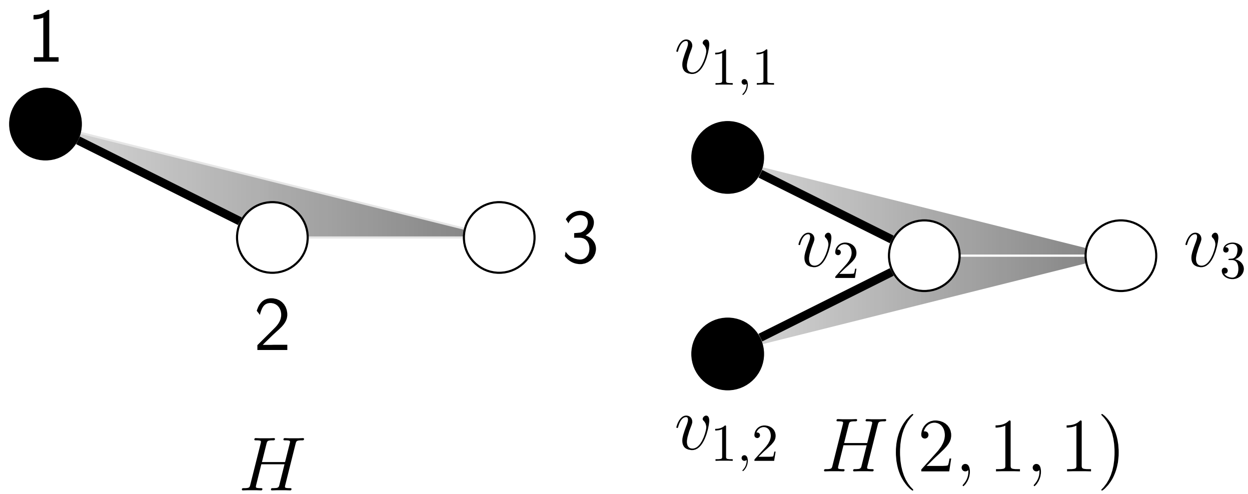

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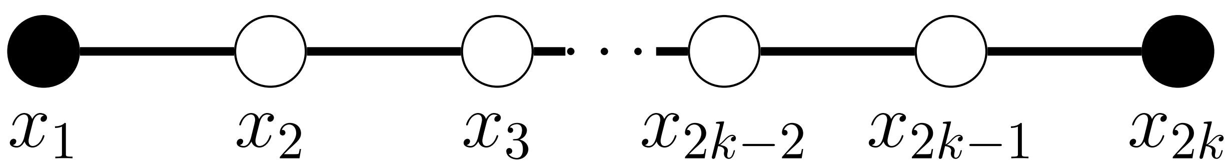
Corollary: If $H_1 \subset H_2 \subset H_1(s)$, then $\pi(H_2) = \pi(H_1)$.

Turán density of $\{1,2\}$-graphs

Theorem [Johnston-Lu 2012+]: For any $\{1,2\}$-graph H, we have

$$\pi(H) = \begin{cases} 2 - \frac{1}{\chi(H^2)-1} & \text{if } \chi(H^2) > 2; \\ \frac{5}{4} & \text{if } \chi(H^2) \text{ and } \bar{P}_2 \subseteq H; \\ \frac{9}{8} & \text{if } H^2 \text{ is bipartite and } \min\{k : \bar{P}_{2k} \subseteq H\} \geq 2; \\ 1 & \text{if } H^2 \text{ is bipartite and } \bar{P}_{2k} \not\subseteq H \text{ for any } k \geq 1. \end{cases}$$

Here H^2 is the level-2 subgraph of H and $\bar{P}_{2k}$ is the following closed even path.



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A value $\alpha \in [0, |R|)$ is a **jump** for R if
$\exists\ c > 0\ \forall\ \epsilon > 0\ \forall\ t \geq \max\{r : r \in R\}$
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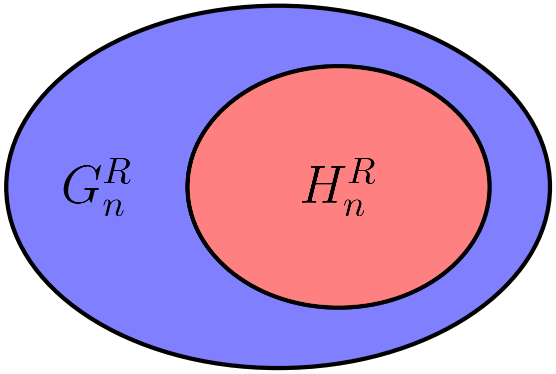

