Symmetric Chains in Quotients of Boolean Lattices

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- P has a *symmetric chain decomposition* if $P = \bigsqcup_{i=0}^{n} C_i$ with each C_i a symmetric, saturated chain in P

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and for chains C_i of distinct lengths and $n_i \in \mathbb{N}$ $(i = 1, 2, \ldots, m)$

$$\mathrm{Aut}(C_1^{n_1} \times C_2^{n_2} \times \cdots \times C_m^{n_m}) \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}.$$

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$B_6/\langle(1\ 2\ 3\ 4\ 5\ 6)\rangle$

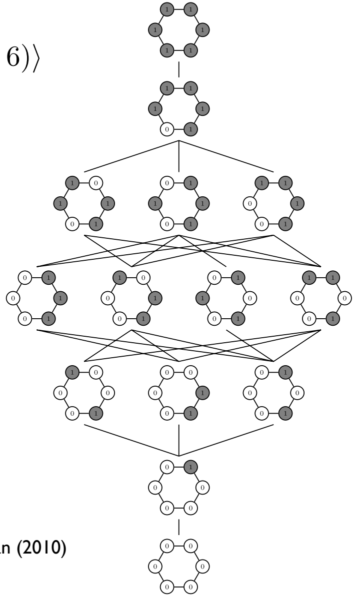

Figure credit: K K Jordan (2010)

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Special cases of this were posed by Stanley but, in this generality, the question was not asked until 20 years after these papers.

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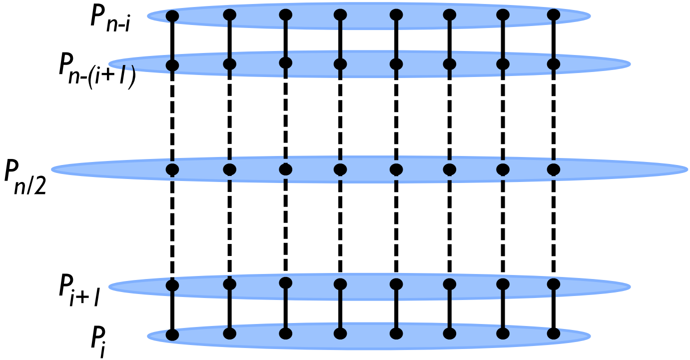

Rank-symmetry, rank-unimodality and strongly Sperner guarantee matchings

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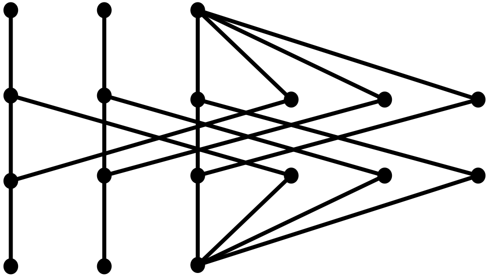
Example: [Griggs]



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Conjecture:

2006 For all $G \leq S_n$, $\mathbf{2}^n/G$ is an SCO. [Canfield & Mason]

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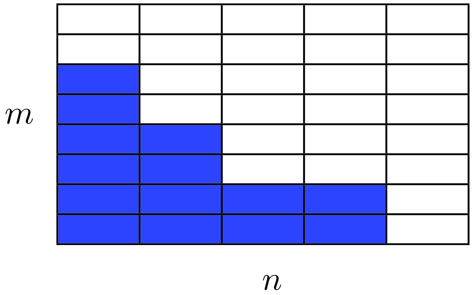

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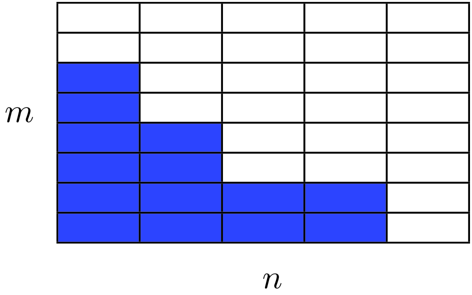

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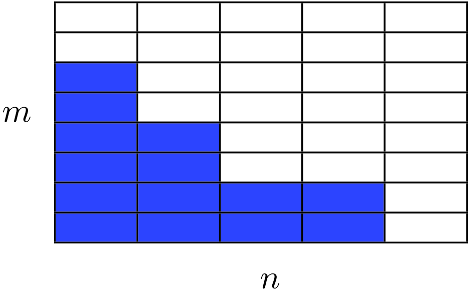

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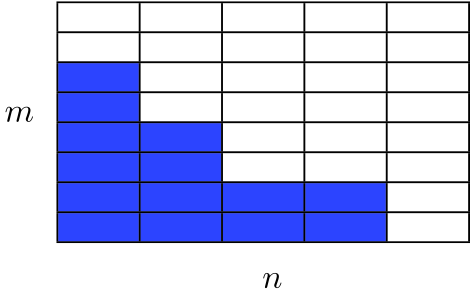

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Question: [Stanley 1980] Is $L(m, n)$ an SCO?

