OPERATOR-VALUED FOURIER MULTIPLIER THEOREMS ON $L_p(X)$ AND GEOMETRY OF BANACH SPACES

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ABSTRACT. This paper gives the optimal order l of smoothness in the Mihlin and Hörmander conditions for operator-valued Fourier multiplier theorems. This optimal order l is determined by the geometry of the underlying Banach spaces (e.g. Fourier type). This requires a new approach to such multiplier theorems, which in turn leads to rather weak assumptions formulated in terms of Besov norms.

1. INTRODUCTION

In recent years, operator-valued multiplier theorems have had many applications in the theory of evolutions equations, in particular in connection with: maximal regularity of parabolic equations [4, 7, 11, 23, 37, 38], stability theory [17, 26, 36], elliptic operators on infinite dimensional state spaces [3], and pseudo differential operators on manifolds with singularities [31]. In these applications one often has a multiplier function m, from $\mathbb{R}^N$ into the space $\mathcal{B}(X)$ of bounded operators on a Banach space X, such that Mihlin's condition holds, that is, the set

$$\tau_l := \left\{ |t|^{|\alpha|} \, D^\alpha m(t) : t \in \mathbb{R}^N \setminus \{0\} \ , \ |\alpha| \leq l \right\} \tag{1.1}$$

is norm bounded in $\mathcal{B}(X)$ and then one wants to conclude that the operator

$$T_m \colon \mathcal{S}\left(\mathbb{R}^N, X\right) \to \mathcal{S}'\left(\mathbb{R}^N, X\right) \quad \text{given by} \quad T_m f := \left[m\widehat{f} \right]^\vee ,$$

defined on the Schwartz class, extends to a bounded operator on $L_q\left(\mathbb{R}^N, X\right)$ for $1 < q < \infty$.

It is a classical result of J. Schwartz (cf. [6, Section 6.1]) that such an extension exists if X is a Hilbert space and $l = [N/2] + 1$. Furthermore, Pisier showed that Hilbert spaces are the only Banach spaces for which the boundedness of τ_l is sufficient for the Mihlin theorem to hold; he showed that such spaces must have type 2 and cotype 2. Bourgain [10] showed that, for a scalar-valued multiplier function m and $N = 1$, Mihlin's theorem holds if and only if X is a UMD Banach space. This result was extended to higher dimensions in [27, 40] with $l = N$. For operator-valued

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multiplier functions, it was first shown in [35, 38] that for a UMD Banach space X and $l = N$, R-boundedness of τ_l is sufficient for Mihlin's theorem to hold. Recall that τ_l is R-bounded if there is a constant C so that for each $n \in \mathbb{N}$, subset $\{T_j\}_{j=1}^n$ of τ_l, and subset $\{x_j\}_{j=1}^n$ of X one has that

$$\mathbb{E}\left\|\sum_{j=1}^{n} r_j\left(\cdot\right) T_j(x_j)\right\| \;\le\; C\,\mathbb{E}\left\|\sum_{j=1}^{n} r_j\left(\cdot\right) x_j\right\|$$

where $\{r_j\}_j$ are the Rademacher functions. For variants of these multiplier theorems, see [4, 11, 16].

This paper presents a new method of proof which allows for the determination of the *optimal* smoothness of the multiplier function; indeed, the best exponent l in (1.1) for a given Banach space X depends on the geometry of X, specifically, on its Fourier type. Recall that a Banach space X has Fourier type $p \in [1,2]$ provided the Fourier transform defines a bounded operator from $L_p(X)$ into $L_{p'}(X)$, i.e. the Hausdorff Young inequality holds for the exponent p.

Corollary 4.4 shows that Mihlin's theorem holds with $l = [N/p] + 1$ if X is a UMD space with Fourier type p. Since a Hilbert space has Fourier type 2, one recovers Schwartz's result. Since each UMD space has Fourier type p for some $p > 1$, one obtains the results in [35, 38] with $l = N$. If X is a subspace of an $L_q(\Omega)$ space, then X has Fourier type $p = \min(q, q')$ and so one may use $l = [N/p]+1$; hence, the l in (1.1) improves (i.e. decreases) as q tends towards 2. Furthermore, the exponent $l = [N/p] + 1$ is best possible for L_p spaces.

The main result of this paper, Theorem 4.1, is a general multiplier theorem. This theorem's assumption, which uses vector-valued Besov spaces, may not look very attractive at first sight; indeed, the assumption is stated in a rather general form. However, this formulation allows as fairly easy corollaries (see Corollaries 4.4, 4.10, 4.11) vector-valued generalizations of several classical multiplier theorems conditions (à la Mihlin, Hörmander, or Lipschitz estimates); the latter corollary gives, for scalar-valued multiplier functions, an improvement of Bourgain's original result in [10]. Theorem 4.1 also gives multiplier theorems in Sobolev spaces, $H^1(X)$, and $\mathrm{BMO}(X)$; see Corollaries 4.6 and 4.9.

Section 3 prepares for the proof of the main result of this paper. Bourgain's work [10] leads to a variant of the Littlewood-Paley decomposition (Corollary 3.3) and some precise norm estimates for scalar multiplier functions (Proposition 3.6). Also needed is a sharping, Corollaries 3.10 and 3.11, of results in [14] on Fourier estimates on Besov spaces. Section 2 collects necessary definitions (such as UMD and Fourier type) and some basic properties of these classes of Banach spaces.

Analogous results for multiplier theorems on Besov spaces $B_{q,r}^{s}\left(\mathbb{R}^N, X\right)$ are contained in [14]. In this setting, the proofs are much more elementary and the UMD assumption and R-boundedness are not necessary. The new technique developed in this paper can also be used to prove boundedness results for singular integral operators (see [19]).

2. DEFINITIONS AND NOTATION

Notation is standard. Throughout this paper X, Y, Z are Banach spaces over the field $\mathbb{C}$ and X^* is the (topological) dual space of X. The space $\mathcal{B}(X,Y)$ of bounded linear operators from X to Y is endowed with the usual uniform operator topology, unless otherwise stated. The Bochner-Lebesgue space $L_p\left(\mathbb{R}^N, X\right)$, where $1 \le p \le \infty$, is endowed with its usual norm topology. The space $C^l\left(\mathbb{R}^N \setminus \{0\}, X\right)$ of functions $f\colon \mathbb{R}^N \setminus \{0\} \to X$ so that $D^\alpha f$ is continuous and bounded for $|\alpha| \le l \in \mathbb{N}_0$ is endowed with the usual supremum norm of f. If convenient and confusion seems unlikely, the various function spaces $E\left(\mathbb{R}^N, X\right)$ in this paper are denoted simply by just $E(X)$ or E, with the exception of the Schwartz class function space. $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers while $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. If s is a positive real number, then $[s] := \max\{n \in \mathbb{N}_0 \colon n \le s\}$. The conjugate exponent p' of $p \in [1, \infty]$ is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Non-numerical subscripts on constants indicate dependency.

The Schwartz class $\mathcal{S}\left(\mathbb{R}^N, X\right)$, or simply $\mathcal{S}(X)$, is the space of X-valued rapidly decreasing smooth functions φ on $\mathbb{R}^N$, equipped with its usual topology generated by seminorms. As customary, $\mathcal{S}\left(\mathbb{R}^N, \mathbb{C}\right)$ is often denoted by just $\mathcal{S}$. Recall $\mathcal{S}(X)$ is norm dense in $L_q(X)$ when $1 \le q < \infty$. The space of X-valued tempered distributions $\mathcal{S}'\left(\mathbb{R}^N, X\right)$ is the space of continuous linear operators $L : \mathcal{S} \to X$, equipped with the bounded convergence topology. Each $m \in L_q\left(\mathbb{R}^N, X\right)$, where $1 \le q \le \infty$, defines an $L_m \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ by $L_m(\varphi) := \int_{\mathbb{R}^N} \varphi(t) m(t)\, dt$; when convenient and confusion seems unlikely, such a function m is identified with L_m.

It is well-known that the Fourier transform $\mathcal{F}\colon \mathcal{S}(X) \to \mathcal{S}(X)$ defined by

$$(\mathcal{F}\varphi)(t) \;\equiv\; \widehat{\varphi}(t) \;:=\; \int_{\mathbb{R}^N} e^{-it\cdot s} \varphi(s)\, ds$$

is an isomorphism whose inverse is given by

$$\left(\mathcal{F}^{-1}\varphi\right)(t) \;\equiv\; \check{\varphi}(t) \;=\; (2\pi)^{-N} \int_{\mathbb{R}^N} e^{it\cdot s} \varphi(s)\, ds\ ,$$

where $\varphi \in \mathcal{S}(X)$ and $t \in \mathbb{R}^N$. Also, the Fourier transform $\mathcal{F}\colon \mathcal{S}'(X) \to \mathcal{S}'(X)$ defined by

$$(\mathcal{F}L)(\varphi) \;\equiv\; \widehat{L}(\varphi) \;:=\; L(\widehat{\varphi}) \qquad \text{where} \quad L \in \mathcal{S}'(X)\ ,\ \varphi \in \mathcal{S}$$

is an isomorphism whose inverse is given by $\left(\mathcal{F}^{-1}L\right)(\varphi) \equiv \check{L}(\varphi) := L(\check{\varphi})$. The set

$$\mathcal{S}_o(X) \;:=\; \{\varphi \in \mathcal{S}(X) : \operatorname{supp}\widehat{\varphi} \text{ is compact}\ ,\ 0 \notin \operatorname{supp}\widehat{\varphi}\}$$

is norm dense in $L_p(X)$ for each $1 < p < \infty$ (cf. [40, Lemma 2.3]).

For completeness a proof of the following fact is included.

Fact 2.1. Let $f \in L_1\left(\mathbb{R}^N, X\right)$ with $N > 1$. Define $F \in L_1\left(\mathbb{R}, L_1\left(\mathbb{R}^{N-1}, X\right)\right)$ by

$$F(t)(u) = f(t,u) \qquad \text{for} \quad u \in \mathbb{R}^{N-1} \qquad \text{and a.e.} \quad t \in \mathbb{R}\ .$$

If the (N-dimensional) Fourier transform $\widehat{f}$ of f has support in

$$K_N(r) := \left\{(v_1, \ldots, v_N) \in \mathbb{R}^N : \ |v_j| \le r\right\} \ ,$$

then the (1-dimensional) Fourier transform $\widehat{F}$ of F has support in $[-r, r]$.

Proof. It suffices to show Fact 2.1 for the case $X = \mathbb{C}$; indeed, just consider functions of the form $x^* \circ f$ for $x^* \in X^*$.