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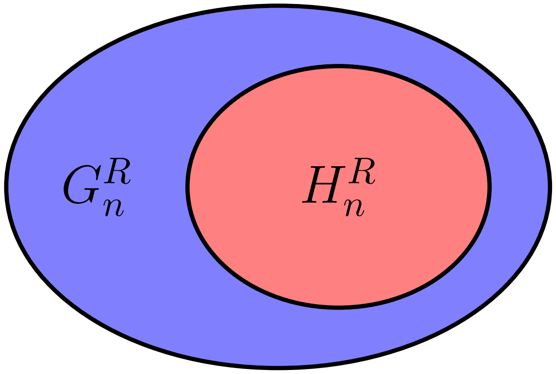

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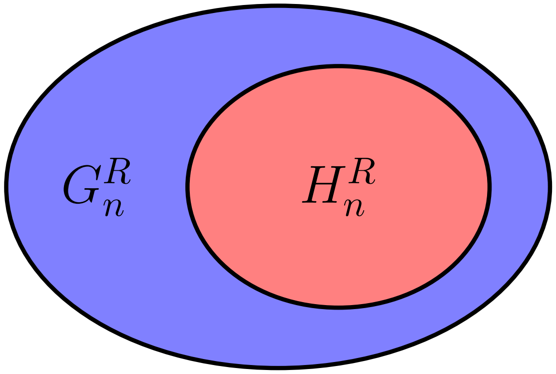

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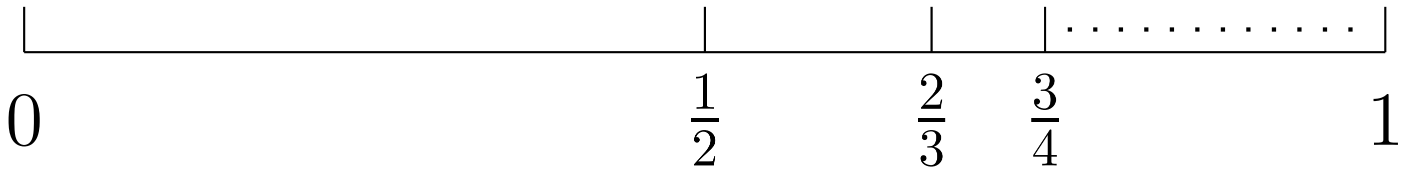

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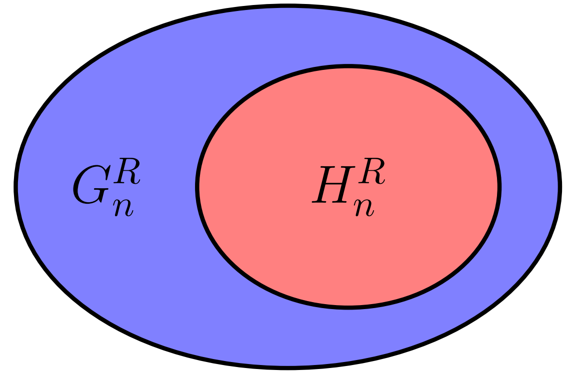

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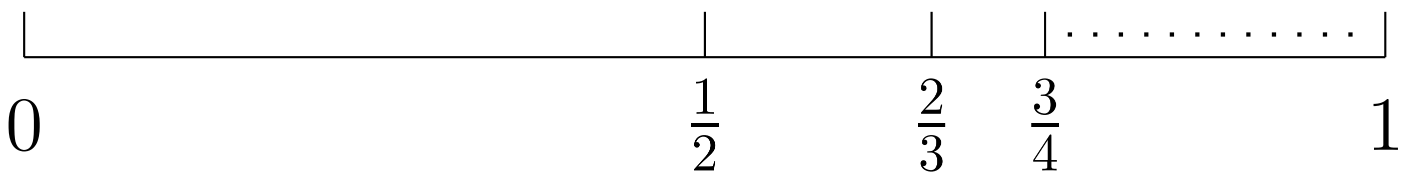

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Erdős asked "Do hypergraphs jump?"