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4. for all n and all SCOs P, $P^n/\mathbb{Z}_n$ is an SCO [Dhand 2011];

5. Let $n = kt$, $G \leq S_n$, $K \leq S_k$, $T \leq S_t$, $G = K \wr T$ via the natural action of T on K^t. If

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   Base case for (a): K is generated by powers of disjoint cycles.

   [Duffus and Thayer(2014$^+$)]

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Problem 2: Show that for all k, t, $L(k, t)$ is an SCO. Equivalently, show that this quotient is an SCO:

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Problem 3: Determine if for every embedding ϕ of a finite abelian group A in S_n, $\mathbf{2}^n/\phi(A)$ is an SCO.

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Problem 4: Determine if the regular representation of an abelian group A produces a quotient of $\mathbf{2}^{|A|}$ with an SCD.

Each embedding of an abelian group A in a symmetric group can be obtained as a product of actions A on factor groups A/H. Here is a test case.

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- for each i, $\widehat{A}_i := \{\widehat{a}_i \mid a \in A\} \cong \mathbb{Z}_p$
- $a \to (\widehat{a}_1, \widehat{a}_2, \ldots \widehat{a}_k)$ is an embedding of A in S_N where $N = \bigcup N_i$, with $N_i := A/H_i$, and so $|N| = k \cdot p$

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- for each $a \in A$, $x + H_i \xrightarrow{\widehat{a}_i} a + x + H_i$ is in S_{A/H_i}
- for each i, $\widehat{A}_i := \{\widehat{a}_i \mid a \in A\} \cong \mathbb{Z}_p$
- $a \to (\widehat{a}_1, \widehat{a}_2, \ldots \widehat{a}_k)$ is an embedding of A in S_N where $N = \bigcup N_i$, with $N_i := A/H_i$, and so $|N| = k \cdot p$
- $\widehat{A}$ is the diagonal subgroup of $\widehat{A}_1 \times \widehat{A}_2 \times \cdots \times \widehat{A}_k$

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Let $\mathbf{C}$ be a member of an SCD of $\mathbf{2}^{N_1}/\widehat{A}_1 \times \cdots \times \mathbf{2}^{N_k}/\widehat{A}_k$ with its rank j element $\mathcal{X}_j = ([X_{1,j}], [X_{2,j}], \ldots, [X_{k,j}])$, $j = r, r+1, \ldots, n-r$. We may assume that representatives are chosen such that

$$X_{i,j} \subseteq X_{i,j+1},\ j = r, r+1, \ldots, n-r-1.$$

Each class $\mathcal{X}_j$ refines into p^{k-1} members of $\mathbf{2}^N/\ \widehat{A}$ as follows: let

$$\bar{\pi} = (1, \pi_2, \pi_3, \ldots, \pi_k),\ \ \pi_i \in \mathbb{Z}_p,\ \text{and}$$

$$\bar{\pi}(\mathcal{X}_j) = X_{1,j} \cup \pi_2(X_{2,j}) \cup \pi_3(X_{3,j}) \cup \cdots \cup \pi_k(X_{k,j}).$$

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Since each $X_{i,j} \subseteq X_{i,j+1}$, $j = r, r+1, \ldots, n-r-1$,

$$\bar{\pi}(\mathcal{X}_r) \subseteq \bar{\pi}(\mathcal{X}_{r+1}) \subseteq \cdots \subseteq \bar{\pi}(\mathcal{X}_{n-r}),\ \text{so}$$

$$[\bar{\pi}(\mathcal{X}_r)] < [\bar{\pi}(\mathcal{X}_{r+1})] < \cdots < [\bar{\pi}(\mathcal{X}_{n-r})]\ \text{in}\ \ \mathbf{2}^N/\ \widehat{A}.$$

Hence, there exist p^{k-1} symmetric chains partitioning all refined classes of $\mathbf{C}$ in $\mathbf{2}^N/\ \widehat{A}$.

A Final Problem

Let S_n have its induced action on the k-element subsets of $[n]$ and let $S_n^{(k)}$ denote the resulting subgroup of $S_{\binom{[n]}{k}}$.

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Problem 5: Determine if the set of unlabelled k-graphs on $[n]$, ordered by subgraph, has an SCD.