Estimation of the Size of P-free Families

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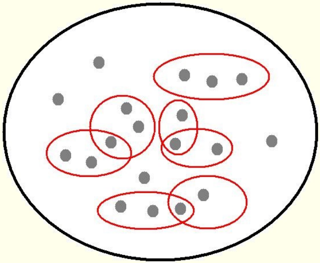
Introduction

Consider a family of subsets of $[n] := \{1, 2, \ldots, n\}$ such that $A \subset B$ is not allowed for any distinct members A and B of this family. Such a family is said to be *inclusion-free*.



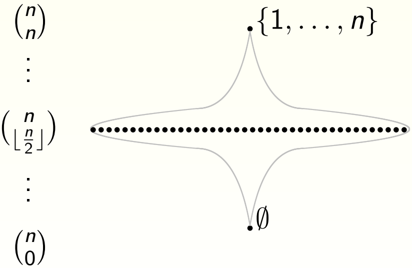
Question: What is the maximum size of such a family?

THEOREM (Sperner, 1928)

Let $\mathcal{F}$ be an inclusion-free family of subsets of $[n]$. Then

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The upper bound is achieved by taking all sets of size $\lfloor \frac{n}{2} \rfloor$.

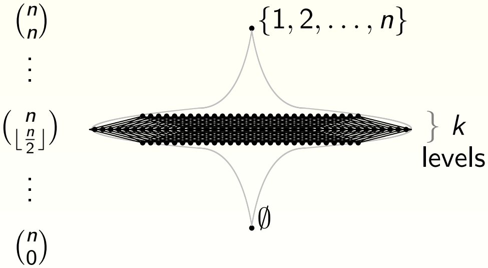


Theorem (Erdős, 1945)

Let $\mathcal{F}$ be a family of subsets of $[n]$ such that no $k+1$ sets in $\mathcal{F}$ satisfy $A_1 \subset \cdots \subset A_{k+1}$. Then

$$|\mathcal{F}| \leq \sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} + i \rfloor}.$$

The upper bound is achieved by taking all sets of middle k sizes.



Families Without a Subposet

A *poset (partially ordered set)* $P = (P, \leq)$ is a set P with a binary partial order relation $\leq$ satisfying

1. For all $x \in P$, $x \leq x$. (reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

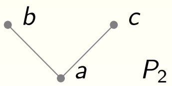
Figure: The Hasse diagrams of some small posets.

The *Boolean lattice* $\mathcal{B}_n = (2^{[n]}, \subseteq)$ is the poset consisting of the power set of $[n]$ and the inclusion relation as the partial order.

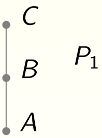
A poset $P_1 = (P_1, \leq_1)$ contains another poset $P_2 = (P_2, \leq_2)$ as a *(weak) subposet* if there exists an injection f from P_2 to P_1, which preserves the order, that is $f(a) \leq_1 f(b)$ whenever $a \leq_2 b$.

Example:

$P_2 = (\{a, b, c\}, \{(a, b), (a, c)\})$

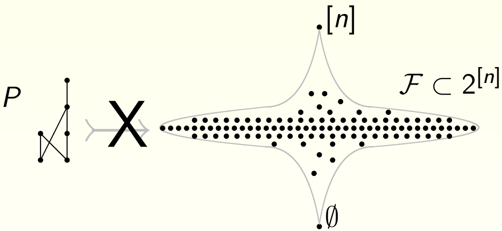


$$f: \begin{array}{l} c \longmapsto C \\ b \longmapsto B \\ a \longmapsto A \end{array}$$



$P_1 = (\{A, B, C\}, \{(A, B), (B, C), (A, C)\})$

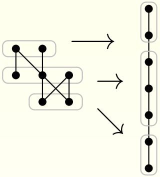
A *P-free family* $\mathcal{F}$ is a collection of subsets of $[n]$ such that it does not contain P as a subsposet.



The largest size of a P-free family of subsets of $[n]$ for a given poset P is denoted by $\mathrm{La}(n, P)$.

The difficulty of solving the problem is to find an upper bound on the size of families $\mathcal{F}$.

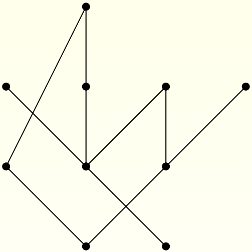
Note that every poset P can be extended as a chain on $|P|$ elements.



Erdős's Theorem implies

$$\mathrm{La}(n, P) \leq \Sigma(n, |P| - 1) \sim (|P| - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

The *height* of a poset P, $h(P)$, is the largest size of any chain in P.