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Strong jump and weak jump

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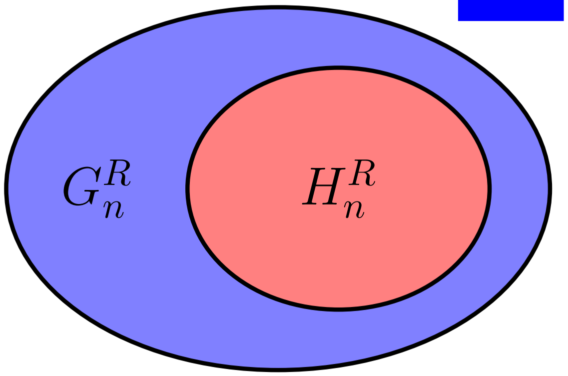
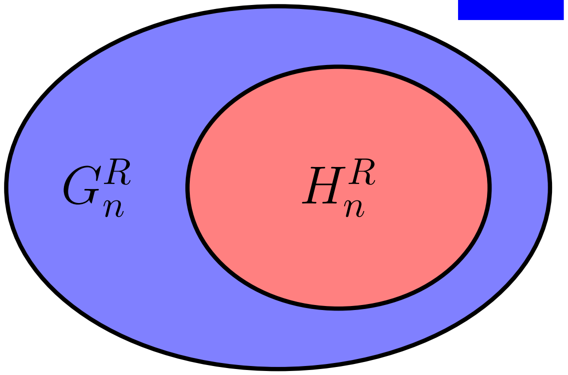


Strong jump and weak jump



A value $\alpha \in (0, |R|)$ is a **strong jump** for R if $\exists\ c > 0\ \forall\ t \geq \max\{r : r \in R\}\ \exists\ n_0$ $\forall\ G_n^R$ with $n \geq n_0$ and $h_n(G_n^R) \geq \alpha - c$ implies $\exists\ H_t^R \subset G_n^R$ with $h_t(H_t^R) \geq \alpha + c$.

- Jump: $\exists c > 0\ \forall \epsilon > 0 \quad \alpha + \epsilon \longrightarrow \alpha + c$.
- Strong jump: $\exists c > 0 \quad \alpha - c \longrightarrow \alpha + c$.

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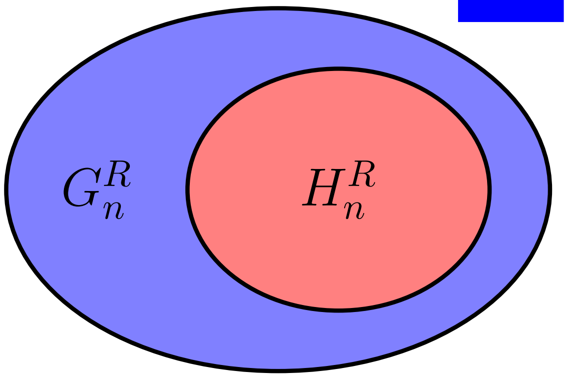

- Jump: $\exists c > 0\ \forall \epsilon > 0 \quad \alpha + \epsilon \longrightarrow \alpha + c$.
- Strong jump: $\exists c > 0 \quad \alpha - c \longrightarrow \alpha + c$. (Not same as Peng-Zhao [2008]'s definition: $\exists c > 0 \quad \alpha \longrightarrow \alpha + c$!)

Properties:

- Strong jump implies jump.
- The set of all strong jumps is **open**.
- If α_i is not a strong jump for (disjoint) R_i, then $\sum_i \alpha_i$ is not a strong jump for $\sqcup R_i$.

α is a **weak jump** if it is a jump but not strong jump.

Characterization of non-jump

A property $\mathcal{P}$ is called **hereditary** if it is closed under taking induced subgraphs. $\pi(\mathcal{P}) := \lim_{n\to\infty} \max_{G\in\mathcal{P}_n} h_n(G)$ exists.

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Theorem [Johnston-Lu 2014+]:

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2. A value $\alpha \in [0, |R|)$ is not a jump for R iff there exists a sequence of values $\{\alpha_i\}$ satisfying
   (a) All α_i are not strong jumps for R.
   (b) The sequence $\{\alpha_i\}$ decreases and goes to the limit α.