If $\widehat{f} \colon \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{C}$ has the special form

$$\begin{aligned} g(t,u) \ &= \ \sum_{j=1}^{m} h_j(t)\ k_j(u) \\ h_j \ \equiv \ h_j\,|_{K_1(r)} \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \quad &\text{and} \quad k_j \ \equiv \ k_j\,|_{K_{N-1}(r)} \in \mathcal{S}\left(\mathbb{R}^{N-1}, \mathbb{C}\right) \ , \end{aligned} \tag{2.1}$$

then $f = \sum_{j=1}^{m} \check{h}_j\, \check{k}_j$ and $\widehat{F}(t)(u) = \sum_{j=1}^{m} h_j(t)\ \check{k}_j(u)$; so, $\operatorname{supp} \widehat{F} \subset [-r, r]$.

For a general $f \in L_1\left(\mathbb{R}^N, \mathbb{C}\right)$ with $\operatorname{supp} \widehat{f} \subset K_N(r)$, approximate $\widehat{f}$ by functions g_n of the special form in (2.1) in the $L_2\left(\mathbb{R}^N, \mathbb{C}\right)$-norm. It then follows from Plancherel's Theorem, first applied in $L_2\left(\mathbb{R}^N, \mathbb{C}\right)$ and then applied in $L_2\left(\mathbb{R}, L_2\left(\mathbb{R}^{N-1}, \mathbb{C}\right)\right)$, that $\widehat{F}$ can be approximated in the $L_2\left(\mathbb{R}, L_2\left(\mathbb{R}^{N-1}, \mathbb{C}\right)\right)$-norm by functions with support in $[-r, r]$. Hence $\operatorname{supp} \widehat{F} \subset [-r, r]$. ■

The derivative, translation, and dilation properties of $\mathcal{F}$ and $\mathcal{F}^{-1}$ that hold in the scalar-valued case also hold in the vector-valued case. However, the Hausdorff-Young inequality need not hold; thus, one has to consider the following class of Banach spaces that was introduced by Peetre [28].

Definition 2.2. Let $1 \le p \le 2$. A Banach space X has *Fourier type* p provided the Fourier transform $\mathcal{F}$ defines a bounded linear operator from $L_p\left(\mathbb{R}^N, X\right)$ to $L_{p'}\left(\mathbb{R}^N, X\right)$ for some (and thus then, by [22], for each) $N \in \mathbb{N}$. The *Fourier type constant* $\mathcal{F}_{p,N}(X)$ of X is then the norm of $\mathcal{F} \in \mathcal{B}\left(L_p\left(\mathbb{R}^N, X\right), L_{p'}\left(\mathbb{R}^N, X\right)\right)$.

Remark 2.3. The simple estimate $\|\mathcal{F}f(t)\|_X \le \|f\|_{L_1(X)}$ shows that each Banach space X has Fourier type 1 with $\mathcal{F}_{1,N}(X) = 1$. The notion becomes more restrictive as p increases to 2. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space [24]. A space $L_q((\Omega, \Sigma, \mu), \mathbb{R})$ has Fourier type $p = \min(q, q')$ [28]. If X has Fourier type $p \in [1, 2]$ and $p \le q \le p'$, then $\mathcal{F}_{p,N}(X) = \mathcal{F}_{p,N}(X^*) = \mathcal{F}_{p,N}\left(L_q\left(\mathbb{R}^N, X\right)\right)$ for each $N \in \mathbb{N}$ (cf. [14]). Each closed subspace (by definition) and each quotient space (by duality) of a Banach space X has the same Fourier type as X. Each uniformly convex Banach space has some non-trivial Fourier type $p > 1$.

Definition 2.4. Let $1 < q < \infty$ and $m : \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X, Y)$ be a bounded measurable function. Consider the $L_\infty\left(\mathbb{R}^N, Y\right)$ *functions*

$$T_m f \ := \ \left[m\,\widehat{f}\,\right]^{\vee} \ \in L_\infty\left(\mathbb{R}^N, Y\right) \qquad \text{for} \qquad f \in \mathcal{S}\left(\mathbb{R}^N, X\right) \ . \tag{2.2}$$

Then m is a *Fourier multiplier* from $L_q\left(\mathbb{R}^N, X\right)$ to $L_q\left(\mathbb{R}^N, Y\right)$ provided there is a constant C so that

$$\|T_m f\|_{L_q(\mathbb{R}^N,Y)} \leq C\, \|f\|_{L_q(\mathbb{R}^N,X)} \qquad \text{for each} \qquad f \in \mathcal{S}\left(\mathbb{R}^N, X\right) \;;$$

in which case, the *operator* $T_m \in \mathcal{B}\left(L_q\left(\mathbb{R}^N, X\right), L_q\left(\mathbb{R}^N, Y\right)\right)$ uniquely determined by (2.2) is the *Fourier multiplier operator induced by* m.

Since multiplier theorems need not, in general, extend from the scalar case to the Banach space case, the class of UMD Banach spaces is considered. There are several (equivalent) formulations of this geometric property of a Banach space; below is one which is pertinent to the setting here.

Definition 2.5. A Banach space X is a UMD space if and only if the Hilbert transform

$$Hf(t) \;=\; \mathrm{PV} \;\; -\!\!\!\!\!\!\int \frac{f(s)}{t-s}\, ds \qquad , \qquad f \in \mathcal{S}\,(X)$$

extends to a bounded operator on $L_p\,(\mathbb{R}, X)$ for some (and thus then for each) $p \in (1, \infty)$.

Thus X is a UMD space if and only if $m\colon \mathbb{R}\setminus\{0\} \to \mathcal{B}\,(X, X)$ given by $m(t) \;=\; \mathrm{sign}\,(t)\; I_X$ is a Fourier multiplier on $L_p\,(\mathbb{R}, X)$ for some (and thus then for each) $p \in (1, \infty)$.

Remark 2.6. a) $L_q\,((\Omega, \Sigma, \mu)\,, \mathbb{R})$ spaces, where $1 < q < \infty$, are examples of UMD spaces. Closed subspaces of, the dual of, and quotient spaces of a UMD space are UMD spaces. If X has UMD and $1 < q < \infty$, then $L_q(\mathbb{R}^N, X)$ has UMD. (cf., e.g., [1, Thm. 4.5.2]).
b) A UMD space has a uniformly convex renorming. A space with a uniformly convex renorming is reflexive and B-convex. Bourgain [8, 10] has shown that each B-convex Banach space has some non-trivial Fourier type $p > 1$.

The notion (cf. [38]) of R-boundedness, which provides a vector-valued substitute for Kahane's contraction principle, is needed to extend scalar-valued multiplier theorems to operator-valued multiplier theorems.

Notation 2.7. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of $\{-1, +1\}$–valued, independent, symmetric random variables on some probability space (Ω, Σ, μ). For example, one can take an enumeration of the sequence $\{r_n\}_{n\in\mathbb{N}}$ of Rademacher functions where $r_n(u) = \mathrm{sign}\;\sin(2^n\pi u)$ on $[0, 1]$. Let $\{\varepsilon_k\}$ and $\{\varepsilon'_k\}$ be independent copies of $\{-1, +1, \}$–valued, independent, symmetric random variables.

Definition 2.8. Let $\tau \subset \mathcal{B}\,(X, Y)$ and $p \in [1, \infty)$. The number $R_p\,(\tau)$ is the smallest of the constants $R \in [0, \infty]$ with the property that, for each $n \in \mathbb{N}$ and subset $\{T_j\}_{j=1}^n$ of τ and subset $\{x_j\}_{j=1}^n$ of X,

$$\left\| \sum_{j=1}^n \varepsilon_j\,(\cdot)\, T_j(x_j) \right\|_{L_p(\Omega, Y)} \;\leq\; R\; \left\| \sum_{j=1}^n \varepsilon_j\,(\cdot)\, x_j \right\|_{L_p(\Omega, X)} \quad .$$

The set τ is **R-bounded** provided $R_p(\tau)$ is finite for some (and thus then, by Kahane's inequality, for each) $p \in [1, \infty)$. In this case, $R_p(\tau)$ is called the **$\mathbf{R}_p$-bound** of τ.

The Banach space notion of $\mathrm{Rad}\,(X)$ provides a convenient way to view R-boundedness.

Definition 2.9. Let X be a Banach space. Then

$$\mathrm{Rad}\,(X) := \{\, \{x_j\}_{j\in\mathbb{Z}} \in X^{\mathbb{Z}} \colon \sum\nolimits_{j=-n}^{n} \varepsilon_j\,(\cdot)\, x_j \colon \Omega \to X \text{ is convergent in } L_2(\Omega, X) \,\}.$$

When equipped with one of the following equivalent norms, where $1 \leq p < \infty$:

$$\|\{x_j\}_{j\in\mathbb{Z}}\|_{\mathrm{Rad}_p(X)} \ := \ \left\| \sum\nolimits_{j\in\mathbb{Z}} \varepsilon_j\,(\cdot)\, x_j \right\|_{L_p(\Omega,X)} \ ,$$

$\mathrm{Rad}_p\,(X)$ is a Banach space. When confusion seems unlikely, $\mathrm{Rad}_p\,(X)$ is denoted by just $\mathrm{Rad}\,(X)$. Much can be found about $\mathrm{Rad}\,(X)$ in the literature (see, e.g. [12]).

Remark 2.10. a) A sequence $\{T_j\}_{j\in\mathbb{Z}}$ from $\mathcal{B}\,(X, Y)$ is R-bounded if and only if the mapping

$$\mathrm{Rad}_p\,(X) \ni \{x_j\}_{j\in\mathbb{Z}} \quad \xrightarrow{\ \widetilde{T}\ } \quad \{T_j x_j\}_{j\in\mathbb{Z}} \in \mathrm{Rad}_p\,(Y)$$

defines an element in $\mathcal{B}\,(\mathrm{Rad}_p(X), \mathrm{Rad}_p(Y))$ for some (or equivalently, for each) $p \in [1, \infty)$; in which case, $R_p\,(\{T_j\}_{j\in\mathbb{Z}}) \ = \ \|\widetilde{T}\|_{\mathcal{B}(\mathrm{Rad}_p(X),\mathrm{Rad}_p(Y))}$.

b) If X has Fourier type p then so does $\mathrm{Rad}\,(X)$. If X has UMD then so does $\mathrm{Rad}\,(X)$. This follows from Remark 2.3, Remark 2.6, and the fact that $\mathrm{Rad}\,(X)$ is a subspace of $L_2\,(X)$.

c) Note that $\{x_j\}_{j\in\mathbb{Z}} \in \mathrm{Rad}\,(X)$ if and only if the series $\sum_{j=-n}^{j=n} \varepsilon_j\,(\cdot)\, x_j$ converges almost surely. If X does not (isomorphically) contain c_0 and $\sup_n \left\| \sum_{j=-n}^{j=n} \varepsilon_j\,(\cdot)\, x_j \right\|_{L_p(X)}$ is finite for some $p \in [1, \infty)$, then $\{x_j\}_{j\in\mathbb{Z}} \in \mathrm{Rad}\,(X)$.