THEOREM (Burcsi and Nagy, 2013)

For any poset P, $\mathrm{La}(n,P) \le \left(\frac{|P| + h(P)}{2} - 1 \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Double Counting Method

Given a family $\mathcal{F} \subset 2^{[n]}$, consider some families $\mathcal{G}_1, \ldots, \mathcal{G}_k$ such that $\mathcal{F} \subset \bigcup_{i=1}^{k} \mathcal{G}_i$. We count the number of pairs $(F, \mathcal{G}_i)$ whenever $F \in \mathcal{F} \cap \mathcal{G}_i$. Then

$$\sum_{F \in \mathcal{F}} (F, \mathcal{G}_i) = \sum_{i=1}^{k} (F, \mathcal{G}_i).$$

This helps us to deduce an upper bound on $|\mathcal{F}|$.

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THEOREM (Spener, 1928)

Let $\mathcal{F}$ be an inclusion-free family of subsets of $[n]$. Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof.(Lubell, 1966)

Let $\mathcal{G}_i$ be a *full chain*: $\emptyset \subset \{a_1\} \subset \{a_1, a_2\} \subset \cdots \subset \{a_1, \ldots, a_n\}$.

On the one hand, each set F is contained in $|F|!(n-F)!$ $\mathcal{G}_i$'s.

On the other hand, since $\mathcal{F}$ is inclusion-free, $|\mathcal{F} \cap \mathcal{G}_i| \leq 1$.

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On the other hand, since $\mathcal{F}$ is inclusion-free, $|\mathcal{F} \cap \mathcal{G}_i| \leq 1$.

$$\sum_{F\in\mathcal{F}} |F|!(n-|F|)! = \sum_{F\in\mathcal{F}} (F, \mathcal{G}_i) = \sum_{i=1}^{n!} (F, \mathcal{G}_i) \leq n!$$

Hence $\displaystyle\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$. This implies $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Main Results

THEOREM (Chen and Li, 2014)

For any poset P, when n is sufficiently large, the inequality

$$\mathrm{La}(n,P) \leq \frac{1}{m+1}\left(|P| + \frac{1}{2}(m^2+3m-2)(h(P)-1)-1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds for any fixed m with $1 \leq m \leq \frac{n}{2}$.

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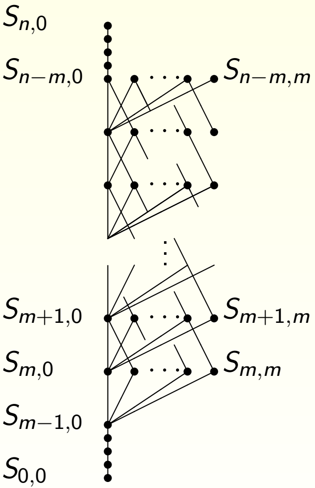
Proof.

An *m-linkage* $\mathcal{L}^{(m)}$ consists of a *main chain* and m *links*.

The main chain is $\{S_{i,0} \mid 0 \leq i \leq n\}$, where $S_{i,0} = \{a_1, \ldots, a_i\}$.

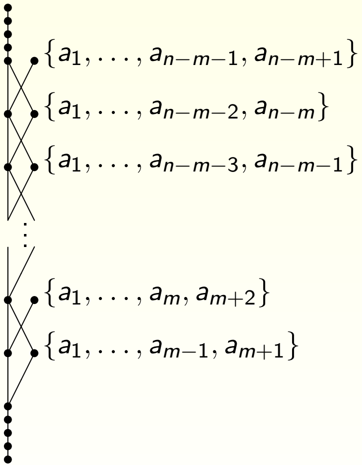
For $1 \leq j \leq m$, the jth-link is $\{S_{i,j} \mid m \leq i \leq n-m\}$, where $S_{i,j} = \{a_1, \ldots, a_{i-1}\} \cup \{a_{i+j}\}$.



Proof.

1st-link: $\{S_{i,1} \mid m \leq i \leq n-m\}$

$S_{i,1} = \{a_1, \ldots, a_{i-1}\} \cup \{a_{i+1}\}$



Proof.

1st-link: $\{S_{i,1} \mid m \leq i \leq n-m\}$

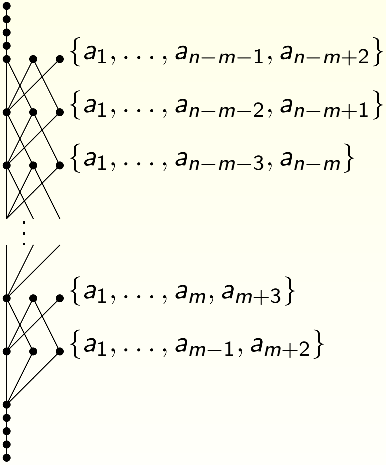
$S_{i,1} = \{a_1, \ldots, a_{i-1}\} \cup \{a_{i+1}\}$

2nd-link: $\{S_{i,1} \mid m \leq i \leq n-m\}$

$S_{i,2} = \{a_1, \ldots, a_{i-1}\} \cup \{a_{i+2}\}$

$\vdots$

For a fixed m, the number of m-linkages is $n!$.