Strong jump for $\{1,2\}$

Theorem [Johnston-Lu 2013+]: The non-strong-jumps for $\{1,2\}$ are precisely

$$0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{k}{k+1}, \ldots, 1, \frac{9}{8}, \frac{7}{6}, \ldots, 1+\frac{k}{4(k+1)}, \ldots, \frac{5}{4},$$
$$\frac{3}{2}, \frac{5}{3}, \ldots, \frac{2k+1}{k+1}, \ldots, 2.$$

All these values are Turán densities. For $k \geq 1$,

$$\frac{k}{k+1} = \pi(K^2_{k+1}), \qquad \frac{2k+1}{k+1} = \pi(K^{\{1,2\}}_{k+1})$$
$$1+\frac{k}{4(k+1)} = \pi(\{K_2^{\{1,2\}}, K^{*2}_{k+2}\}).$$

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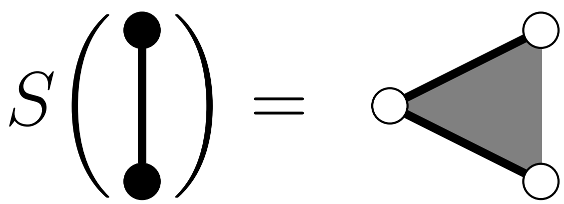
For any finite family $\mathcal{H}$ of $\{1,2\}$-graphs, $\pi(\mathcal{H})$ must be one of the values listed above.

Suspension

Suspension $S(H)$:

- $V(S(H)) = V(H) \bigcup \{*\}$,
- $E(S(H)) = \{F \bigcup \{*\} : F \in E(H)\}$.



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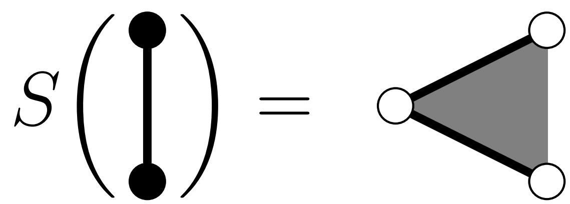

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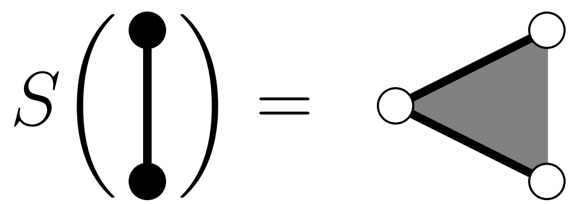

Lemma [Johnston-Lu 2012+]: For any hypergraph H we have that $\pi(S(H)) \le \pi(H)$.

Corollary: Let $S^t(H) := S(S(\cdots S(H)))$: iterating t times. Then the limit $\lim_{t\to\infty} \pi(S^t(H))$ always exists.

Connection to $\pi_c(P)$

Theorem [Johnston-Lu 2014+]: For any poset P, limit $\pi_c(P) := \lim_{n\to\infty} \frac{\mathrm{La}_c(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists.

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- Thus, $\pi_c(P) = \lim_{t\to\infty} \pi(S^t(H))$. $\square$

Suspension conjecture

Conjecture: For any t and any hypergraph H,

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- The special case with $H = K_n$ was conjectured at the AIM workshop on "Hypergraph Turán problem" in 2011.
- This conjecture implies the consecutive-layer version of Griggs-Lu's conjecture:

For any poset P, $\pi_c(P) = e(P)$.

Partial result

Theorem [Johnston-Lu 2012+]: Suppose that H is a subgraph of the blowup of a chain. Let k_1 be the minimum number in $R(H)$. Suppose $k_1 \geq 2$, and H' is a new hypergraph obtained by adding finitely many edges of type $k_1 - 1$ arbitrarily to H. Then

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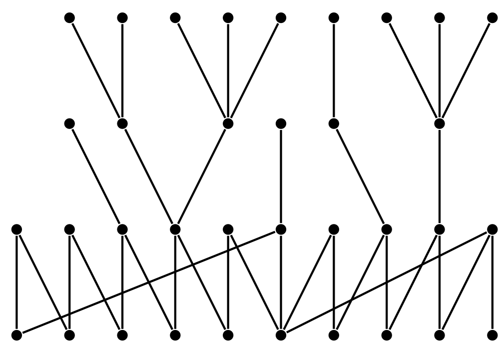
Corollary: If a poset P has a representation of a hypergraph H described above, then the consecutive-layer version of Griggs-Lu's conjecture holds for P.



Open questions

General questions:

- For any finite poset P, is $\pi(P) = e(P)$?
- For any hypergraph H, is $\lim_{t\to\infty} \pi(S^t(H)) = |R(H)| - 1$?
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Specific questions:

- For posets, determine $\pi(\mathcal{D}_2)$, $\pi(\mathcal{O}_6)$, and $\pi(\mathcal{O}_{10})$.
- For non-uniform hypergraphs:
  Determine $\pi(S^2(K_2^{\{0,1,2\}}))$, $\pi(S(K_3^{\{1,2\}}))$, and $\pi(S(C_5^{\{1,2\}}))$.

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Thank You