Besov spaces serve as a tool in this paper (see [15] for further details). Among the many equivalent descriptions of Besov spaces, the most useful one in this context is given in terms of the so-called *Littlewood-Paley decomposition*. Roughly speaking this means that one considers $f \in \mathcal{S}'\,(X)$ as a distributional sum $f = \sum_k f_k$ of analytic functions $f_k := \check{\varphi}_k * f$ whose Fourier transforms have support in *dyadic-like* intervals $\{\mathrm{supp}\,\varphi_k\}_k$ and then defines the Besov norm in terms of the f_k's. Here, $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is a partition of unity chosen as follows.

Notation 2.11. Take a nonnegative function $\psi \in \mathcal{S}\,(\mathbb{R}, \mathbb{R})$ that has support in $[2^{-1}, 2]$ and satisfies $\sum_{k\in\mathbb{Z}} \psi(2^{-k}s) = 1$ for each $s \in \mathbb{R}\backslash\{0\}$. Let $\varphi_k(t) := \psi\left(2^{-k}|t|\right)$ if $k \in \mathbb{N}$ and $\varphi_0(t) := 1 - \sum_{k\in\mathbb{N}} \varphi_k(t)$. Note that $\varphi_k \in \mathcal{S}\left(\mathbb{R}^N, \mathbb{R}\right)$ for each $k \in \mathbb{N}_0$.

Definition 2.12. Let $1 \leq q, r \leq \infty$ and the *smoothness index* $s \in \mathbb{R}$. The *Besov space* $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ is the space of all $f \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ for which

$$\|f\|_{B^s_{q,r}(\mathbb{R}^N,X)} \ := \ \left\| \left\{ 2^{ks}\ (\check{\varphi}_k * f) \right\}_{k=0}^{\infty} \right\|_{\ell_r(L_q(X))} \tag{2.3}$$

is finite. $B^s_{q,r}\left(\mathbb{R}^N, X\right)$, together with the norm in (2.3), is a Banach space.

Different choices of $\{\varphi_k\}$'s lead to equivalent norms on $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ (cf. [29, Lemma 3.2]). Weighted Besov spaces will give a precise estimate for the Fourier transform on Besov spaces.

Definition 2.13. Let $1 \le q \le \infty$ and $A = \{a_k\}_{k\in\mathbb{N}_0}$ be a sequence of non-negative real numbers. Then $B^A_q\left(\mathbb{R}^N, X\right)$ is the space of all $f \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ for which $\check{\varphi}_k * f \in L_q\left(\mathbb{R}^N, X\right)$ for each $k \in \mathbb{N}_0$ and

$$\|f\|_{B^A_q(\mathbb{R}^N,X)} := \sum_{k\in\mathbb{N}_0} a_k \|\check{\varphi}_k * f\|_{L_q(\mathbb{R}^N,X)} \tag{2.4}$$

is finite. $B^A_q\left(\mathbb{R}^N, X\right)$, endowed with the norm in (2.4), is normed linear space.

Other function spaces are also considered in this paper.

Definition 2.14. Let $1 \le q \le \infty$ and $m \in \mathbb{N}_0$. The *Sobolev space* $W^m_q\left(\mathbb{R}^N, X\right)$ is

$$W^m_q\left(\mathbb{R}^N, X\right) := \left\{f \in \mathcal{S}'\left(\mathbb{R}^N, X\right) : D^\alpha f \in L_q\left(\mathbb{R}^N, X\right) \text{ for each } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \le m\right\},$$

equipped with the norm

$$\|f\|_{W^m_q(\mathbb{R}^N,X)} := \sum_{0\le|\alpha|\le m} \|D^\alpha f\|_{L_q(\mathbb{R}^N,X)} .$$

It is well-known that the Sobolev spaces are Banach spaces. For more information regarding Besov and Sobolev spaces, see [2, 14, 29, 30].

Definition 2.15. Let $1 < q < \infty$ and $s \in \mathbb{R}$. The *Bessel potential* $\mathcal{J}^s : \mathcal{S}\left(\mathbb{R}^N, X\right) \to \mathcal{S}\left(\mathbb{R}^N, X\right)$ is given by $\mathcal{J}^s f := \left[\left(1 + |\cdot|^2\right)^{s/2} \widehat{f}(\cdot)\right]^\vee$. The *fractional Sobolev space* $H^s_q\left(\mathbb{R}^N, X\right)$ is the completion of $\mathcal{S}\left(\mathbb{R}^N, X\right)$ with respect to the norm $\|f\|_{H^s_q(\mathbb{R}^N,X)} := \|\mathcal{J}^s f\|_{L_q(\mathbb{R}^N,X)}$.

The Banach spaces $H^s_q\left(\mathbb{R}^N, X\right)$ are also called *Liouville spaces* and *Bessel potential spaces.* If X has UMD then $H^n_q\left(\mathbb{R}^N, X\right) = W^n_q\left(\mathbb{R}^N, X\right)$ for $q \in (1, \infty)$ and $n \in \mathbb{N}$ (see [34, (15.55)]).

Definition 2.16. Let X be a Banach space.
a) For a locally integrable function $f : \mathbb{R}^N \to X$ let

$$\|f\|_{\mathrm{BMO}(X)} := \sup \frac{1}{|Q|} \int_Q \|f(t) - f_Q\|_X \, dt \tag{2.5}$$

where the sup is taken over all cubes of $\mathbb{R}^N$ and $f_Q = |Q|^{-1} \int_Q f(t)\, dt$. $\mathrm{BMO}(X)$ is the space of such functions f for which $\|f\|_{\mathrm{BMO}(X)}$ is finite, endowed with the seminorm given by (2.5).
b) A function $a \in L_\infty\left(\mathbb{R}^N, X\right)$ is an *atom* if a is supported in a cube Q and $\int a(t)\, dt = 0$ and $\|a\|_{L_\infty} \le |Q|^{-1}$. The Hardy space $H_1\left(\mathbb{R}^N, X\right)$ is the space of all $f \in L_1\left(\mathbb{R}^N, X\right)$ which can be represented as $f = \sum_n \lambda_n a_n$ where $\{\lambda_n\}_n \in \ell_1$ and each a_n is an atom. The norm $\|f\|_{H_1}$ is the

infimum of $\sum_n |\lambda_n|$ over all such representations.

c) The weak-L_1 space $L_1^{\mathrm{wk}}\left(\mathbb{R}^N, X\right)$ consists of all measurable functions $f\colon \mathbb{R}^N \to X$ that satisfy

$$\|f\|_{L_1^{\mathrm{wk}}(\mathbb{R}^N,X)} \;:=\; \sup_{\lambda>0} \lambda\, \mu\left(\left\{t \in \mathbb{R}^N\colon\; \|f(t)\|_X > \lambda\right\}\right) \;<\; \infty\,. \tag{2.6}$$

It is well-known that the expression $\|\cdot\|_{L_1^{\mathrm{wk}}}$ in (2.6) is a quasi-norm on $L_1^{\mathrm{wk}}\left(\mathbb{R}^N, X\right)$ with

$$\|f+g\|_{L_1^{\mathrm{wk}}(\mathbb{R}^N,X)} \;\le\; 2\left[\|f\|_{L_1^{\mathrm{wk}}(\mathbb{R}^N,X)} \;+\; \|g\|_{L_1^{\mathrm{wk}}(\mathbb{R}^N,X)}\right]\,.$$

The balls with respect to $\|\cdot\|_{L_1^{\mathrm{wk}}}$ define a linear topology on $L_1^{\mathrm{wk}}\left(\mathbb{R}^N, X\right)$ and $L_1^{\mathrm{wk}}\left(\mathbb{R}^N, X\right)$, endowed with this topology, is a quasi-Banach space.

3. STEPS TOWARDS MULTIPLIER THEOREMS

Bourgain [9, N=1], and in the higher dimensional case McConnell [27] and Zimmermann [40], showed a generalization of the Mihlin's multiplier theorem: Theorem 3.2. The following notation simplifies the statements of their result and results to follow.

Notation 3.1. Let $\mathcal{M}_l^N(X)$ be the set of all measurable functions $m\colon \mathbb{R}^N \setminus \{0\} \to X$ whose distributional derivatives $D^\alpha m$ are represented by measurable functions for each $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \le l$ and

$$\|m\|_{\mathcal{M}_l^N(X)} \;:=\; \sup\left\{|t|^{|\alpha|}\,\|D^\alpha m(t)\|_X : t \in \mathbb{R}^N \setminus \{0\}\;,\; \alpha \in \mathbb{N}_0^N\;,\; |\alpha| \le l\right\} \;< \infty$$

where $l \in \mathbb{N}_0$.

Theorem 3.2 ([9, 27, 40])**.** *Let X be a UMD space and $1 < q < \infty$. If $m \in \mathcal{M}_N^N(\mathbb{C})$ then $m(\cdot)\, I_X$ is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, X)$ with $\|T_m\| \le C_{X,N,q}\, \|m\|_{\mathcal{M}_N^N(\mathbb{C})}$.*

Multiplier theorems such as Theorem 3.2 imply Littlewood-Paley decompositions such as Corollary 3.3. (Recall that $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ was defined in Notation 2.7).

Corollary 3.3. *Let X be a UMD space and $1 < q < \infty$. Let $\vartheta_0 \in \mathcal{S}\left(\mathbb{R}^N, \mathbb{R}\right)$ be a nonnegative function that satisfies, for some $n \in \mathbb{N}$,*

$$\mathit{supp}\,\vartheta_0 \subset \left\{t \in \mathbb{R}^N : 2^l \le |t| \le 2^L\right\} \qquad \text{for some } l, L \in \mathbb{Z}$$

$$\sum\nolimits_{k\in\mathbb{Z}} \vartheta_k(t) = 1 \quad \text{for each } t \in \mathbb{R}^N \setminus \{0\} \qquad \text{where } \vartheta_k(t) := \vartheta_0\left(2^{-nk} t\right)\,.$$

Then there is a constant $C = C_{X,N,q,\vartheta_0}$ so that

$$\frac{1}{C}\,\|f\|_{L_q(\mathbb{R}^N,X)} \;\le\; \mathbb{E}_\varepsilon \left\|\sum_{k\in\mathbb{Z}} \varepsilon_k\left(\check{\vartheta}_k * f\right)\right\|_{L_q(\mathbb{R}^N,X)} \;\le\; C\,\|f\|_{L_q(\mathbb{R}^N,X)} \tag{3.1}$$

for each $f \in L_q\left(\mathbb{R}^N, X\right)$.