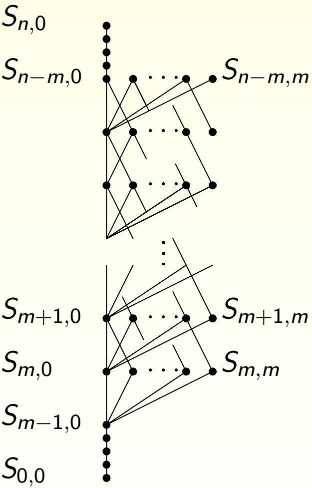


We count the number of pairs $(F, \mathcal{L}^{(m)})$.

For a set $F \subseteq [n]$, the number of pairs $(F, \mathcal{L}^{(m)})$ is equal to

$$\sum_{\substack{F \in \mathcal{F} \\ |F|<m \text{ or } |F|>n-m}} |F|!(n-|F|)!$$

$$+ \sum_{\substack{F \in \mathcal{F} \\ m \le |F| \le n-m}} (m+1)|F|!(n-|F|)!$$

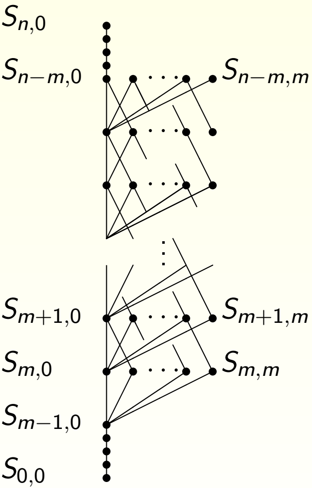


Let $\mathcal{F} \subseteq 2^{[n]}$ be a P-free family.

For any m-linkage $\mathcal{L}^{(m)}$,

$$|\mathcal{L}^{(m)} \cap \mathcal{F}| \le |P| + \frac{1}{2}(M)(H) - 1,$$

where $M = m^2 + 3m - 2$ and $H = h(P) - 1$.



Combine

$$\sum_{\substack{F\in\mathcal{F}\\ |F|<m \,\mathrm{or}\, |F|>n-m}} |F|!(n-|F|)! + \sum_{\substack{F\in\mathcal{F}\\ m\le|F|\le n-m}} (m+1)|F|!(n-|F|)!$$

and

$$|\mathcal{L}^{(m)}\cap\mathcal{F}|n! \le (|P| + \frac{1}{2}(m^2+3m-2)(h(P)-1)-1)n!.$$

We obtain

$$\boxed{|\mathcal{F}| \le \frac{1}{m+1}\left(|P| + \frac{1}{2}(m^2+3m-2)(h(P)-1)-1\right)\binom{n}{\lfloor\frac{n}{2}\rfloor}}$$

when n is sufficiently large. □

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COROLLARY (Chen and Li, 2014)

For any poset P and any sufficiently large n,

$$\mathrm{La}(n, P) \leq \left(\frac{1}{2}(|P| + h(P)) - 1\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

In particular, if $|P| \geq (5 + 2\sqrt{2})(h(P) - 1) + 1$, then we have a better bound

$$\mathrm{La}(n, P) \leq \left(\sqrt{2(h(P) - 1)(|P| - 2h(P) + 1)} + h(P) - 1\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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Proof. Find the value of m that minimizes

$$f(m) = \frac{1}{m+1}(|P| + \frac{1}{2}(m^2+3m-2)(h(P)-1) - 1). \qquad \square$$

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Remark It is not hard to see

$$\mathrm{La}(n,P) = O(\sqrt{h(P)|P|}).$$

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Question: When $m = 1$,

$$\mathrm{La}(n, P) \le \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Bursi and Nagy found many posets having

$$\mathrm{La}(n, P) \sim \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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$$\mathrm{La}(n, P) \sim \left(\frac{|P| + h(P)}{2} - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

For $m \ge 2$, does there exist P such that

$$\mathrm{La}(n, P) \sim \frac{1}{m+1}\left(|P| + \frac{1}{2}(m^2 + 3m - 2)(h(P) - 1) - 1\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}?$$

Question: Can we use more parameters of a poset P, such as width, dimension etc., to improve the upper bound of $\mathrm{La}(n, P)$?

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Thank you for your attention!!