Proof. For $K \in \mathbb{N}$ and $u \in \Omega$, let

$$m_{u,K}(\cdot) \;:=\; \sum_{k=-K}^{K} \varepsilon_k(u)\vartheta_k(\cdot) \quad : \quad \mathbb{R}^N \setminus \{0\} \to \mathbb{C}\ . \tag{3.2}$$

Note that, for a fixed $\cdot \in \mathbb{R}^N \setminus \{0\}$, there are at most $\frac{L-l}{n}+1$ non-zero summands in (3.2). Thus $m_{u,K} \in \mathcal{M}_N^N(\mathbb{C})$ with

$$\|m_{u,K}\|_{\mathcal{M}_N^N(\mathbb{C})} \;\le\; \left(\frac{L-l}{n}+1\right)\ \|\vartheta_0\|_{\mathcal{M}_N^N(\mathbb{C})}\ .$$

So by Theorem 3.2, there is a constant $C_{X,N,q}$ so that

$$\sup_{\substack{u\in\Omega\\ K\in\mathbb{N}}} \left\| \sum_{k=-K}^{K} \varepsilon_k(u)\left(\check{\vartheta}_k * f\right) \right\|_{L_q(X)} \;\le\; C_{X,N,q}\ \left(\frac{L-l}{n}+1\right)\ \|\vartheta_0\|_{\mathcal{M}_N^N(\mathbb{C})}\ \|f\|_{L_q(X)} \tag{3.3}$$

for $f \in \mathcal{S}(X)$. This gives the desired upper estimate in (3.1).

As usual one obtains the lower estimate from (3.3) and the corresponding inequality for X^*. Since $\operatorname{supp}\vartheta_k$ and $\operatorname{supp}\vartheta_{k+j}$ can overlap only for $|j| < \frac{L-l}{n}$, for each $f \in \mathcal{S}_o(X)$ and $g \in \mathcal{S}_o(X^*)$

$$\begin{aligned}
\langle g, f\rangle_{L_q(X)} &= \sum_{|j|<\frac{L-l}{n}} \sum_{k\in\mathbb{Z}} \left\langle \check{\vartheta}_{k+j} * g\ ,\ \check{\vartheta}_k * f \right\rangle_{L_q(X)} \\
&= \sum_{|j|<\frac{L-l}{n}} \mathbb{E}\left(\left\langle \sum_{m\in\mathbb{Z}} \varepsilon_m\left(\check{\vartheta}_{m+j} * g\right)\ ,\ \sum_{k\in\mathbb{Z}} \varepsilon_k\left(\check{\vartheta}_k * f\right) \right\rangle_{L_q(X)} \right) \\
&\le \sum_{|j|<\frac{L-l}{n}} \left(\mathbb{E} \left\| \sum_{m\in\mathbb{Z}} \varepsilon_m\left(\check{\vartheta}_{m+j} * g\right) \right\|^2_{L_{q'}(X^*)} \right)^{1/2} \left(\mathbb{E} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k\left(\check{\vartheta}_k * f\right) \right\|^2_{L_q(X)} \right)^{1/2} \\
&\le \widetilde{C}_{X,N,q,\vartheta_0}\ \left(\frac{2(L-l)}{n}+1\right)\ \|g\|_{L_{q'}(X^*)}\ \mathbb{E} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k\left(\check{\vartheta}_k * f\right) \right\|_{L_q(X)}
\end{aligned}$$

by Kahane's inequality and (3.3) applied to $g \in L_{q'}(X^*)$. Supping over all $g \in \mathcal{S}_o(X^*)$ with $\|g\|_{L_{q'}(X^*)} \le 1$ gives the lower estimate in (3.1). ■

Notation 3.4. There is (cf., eg. [6, Lemma 6.1.7]) a nonnegative function $\phi_0 \in \mathcal{S}\left(\mathbb{R}^N, \mathbb{R}\right)$ satisfying

$$\operatorname{supp}\phi_0 \;\subset\; \left\{t \in \mathbb{R}^N : 2^{-1} \le |t| \le 2^1\right\}$$

$$\textstyle\sum_{k\in\mathbb{Z}} \phi_k(t) = 1 \quad \text{for each } t \in \mathbb{R}^N \setminus \{0\} \qquad \text{where } \phi_k(t) := \phi_0\left(2^{-k}t\right)\ .$$

Thus one can take $\vartheta_0 \equiv \phi_0$ in Corollary 3.3.

Note that

$$\operatorname{supp}\phi_k \subset \left\{t \in \mathbb{R}^N : 2^{k-1} \le |t| \le 2^{k+1}\right\}\ ;$$

thus, $\operatorname{supp}\phi_k$ and $\operatorname{supp}\phi_{k+j}$ can overlap only for $j \in \{-1,0,1\}$. So fix $j \in \{-1,0,1\}$ and let

$$\begin{aligned} \psi_0 \ &:= \ \phi_{j-1} + \phi_j + \phi_{j+1} \\ \psi_k(\cdot) \ &:= \ \psi_0\left(2^{-3k}\ \cdot\right) \qquad \text{for each } k \in \mathbb{Z} \ . \end{aligned}$$

Then

$$\operatorname{supp}\psi_0 \subset \left\{t \in \mathbb{R}^N \colon 2^{j-2} \le |t| \le 2^{j+2}\right\}$$
$$\psi_k \ = \ \phi_{j+3k-1} \ + \ \phi_{j+3k} \ + \ \phi_{j+3k+1} \qquad \text{for each } k \in \mathbb{Z} \ .$$

Thus one can also take $\vartheta_0 \equiv \psi_0$ in Corollary 3.3. Note that if

$$\operatorname{supp} h \subset \left\{t \in \mathbb{R}^N \colon 2^{j+3n-1} \le |t| \le 2^{j+3n+1}\right\} \ ,$$

then

$$h\ \psi_k = \begin{cases} h & \text{if } k = n \\ 0 & \text{if } k \ne n \end{cases}$$

for each $k \in \mathbb{Z}$.

The next lemma transfers [9, Lemma 10] from $\mathbb{T}$ to $\mathbb{R}^N$.

Lemma 3.5. *Let X be a UMD space and $1 < p < \infty$. Consider translations $\left\{\tau_{\theta_j s}\right\}_{j=1}^m$,*

$$L_p\left(\mathbb{R}^N, X\right) \ \ni \ f \quad \xrightarrow{\tau_{\theta_j s}} \quad \left(\tau_{\theta_j s} f\right)(\cdot) \ := \ f\left(\cdot \ + \ \theta_j s\right) \ \in L_p\left(\mathbb{R}^N, X\right) \ ,$$

for $\theta_j \in \mathbb{R}$ and a fixed unit (in Euclidean norm) vector $s \in \mathbb{R}^N$. Let $f_j \in L_p\left(\mathbb{R}^N, X\right)$ satisfy

$$\mathit{supp}\, \widehat{f_j} \ \subset \ B_N(r_j) \ := \ \left\{(v_1, \ldots, v_N) \in \mathbb{R}^N \colon \ \left[\sum\nolimits_{j=1}^N |v_j|^2\right]^{1/2} \ \le \ r_j\right\} \ .$$

Assume that $|\theta_j| < \frac{K}{r_j}$ for a fixed constant $K > 2$ and $\frac{r_{j+1}}{r_j} \ge 2$. Then

$$\mathbb{E}\ \left\|\sum\nolimits_{j=1}^m \varepsilon_j \tau_{\theta_j s} f_j\right\|_{L_p(\mathbb{R}^N, X)} \ \le \ C_{X,N,p}\ (\ln K)\ \mathbb{E}\ \left\|\sum\nolimits_{j=1}^m \varepsilon_j f_j\right\|_{L_p(\mathbb{R}^N, X)} \ .$$

Proof. Without loss of generality, $\{f_j\}_{j=1}^m \subset \mathcal{S}\left(\mathbb{R}^N, X\right)$. Indeed, for each $f \in L_p\left(\mathbb{R}^N, X\right)$ with $\operatorname{supp}\widehat{f} \subset B_N(r)$ there is a sequence $g_n \in \mathcal{S}\left(\mathbb{R}^N, X\right)$, with $g_n \to f$ in L_p-norm, along with $\varphi \in \mathcal{S}\left(\mathbb{R}^N, \mathbb{C}\right)$ such that $\operatorname{supp}\varphi \subset B_N(r+\varepsilon)$ and $\varphi \equiv 1$ on $B_N(r)$; thus, $\check{\varphi} * g_n \to f$ in L_p-norm and $\operatorname{supp}\left(\check{\varphi} * g_n\right)\widehat{\ } \subset B_N(r+\varepsilon)$.

<u>Step 1</u>: reduction to the one-dimensional case.

Since the $L_p\left(\mathbb{R}^N, X\right)$-norm is invariant under rotations, it suffices to take $s = e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^N$. If $N > 1$, then define $F_j \in L_p\left(\mathbb{R}, L_p\left(\mathbb{R}^{N-1}, X\right)\right) \cap L_1\left(\mathbb{R}, L_1\left(\mathbb{R}^{N-1}, X\right)\right)$ by

$$F_j(t)(u) = f_j(t, u) \qquad \text{for} \quad u \in \mathbb{R}^{N-1} \qquad \text{and a.e.} \quad t \in \mathbb{R} \ .$$

Then $\operatorname{supp}\widehat{F_j} \subset [-r_j, r_j]$ by Fact 2.1. If Lemma 3.5 holds in the one-dimensional case, then, with the help of Remark 2.6a,

$$\begin{aligned}\mathbb{E}\left\|\sum\nolimits_j \varepsilon_j \tau_{\theta_j e_1} f_j\right\|_{L_p(\mathbb{R}^N,X)} &= \mathbb{E}\left\|\sum\nolimits_j \varepsilon_j \tau_{\theta_j} F_j\right\|_{L_p(\mathbb{R},L_p(\mathbb{R}^{N-1},X))} \\ &\le C_{X,N,p}\,(\ln K)\ \mathbb{E}\left\|\sum\nolimits_j \varepsilon_j F_j\right\|_{L_p(\mathbb{R},L_p(\mathbb{R}^{N-1},X))} \\ &= C_{X,N,p}\,(\ln K)\ \mathbb{E}\left\|\sum\nolimits_j \varepsilon_j f_j\right\|_{L_p(\mathbb{R}^N,X)} .\end{aligned}$$

Thus, without loss of generality, $N = 1$ and $s = e_1 \in \mathbb{R}^1$.

<u>STEP 2</u>: THE TRANSFERENCE.

For $f \in \mathcal{S}(\mathbb{R}, X)$ and $\varepsilon > 0$, define the 2π-periodic function

$$F_\varepsilon(s) := \sum_{n\in\mathbb{Z}} \frac{1}{\varepsilon} f\left(\frac{s+2\pi n}{\varepsilon}\right) . \tag{3.4}$$

By the Poisson summation formula (cf. eg. [20])

$$F_\varepsilon(s) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \widehat{f}(\varepsilon n)\, e^{ins} , \tag{3.5}$$

i.e., the n^{th} Fourier coefficient $\widehat{F_\varepsilon}(n)$ of F_ε with respect to the discrete Fourier transform on $L_1(-\pi,\pi)$ is $\frac{1}{2\pi}\widehat{f}(\varepsilon n)$.

For a fixed $\theta \in \mathbb{R}$

$$\int_{-\pi}^{\pi} \left\|\varepsilon^{1-\frac{1}{p}} F_\varepsilon(s+\varepsilon\theta)\right\|_X^p ds \xrightarrow{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \|f(t+\theta)\|_X^p\, dt. \tag{3.6}$$

To see (3.6), first note that, by (3.4),

$$\int_{-\pi}^{\pi} \left\|\varepsilon^{1-\frac{1}{p}} F_\varepsilon(s+\varepsilon\theta)\right\|_X^p ds = \tag{3.7}$$

$$\int_{-\pi}^{\pi} \left\|\varepsilon^{-\frac{1}{p}} f\left(\frac{s+\varepsilon\theta}{\varepsilon}\right) + \sum\nolimits_{0\neq n\in\mathbb{Z}} \varepsilon^{-\frac{1}{p}} f\left(\frac{s+\varepsilon\theta+2\pi n}{\varepsilon}\right)\right\|_X^p ds .$$

For the $n = 0$ term in (3.7)

$$\int_{-\pi}^{\pi} \left\|\varepsilon^{-\frac{1}{p}} f\left(\frac{s+\varepsilon\theta}{\varepsilon}\right)\right\|_X^p ds = \int_{-\frac{\pi}{\varepsilon}}^{+\frac{\pi}{\varepsilon}} \|f(t+\theta)\|_X^p\, dt \xrightarrow{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \|f(t+\theta)\|_X^p\, dt.$$

Next estimate the remaining terms in (3.7) for $\varepsilon|\theta| < \pi/2$ using the fact that, since $f \in \mathcal{S}(X)$, there is a constant C so that $\|f(t)\|_X \le C|t|^{-2}$:

$$\begin{aligned}\left[\int_{-\pi}^{\pi} \left\|\varepsilon^{-1/p} \sum_{0\neq n\in\mathbb{Z}} f\left(\frac{s+2\pi n+\varepsilon\theta}{\varepsilon}\right)\right\|_X^p ds\right]^{1/p} &\le C\,\varepsilon^{2-\frac{1}{p}} \sum_{0\neq n\in\mathbb{Z}} \left[\int_{-\pi}^{\pi} |s+2\pi n+\varepsilon\theta|^{-2p}\, ds\right]^{1/p} \\ &\le C\,\varepsilon^{2-\frac{1}{p}}\,(2\pi)^{1/p}\,\frac{2}{\pi^2} \sum_{n=1}^{\infty} \left(2n-\frac{3}{2}\right)^{-2} \\ &\xrightarrow{\varepsilon \searrow 0} 0 .\end{aligned}$$

Thus (3.6) holds. Furthermore (3.6) clearly extends to linear combinations:

$$\left\| \sum_{j=1}^{m} \varepsilon_j \, \varepsilon^{1-\frac{1}{p}} \, (F_j)_\varepsilon \, (\cdot + \varepsilon\theta_j) \right\|_{L_p(\mathbb{T},X)} \xrightarrow{\ \varepsilon \searrow 0\ } \left\| \sum_{j=1}^{m} \varepsilon_j \, f_j \, (\cdot + \theta_j) \right\|_{L_p(\mathbb{R},X)}$$

for each choice $\{\varepsilon_j\}_{j=1}^m$ of signs ± 1. Thus

$$\mathbb{E} \left\| \sum_{j=1}^{m} \varepsilon_j \varepsilon^{1-\frac{1}{p}} \, (F_j)_\varepsilon \, (\cdot + \varepsilon\theta_j) \right\|_{L_p(\mathbb{T},X)} \xrightarrow{\ \varepsilon \searrow 0\ } \mathbb{E} \left\| \sum_{j=1}^{m} \varepsilon_j f_j \, (\cdot + \theta_j) \right\|_{L_p(\mathbb{R},X)} . \tag{3.8}$$

If $\operatorname{supp} \widehat{f_j} \subset [-r_j, r_j]$, then by (3.5), $\operatorname{supp} \left((F_j)_\varepsilon\right)^\frown \subset \left[-\frac{r_j}{\varepsilon}, \frac{r_j}{\varepsilon}\right]$. If $|\theta_j| < \frac{K}{r_j}$ and $\frac{r_{j+1}}{r_j} \geq 2$, then, for ε small enough, there is a strictly increasing sequence $\{n_j\}_{j=1}^m$ of positive integers so that $2^{n_j - 1} < \frac{r_j}{\varepsilon} \leq 2^{n_j}$ and thus $|\varepsilon\theta_j| < \frac{K}{r_j/\varepsilon} \leq \frac{2K}{2^{n_j}}$. Hence by [9, Lemma 10] applied to $\left\{(F_j)_\varepsilon\right\} \in L_p(\mathbb{T}, X)$ and $\{\varepsilon\theta_j\}$,

$$\mathbb{E} \left\| \sum_{j=1}^{m} \varepsilon_j \varepsilon^{1-\frac{1}{p}} \, (F_j)_\varepsilon \, (\cdot + \varepsilon\theta_j) \right\|_{L_p(\mathbb{T},X)} \leq C \, (\ln K) \; \mathbb{E} \left\| \sum_{j=1}^{m} \varepsilon_j \varepsilon^{1-\frac{1}{p}} \, (F_j)_\varepsilon \right\|_{L_p(\mathbb{T},X)} \tag{3.9}$$

for ε small enough. Letting $\varepsilon \searrow 0$ in (3.8) and (3.9) finishes the proof. ■

Lemma 3.5 leads to the following corollary to Theorem 3.2.

Proposition 3.6. *Let X be a UMD space, $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence from $\mathbb{C}$ with $|a_k| \leq 1$, and $1 < q < \infty$. Fix $h \in \mathcal{M}_N^N(\mathbb{C})$.*

(a) *Assume that $\operatorname{supp} h \subset \left\{t \in \mathbb{R}^N : b^{-d} \leq |t| \leq b^d\right\}$ with some $b > 1$ and $d \in \mathbb{N}$. Then*

$$n(t) \ := \ \sum_{k\in\mathbb{Z}} a_k h\left(b^{-k} t\right)$$

is a Fourier multiplier on $L_q\left(\mathbb{R}^N, X\right)$ with $\|T_n\| \leq C_{X,N,q} \; d \; \|h\|_{\mathcal{M}_N^N(\mathbb{C})}$.

(b) *Assume that $\operatorname{supp} h \subset \left\{t \in \mathbb{R}^N : 2^{-1} \leq |t| \leq 2\right\}$ and $s \in \mathbb{R}^N$. Then*

$$m_s(t) \ := \ \sum_{k\in\mathbb{Z}} a_k \exp\left(is \cdot 2^{-k}\, t\right) h\left(2^{-k} t\right)$$

is a Fourier multiplier on $L_q\left(\mathbb{R}^N, X\right)$ with $\|T_{m_s}\| \leq C_{X,N,q} \; \ln(2 + |s|) \; \|h\|_{\mathcal{M}_N^N(\mathbb{C})}$.

Proof. Throughout this proof, the C_i's are constants that depend on at most: X, N, and q. To simplify notation, let $\mathcal{M} := \mathcal{M}_N^N(\mathbb{C})$. Note that $h(a\,\cdot) \in \mathcal{M}$ and $\|h(a\,\cdot)\|_{\mathcal{M}} = \|h\|_{\mathcal{M}}$ for each $a > 0$.

Part (a) follows easily from Theorem 3.2: indeed, the support of $h\left(b^{-k}\cdot\right)$ overlaps with the support of $h\left(b^{-m}\cdot\right)$ only if $|m - k| < 2d$; hence, $n \in \mathcal{M}$ and $\|n\|_{\mathcal{M}} \leq 4d\,\|h\|_{\mathcal{M}}$.

Now to show part (b). Note that the function $h_s(\circ) := \exp(is \cdot \circ)\, h(\circ)$ is in $\mathcal{M}$ with $\|h_s\|_{\mathcal{M}} \leq C_1 \left[\max\left\{1, |s|^N\right\}\right] \|h\|_{\mathcal{M}}$ for some constant C_1 that is *independent* of s. So the desired conclusion for $|s| \leq 1$ follows from part (a). Thus assume that $|s| > 1$.

Fix $f \in \mathcal{S}_o\left(\mathbb{R}^N, X\right)$. Then

$$T_{m_s} f \;=\; \sum_{k\in\mathbb{Z}} a_k\, \tau_k\left(\check{h}_k * f\right)$$

where

$$L_q\left(\mathbb{R}^N, X\right) \;\ni\; g \quad \xrightarrow{\tau_k} \quad (\tau_k g)(\cdot) \;:=\; g\Big(\cdot \;+\; 2^{-k}\, s\Big) \;\in L_q\left(\mathbb{R}^N, X\right) \;,$$

and

$$h_k(\cdot) \;:=\; h\left(2^{-k}\,\cdot\right) \;,$$

which is a Fourier multiplier on $L_q\left(\mathbb{R}^N, X\right)$ by part (a). Thus, by the Littlewood-Paley decomposition Corollary 3.3, for an appropriate choice of ϑ_0 (eg., $\vartheta_0 = \psi_0$ of Notation 3.4 so that for then $\vartheta_k h_{3n+j} = \delta_{k,n} h_{3n+j}$),

$$\begin{aligned}
\|T_{m_s} f\|_{L_q(\mathbb{R}^N,X)} \;&\leq\; \sum_{j=-1}^{1} \left\| \sum_{n\in\mathbb{Z}} a_{3n+j}\, \tau_{3n+j}\left(\check{h}_{3n+j} * f\right) \right\|_{L_q(\mathbb{R}^N,X)} \\
&\leq\; C_2 \sum_{j=-1}^{1} \mathbb{E} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k\, a_{3k+j}\, \tau_{3k+j}\left(\check{h}_{3k+j} * f\right) \right\|_{L_q(\mathbb{R}^N,X)} \;.
\end{aligned}$$

Thus, by Kahane's contraction principle, Lemma 3.5, and then part (a),

$$\begin{aligned}
\|T_{m_s} f\|_{L_q(\mathbb{R}^N,X)} \;&\leq\; 6C_2\, \mathbb{E} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k\, \tau_k\left(\check{h}_k * f\right) \right\|_{L_q(\mathbb{R}^N,X)} \\
&\leq\; C_3\, \ln\left(2\,|s|\right)\, \mathbb{E} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k\, \left(\check{h}_k * f\right) \right\|_{L_q(\mathbb{R}^N,X)} \\
&\leq\; C_4\, \ln\left(2 + |s|\right)\, \|h\|_{\mathcal{M}}\, \|f\|_{L_q(\mathbb{R}^N,X)} \;.
\end{aligned}$$

This completes the proof of Proposition 3.6. ∎

The logarithmic estimate in Proposition 3.6b enters the proof of the next result, which is a centerpiece of the proof of Theorem 4.1.

Proposition 3.7. *Let X have UMD and $1 < q < \infty$. Let $k : \mathbb{R}^N \to \mathcal{B}(X,Y)$ be a strongly integrable function such that*

$$\mathit{supp}\, \widehat{k} \;\subset\; \mathit{supp}\, \phi_0 \tag{3.10}$$

and

$$\int_{\mathbb{R}^N} \|[k(s)]^* g\|_{L_{q'}(\mathbb{R}^N,X^*)}\, w(s)\, ds \;\leq A\; \|g\|_{L_{q'}(\mathbb{R}^N,Y^*)} \qquad \text{for each} \quad g \in L_{q'}\left(\mathbb{R}^N, Y^*\right) \tag{3.11}$$

where $w(\cdot) := \ln\left(2 + |\cdot|\right)$. Set $k_j(\cdot) := 2^{Nj} k\left(2^j\, \cdot\right)$.

(a) *For each finitely supported scalar sequence $\{a_j\}_{j\in\mathbb{Z}}$ with $|a_j| \leq 1$ and*

$$L(\cdot) \;:=\; \sum_{j\in\mathbb{Z}} a_j k_j(\cdot) \quad : \quad \mathbb{R}^N \to \mathcal{B}(X,Y) \tag{3.12}$$

the operator

$$Tf \ := \ L * f \qquad \textit{for} \qquad f \in \mathcal{S}\left(\mathbb{R}^N, X\right)$$

extends to a bounded operator $T : L_q\left(\mathbb{R}^N, X\right) \to L_q\left(\mathbb{R}^N, Y\right)$ *with* $\|T\| \leq C_{X,N,q}\, A$.

(b) *For each* $f \in \mathcal{S}\left(\mathbb{R}^N, X\right)$ *and* $n \in \mathbb{N}$

$$\mathbb{E} \left\| \sum_{j=-n}^{n} \varepsilon_j k_j * f \right\|_{L_q(\mathbb{R}^N, Y)} \ \leq \ C_{X,N,q}\, A \, \|f\|_{L_q(\mathbb{R}^N, X)} \ . \tag{3.13}$$

Also, if Y *does not contain* c_0*, then the summand in* (3.13) *can be taken over* $j \in Z$.

Proof. Recall that the ϕ_k's were defined in Notation 3.4.

Since k is strongly integrable, $f \to k_j * f$ is a bounded operator from $L_1\left(\mathbb{R}^N, X\right)$ to $L_1\left(\mathbb{R}^N, Y\right)$. Fix $q \in (1, \infty)$.

Fix $f = \sum_{i=1}^m x_i g_i$ with $x_i \in X$ and $g_i \in \mathcal{S}\left(\mathbb{R}^N, \mathbb{C}\right)$ and $m \in \mathbb{N}$. Let $h_j := \phi_{j-1} + \phi_j + \phi_{j+1}$; note that $h_j = 1$ on $\operatorname{supp} \phi_j$. Put $f_j := \check{h}_j * f$. Note that $[k_j * f_j]^\wedge = \widehat{k}_j \, h_j \, \widehat{f} = \widehat{k}_j \, \widehat{f} = [k_j * f]^\wedge$ by (3.10). Thus, for a.e. $t \in \mathbb{R}^N$,

$$\begin{aligned} (L * f)(t) \ &= \ \sum_{j \in \mathbb{Z}} a_j \left(k_j * f\right)(t) && = \ \sum_{j \in \mathbb{Z}} a_j \int_{\mathbb{R}^N} 2^{Nj} k(2^j s) f_j(t - s) ds \\ &= \ \int_{\mathbb{R}^N} k(s) \sum_{j \in \mathbb{Z}} a_j \, f_j(t - 2^{-j} s) ds && = \ \int_{\mathbb{R}^N} k(s) \sum_{j \in \mathbb{Z}} a_j \, (\tau_{2^{-j} s} f_j)(t) ds \\ &= \ \int_{\mathbb{R}^N} k(s) (\check{m}_s * f)(t) ds \end{aligned} \tag{3.14}$$

where $\tau_a g(\cdot) := g(\cdot - a)$ and $m_s(t) := \sum_j a_j \, \exp\left(-is \cdot 2^{-j} t\right) \, h_0(2^{-j} t)$. By Proposition 3.6

$$\|\check{m}_s * f\|_{L_q(\mathbb{R}^N, X)} \leq C w(s) \, \|f\|_{L_q(\mathbb{R}^N, X)} \tag{3.15}$$

for some constant $C = C_{X,N,q} \, \|\phi_0\|_{\mathcal{M}_N^N(\mathbb{C})}$. If $g \in L_{q'}\left(\mathbb{R}^N, Y^*\right)$, then by (3.14), (3.15), and (3.11)

$$\begin{aligned} \left|\langle g, L * f \rangle_{L_q(Y)}\right| \ &\leq \ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\ \langle\ g(t), k(s)\, (\check{m}_s * f)(t)\ \rangle_Y |\ ds\, dt \\ &\leq \ \int \|[k(s)]^* \ g\|_{L_{q'}(X^*)} \, \|\check{m}_s * f\|_{L_q(X)} \ ds \ \leq \ C\, A \, \|g\|_{L_{q'}(Y^*)} \, \|f\|_{L_q(X)} \ . \end{aligned} \tag{3.16}$$

Thus $\|L * f\|_{L_q(\mathbb{R}^N, Y)} \leq C A \, \|f\|_{L_q(\mathbb{R}^N, X)}$. Hence part (a) holds.

Part (b) follows formally from part (a). Indeed, for a fixed $n \in \mathbb{N}$ and $u \in \Omega$, define $L_{u,n}(\cdot) := \sum_{j=-n}^{n} \varepsilon_j(u) \, k_j(\cdot)$. Then by part (a)

$$\mathbb{E} \left\| \sum_{j=-n}^{n} \varepsilon_j k_j * f \right\|_{L_q(\mathbb{R}^N, Y)} \ = \ \int_{\mathbb{R}^N} \|L_{u,n} * f\|_{L_q(\mathbb{R}^N, Y)} \ du \ \leq \ C_{X,N,q}\, A \, \|f\|_{L_q(\mathbb{R}^N, X)} \ . \tag{3.17}$$

If Y does not contain c_0, then $L_q\left(\mathbb{R}^N, Y\right)$ does not contain c_0 [25] and so, by Remark 2.10c, the sum in (3.17) can be taken over $j \in \mathbb{Z}$. Thus part (b) holds. ■

Remark 3.8. a) In Proposition 3.7, if (3.11) is replaced by $\int_{\mathbb{R}^N} \|k(s)\|_{\mathcal{B}(X,Y)}\ w(s)ds \leq A$, then this modified version of Proposition 3.7 remains true.

b) In Proposition 3.7, if:

(1) in addition, Y also has UMD,
(2) in addition, the function $k^*(\cdot) := [k(\cdot)]^* : \mathbb{R}^N \to \mathcal{B}(Y^*, X^*)$ is strongly integrable,
(3) the condition (3.11) is replaced by:
  (i) $\int_{\mathbb{R}^N} \|k(s)f\|_{L_r(Y)}\ w(s)\,ds \ \leq\ A\,\|f\|_{L_r(X)}$ for each $f \in L_r\left(\mathbb{R}^N, X\right)$ and $1 < r \leq 2$
  (ii) $\int_{\mathbb{R}^N} \|k^*(s)g\|_{L_r(X^*)}\ w(s)\,ds \ \leq\ A\,\|g\|_{L_r(Y^*)}$ for each $g \in L_r\left(\mathbb{R}^N, Y^*\right)$ and $1 < r \leq 2$,
(4) $C_{X,N,q}$ is replaced by $C_{X,Y,N,q}$,

then this modified version of Proposition 3.7 remains true (for each $q \in (1, \infty)$).

Proof of Remark 3.8. a) Indeed, the calculation in (3.16) can then be replaced by

$$\begin{aligned}\|L * f\|_{L_q(Y)} &= \left\|\int_{\mathbb{R}^N} k\,(s)\,(\check{m}_s * f)\,(\cdot)\ ds\right\|_{L_q(Y)} \leq \int_{\mathbb{R}^N} \|k\,(s)\|_{\mathcal{B}(X,Y)}\,\|\check{m}_s * f\|_{L_q(X)}\ ds \\ &\leq\ C\,\|f\|_{L_q(X)} \int_{\mathbb{R}^N} \|k\,(s)\|_{\mathcal{B}(X,Y)}\,w\,(s)\ ds\ \leq\ CA\,\|f\|_{L_q(X)}\end{aligned}$$

with the help of (3.14) and (3.15).

b) If $q \in [2, \infty)$, then the modified Proposition 3.7 follows from the original Proposition 3.7 since condition (ii) implies (3.11). So fix $q \in (1, 2]$ and consider a function L of the form (3.12). Since X and Y are UMD spaces, they are reflexive. Applying the original Proposition 3.7 to k^* gives, by condition (i), that

$$Kf\ :=\ [L\,(\cdot)]^* * [f\,(-\,\cdot)] \qquad \text{for } f \in \mathcal{S}\left(\mathbb{R}^N, Y^*\right)$$

extends to a bounded operator $K\colon L_{q'}\left(\mathbb{R}^N, Y^*\right) \to L_{q'}\left(\mathbb{R}^N, X^*\right)$ with $\|K\| \leq C_{Y^*,N,q'}\,A$. Since X and Y are reflexive, K^* is in $\mathcal{B}\left(L_q\left(X\right), L_q\left(Y\right)\right)$ and K^* restricted to $\mathcal{S}\left(\mathbb{R}^N, X\right)$ is $L\,(\cdot) * f\,(-\,\cdot)$. ■

In the proof of Theorem 4.1, the assumptions Remark 3.8b will be checked by applying estimates of the operator norm of the Fourier transform on Besov spaces. The Hausdorff-Young inequality holding in spaces with Fourier type gives a starting point for such norm estimates.

Corollary 3.9 ([14])**.** *Let X have Fourier type $p \in [1, 2]$. Then the Fourier transform defines a bounded operator*

$$\mathcal{F}\ :\ B^{N/p}_{p,1}\left(\mathbb{R}^N, X\right)\ \longrightarrow\ L_1\left(\mathbb{R}^N, X\right)\ . \tag{3.18}$$

Furthermore, the norm of $\mathcal{F}$ is bounded above by a constant depending only on $\mathcal{F}_{p,N}(X)$.

A sharping of this result, in the spirit of the logarithmic estimate in Proposition 3.6, is needed in this paper. Recall that B^A_p was given in Definition 2.13 and $\{\varphi_k\}_{k\in\mathbb{N}_0}$ was given in Notation 2.11.

Corollary 3.10. *Let $w\colon \mathbb{R}^N \to [0,\infty)$ be a measurable function so that*

$$a_k \;:=\; 2^{Nk/p}\;\left\| w\chi_{supp\,\varphi_k}\right\|_{L_\infty(\mathbb{R}^N,\mathbb{R})} \;<\; \infty \qquad \text{for each} \quad k\in\mathbb{N}_0\ .$$

Set $A := \{a_k\}_{k\in\mathbb{N}_0}$. If X has Fourier type $p\in[1,2]$ then the Fourier transform defines a bounded operator

$$\mathcal{F} \;:\; B_p^A\left(\mathbb{R}^N, X\right) \;\longrightarrow\; L_1\left(\left(\mathbb{R}^N, w\,(t)\ dt\right), X\right)\ . \tag{3.19}$$

Furthermore, the norm of $\mathcal{F}$ is bounded above by a constant depending only on $\mathcal{F}_{p,N}(X)$.

Note that if $w\equiv 1$ in Corollary 3.10, then (3.19) reduces to (3.18). Corollary 3.9 and Corollary 3.10 remain valid if $\mathcal{F}$ is replaced with $\mathcal{F}^{-1}$.

Proof of Corollary 3.10. Let $\{J_k\}_{k\in\mathbb{N}_0}$ be the partitioning of $\mathbb{R}^N$ given by

$$J_k := \left\{t\in\mathbb{R}^N : 2^{k-1} < |t| \le 2^k\right\} \quad \text{for } k\in\mathbb{N} \quad \text{and} \quad J_0 := \left\{t\in\mathbb{R}^N : |t|\le 1\right\}$$

and set $\varphi_{-1}\equiv 0$ and $J_{-1}=\emptyset$ to simplify notation.

Fix $f\in B_p^A\left(\mathbb{R}^N, X\right)$. For each $k\in\mathbb{N}_0$, since $\check{\varphi}_k * f\in L_p\left(\mathbb{R}^N, X\right)$ and X has Fourier type p, one has that $\varphi_k\cdot\widehat{f} \;=\; \mathcal{F}\left(\check{\varphi}_k * f\right) \;\in\; L_{p'}\left(\mathbb{R}^N, X\right)$; in particular, the distributional Fourier transform of f is represented as a measurable function. Thus for each $m\in\mathbb{N}_0$

$$\widehat{f}\cdot\chi_{J_m} \;=\; \sum_{k=m-1}^{m} \varphi_k\,\widehat{f}\,\chi_{J_m}$$

and so (with $a_{-1}:=0$)

$$\begin{aligned}
\left\|\widehat{f}\cdot\chi_{J_m}\cdot w\right\|_{L_1(X)} \;&\le\; \sum_{k=m-1}^{m}\left\|\widehat{f}\varphi_k\left[\frac{1+|\cdot|}{4}\right]^{N/p}\chi_{J_m}\right\|_{L_{p'}(X)}\left\|\left[\frac{1+|\cdot|}{4}\right]^{-N/p}\chi_{J_m}w\chi_{\mathrm{supp}\,\varphi_k}\right\|_{L_p(\mathbb{R})}\\
&\le\; \sum_{k=m-1}^{m}\left\|\left[\frac{1+|\cdot|}{4}\right]^{N/p}\chi_{J_m}\right\|_{L_\infty(\mathbb{R})}\left\|\widehat{f}\varphi_k\right\|_{L_{p'}(X)}\left\|w\chi_{\mathrm{supp}\,\varphi_k}\right\|_{L_\infty(\mathbb{R})}\left\|\left[\frac{1+|\cdot|}{4}\right]^{-N/p}\chi_{J_m}\right\|_{L_p(\mathbb{R})}\\
&\le\; \sum_{k=m-1}^{m}\left(2^k\right)^{N/p}\left\|\widehat{f}\varphi_k\right\|_{L_{p'}(X)}\,a_k\,2^{-Nk/p}\left[\int_{J_m}\left(\frac{1+|t|}{4}\right)^{-N}dt\right]^{1/p}\\
&\le\; C\sum_{k=m-1}^{m} a_k\left\|\widehat{f}\varphi_k\right\|_{L_{p'}(X)} \;\le\; C\,\mathcal{F}_{p,N}(X)\sum_{k=m-1}^{m} a_k\;\left\|\check{\varphi}_k * f\right\|_{L_p(X)}
\end{aligned}$$

for some universal constant C. Thus

$$\int_{\mathbb{R}^N}\left\|\widehat{f}\,(t)\right\|_X\,w\,(t)\ dt \;\le\; 2\,C\,\mathcal{F}_{p,N}(X)\;\|f\|_{B_p^A(\mathbb{R}^N,X)}\ ,$$

as needed. ∎

Let X and Y have Fourier type p and $m\in B_{p,1}^{N/p}\left(\mathbb{R}^N,\mathcal{B}\left(X,Y\right)\right)$. Then Corollary 3.9 implies that $\widehat{m}\in L_1\left(\mathbb{R}^N,\mathcal{B}\left(X,Y\right)\right)$ only if $p=1$ since if $p>1$ then $\mathcal{B}\left(X,Y\right)$ usually does not also have Fourier type p. However, for $p>1$, one can at least conclude from Corollary 3.9 that for each $x \in X$

$$\int_{\mathbb{R}^N} \|\, [m(\cdot)\, x]\widehat{\ }(t)\, \|_Y \; dt \;\le\; \|\mathcal{F}\|_{B^{N/p}_{p,1}(\mathbb{R}^N,Y)\to L_1(\mathbb{R}^N,Y)} \;\|m\|_{B^{N/p}_{p,1}(\mathbb{R}^N,\mathcal{B}(X,Y))} \;\|x\|_X \;.$$

A somewhat stronger statement than this, which is made possible by Corollary 3.10, is needed.

Corollary 3.11. *Let X and Y have Fourier type $p \in (1,2]$ and $1<q\le p$. Let $w\colon \mathbb{R}^N \to [0,\infty)$ be a measurable function so that*

$$a_k \;:=\; 2^{Nk/q} \;\|w \chi_{supp\,\varphi_k}\|_{L_\infty(\mathbb{R}^N,\mathbb{R})} \;<\; \infty \qquad \textit{for each} \quad k \in \mathbb{N}_0 \;.$$

Set $A \;:=\; \{a_k\}_{k\in\mathbb{N}_0}$ and let $m \in B^A_q\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right) \cap L_1\left(\mathbb{R}^N,\mathcal{B}(X,Y)\right)$ have norm M in B^A_q. Then then there is a constant C, depending only on $\mathcal{F}_{q,N}(X)$ and $\mathcal{F}_{q,N}(Y)$, so that

$$\int_{\mathbb{R}^N} \|\, \widehat{m}(t)\,[f(\cdot)]\,\|_{L_r(\mathbb{R}^N,Y)} \; w(t)\; dt \;\le\; C\; M \;\; \|f\|_{L_r(\mathbb{R}^N,X)} \;\; \textit{for each } f \in L_r\left(\mathbb{R}^N, X\right) \tag{3.20}$$

$$\int_{\mathbb{R}^N} \|\, [\widehat{m}(t)]^* \;[g(\cdot)]\,\|_{L_r(\mathbb{R}^N,X^*)} \; w(t)\; dt \;\le\; C\; M \;\; \|g\|_{L_r(\mathbb{R}^N,Y^*)} \;\; \textit{for each } g \in L_r\left(\mathbb{R}^N, Y^*\right) \tag{3.21}$$

for each $1 \le r \le q'$.

Note that if $w \equiv 1$, then $B^A_q(\mathcal{B}(X,Y)) = B^{N/q}_{q,1}(\mathcal{B}(X,Y))$. Also, Corollary 3.11 remains valid if one replaces $\widehat{m}$ with $\check{m}$ in (3.20) and (3.21).

Proof. The first step is to show that the operator

$$L_r\left(\mathbb{R}^N, X\right) \;\ni\; f \xrightarrow{\;T_r\;} \widehat{m}(\cdot)\, f w(\cdot) \;\in\; L_1\left(\mathbb{R}^N, L_r\left(\mathbb{R}^N, Y\right)\right)$$

is bounded for $r=1$ and $r=q'$.

Fix $x \in X$. Note that $m(\cdot)\,x \in B^A_q(Y)$ has norm at most $M\,\|x\|_X$ since $m \in B^A_q(\mathcal{B}(X,Y))$ has norm M and that $[m(\cdot)\,x]\widehat{\ }(t) \;=\; [\widehat{m}(t)]\,(x)$ since $m \in L_1(\mathcal{B}(X,Y))$. Thus by Corollary 3.10 applied to $m(\cdot)\,x$, there is a constant C_1, depending only on $\mathcal{F}_{q,N}(Y)$, so that

$$\int_{\mathbb{R}^N} \|\, [\widehat{m}(t)]\, x\, \|_Y \, w(t)\, dt \;\le\; C_1\, M \;\|x\|_X \;.$$

Thus, for simple functions $f \in L_1(X)$,

$$\int_{\mathbb{R}^N} \|\, [\,\widehat{m}(t)]\, f(\cdot)\,\|_{L_1(Y)}\, w(t)\, dt \;=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \|\, [\widehat{m}(t)]\, f(s)\,\|_Y\, w(t)\, dt\, ds \;\le\; C_1\, M\; \|f\|_{L_1(X)} \;.$$

Thus $\|T_1\| \le C_1 M$.

Let $\overline{X} := L_{q'}\left(\mathbb{R}^N, X\right)$ and $\overline{Y} := L_{q'}\left(\mathbb{R}^N, Y\right)$. Then $\mathcal{F}_{q,N}(X) = \mathcal{F}_{q,N}(X^*) = \mathcal{F}_{q,N}\left(\overline{X}\right)$, and likewise for Y, by Remark 2.3. Extend m to an operator $\mathbb{R}^N \ni t \to \overline{m}(t) \in \mathcal{B}\left(\overline{X},\overline{Y}\right)$ by

$$[\,\overline{m}(t)\;\, f\,]\,(s) \;:=\; [m(t)]\,(f(s)) \quad \text{ for } \; f \in \overline{X}\;,\;\; s \in \mathbb{R}^N \;.$$

Note that $\overline{m} \in B^A_q\left(\mathcal{B}\left(\overline{X},\overline{Y}\right)\right) \cap L_1\left(\mathcal{B}\left(\overline{X},\overline{Y}\right)\right)$ has norm M in B^A_q since for each $t \in \mathbb{R}^N$

$$\|\overline{m}(t)\|_{\mathcal{B}(\overline{X},\overline{Y})} \;=\; \|m(t)\|_{\mathcal{B}(X,Y)} \qquad \text{and} \qquad \|(\check{\varphi}_k * \overline{m})\,(t)\|_{\mathcal{B}(\overline{X},\overline{Y})} \;=\; \|(\check{\varphi}_k * m)\,(t)\|_{\mathcal{B}(X,Y)} \;.$$

Fix a simple function $f \in \overline{X}$. Note $\overline{m}(\cdot) f \in B_q^A\left(\overline{Y}\right)$ has norm at most $M \|f\|_{\overline{X}}$ and

$$[\overline{m}(\cdot) f]^{\widehat{}}(t) \; = \; \widehat{m}(t)[f(\cdot)] \; .$$

Thus by Corollary 3.10 applied to $\overline{m}(\cdot) f$, there is a constant C_2, depending only on $\mathcal{F}_{q,N}\left(\overline{Y}\right)$, so that

$$\int_{\mathbb{R}^N} \|\widehat{m}(t)[f(\cdot)]\|_{\overline{Y}} \, w(t) \, dt \; \leq \; C_2 \; M \; \|f\|_{\overline{X}} \; .$$

Thus $\left\|T_{q'}\right\| \leq C_2 M$.

Thus (3.20) holds for $1 \leq r \leq q'$ by interpolation in the scale of Bochner-spaces (see [6, Thm. 5.1.2]). Claim (3.21) is obtained by applying the same argument to $[m(t)]^* \in \mathcal{B}(Y^*, X^*)$. ■

One more preparatory result is needed for the proof of Theorem 4.1. The proof of the following lemma is due to N.J. Kalton and replaces our original (more complicated) proof. Recall that $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ and $\{\varepsilon'_k\}_{k\in\mathbb{Z}}$ were given in Notation 2.7.

Lemma 3.12. *For any Banach space X and any array $\{x_{kl}\}_{k,l\in\mathbb{Z}}$ from X*

$$\mathbb{E}_{\,\varepsilon} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k x_{kk} \right\|_X \; \leq \; \mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \sum_{k,l\in\mathbb{Z}} \varepsilon_k \varepsilon'_l x_{kl} \right\|_X \; .$$

Proof. For $\alpha_k \in \{-1, 1\}$, the sequences of random variables $\{\varepsilon_k\}$, $\{\varepsilon'_k\}$, $\{\alpha_k \, \varepsilon_k\}$, and $\{\alpha_k \, \varepsilon'_k\}$ have the same distribution. Hence by Fubini's theorem

$$\mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \sum_{k,l\in\mathbb{Z}} \varepsilon_k \varepsilon'_l x_{kl} \right\|_X \; = \; \mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \sum_{k,l\in\mathbb{Z}} \alpha_k \, \alpha_l \, \varepsilon_k \varepsilon'_l x_{kl} \right\|_X \; . \tag{3.22}$$

Let $\{\eta_k\}$ be a sequence of $\{-1, +1\}$–valued, independent, symmetric random variables that is independent of $\{\varepsilon_k\}$ and $\{\varepsilon'_k\}$. Integrating (3.22) gives

$$\begin{aligned} \mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \sum_{k,l\in\mathbb{Z}} \varepsilon_k \varepsilon'_l x_{kl} \right\|_X \; &= \; \mathbb{E}_{\,\varepsilon,\varepsilon'} \, \mathbb{E}_{\,\eta} \left\| \sum_{k,l\in\mathbb{Z}} \eta_k \eta_l \varepsilon_k \varepsilon'_l x_{kl} \right\|_X \; \geq \; \mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \mathbb{E}_{\,\eta} \left(\sum_{k,l\in\mathbb{Z}} \eta_k \eta_l \varepsilon_k \varepsilon'_l x_{kl} \right) \right\|_X \\ &= \; \mathbb{E}_{\,\varepsilon,\varepsilon'} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k \varepsilon'_k x_{kk} \right\|_X \; = \; \mathbb{E}_{\,\varepsilon} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k x_{kk} \right\|_X \; , \end{aligned}$$

again since $u \to \varepsilon_k(u) \cdot \varepsilon'_k(v)$, for each fixed v, is a sequence of independent random variables with the same distribution as $\{\varepsilon_k\}$. ■

4. FOURIER MULTIPLIER THEOREMS

This section explores operator-valued Fourier multiplier operators. The assumptions of the main result, Theorem 4.1, may be somewhat awkward looking. However, with respect to the smoothness of the multiplier function, it is a rather weak assumption that will make it easy to derive, in a unified way, several results modeled after classical conditions, such as of Mihlin and Hörmander, as shown by the corollaries to follow.

Theorem 4.1. *Let X and Y be UMD spaces with Fourier type $p \in (1,2]$ and $1 < q < \infty$. Set $\widetilde{X} := Rad_q(X)$ and $\widetilde{Y} := Rad_q(Y)$. Let $m : \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$ be a measurable function that induces a mapping $M\colon \mathbb{R}^N \to \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)$,*

$$M(s) \ := \ \left\{\phi_0(s)m(2^k s)\right\}_{k\in\mathbb{Z}} \qquad \text{for } s \neq 0 \qquad \text{and} \qquad M(0) := 0 \ ,$$

that satisfies $M \in B_p^A\left(\mathbb{R}^N, \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$ with norm D where $A = \left\{(k\vee 1)\,2^{Nk/p}\right\}_{k\in\mathbb{N}_0}$. Then m is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, Y)$. Furthermore, $\|T_m\|_{L_q\to L_q} \le C\,D$ for some constant C depending on: X, Y, N, p, q, and ϕ_0.

Remark 4.2. a) Recall that each UMD space has Fourier type p for some $p > 1$.
b) Remark 2.10 provides a convenient way to compute the norm of M; indeed, $\|M(s)\|_{\mathcal{B}(\widetilde{X},\widetilde{Y})} = \phi_0(s)R_q\left(\left\{m(2^k s)\colon k\in\mathbb{Z}\right\}\right)$ for each $s \in \mathbb{R}^N$.
c) For the A in Theorem 4.1, there are continuous embeddings

$$W_p^l\left(\mathbb{R}^N, Z\right) \ \subset \ B_{p,r}^s\left(\mathbb{R}^N, Z\right) \ \subset \ B_p^A\left(\mathbb{R}^N, Z\right) \ \subset \ B_{p,1}^{N/p}\left(\mathbb{R}^N, Z\right)$$

where $N/p < s < l \in \mathbb{N}$ and $r \in [1,\infty]$.
d) The exponent N/p is best possible in the following sense. Let $A_p := \left\{(k\vee 1)\,2^{Nk/p}\right\}_{k\in\mathbb{N}_0}$. Note that the spaces $B_p^{A_p}$ form a scale since, for $1 \le p < q < \infty$,

$$B_p^{A_p}\left(\mathbb{R}^N, Z\right) \quad \subset \quad B_q^{A_q}\left(\mathbb{R}^N, Z\right) \ .$$

In general it is not possible to replace $B_p^{A_p}$ in Theorem 4.1 by a larger space $B_q^{A_q}$ for some $q > p$. This follows as in [14, Remark 4.9].

Proof of Theorem 4.1. Throughout this proof, the C_i's denotes constants which depend on at most: X, Y, N, p, q, and ϕ_0. Note that $\widetilde{X}$ and $\widetilde{Y}$ are UMD spaces with Fourier type p by Remark 2.10.

Since $M \in L_\infty\left(\mathbb{R}^N, \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$ has bounded support, $M \in L_1\left(\mathbb{R}^N, \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$. Define the function K by:

$$\mathbb{R}^N \ni s \quad \to \quad K(s) \ := \ \left(\mathcal{F}^{-1}M\right)(s) \in \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right) \ .$$

Corollary 3.11 applied to $M \in B_p^A\left(\mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$ with $w(t) = \ln(2+|t|)$ gives that for $1 \le r \le 2$

$$\int_{\mathbb{R}^N} \|K(t)F(\cdot)\|_{L_r(\widetilde{Y})}\, w(t)\, dt \ \le \ C_1 D\, \|F\|_{L_r(\widetilde{X})} \qquad \text{for each} \quad F \in L_r\left(\widetilde{X}\right)$$
$$\int_{\mathbb{R}^N} \|[K(t)]^*\, G(\cdot)\|_{L_r(\widetilde{X}^*)}\, w(t)\, dt \ \le \ C_1 D\, \|G\|_{L_r(\widetilde{Y}^*)} \qquad \text{for each} \quad G \in L_r\left(\widetilde{Y}^*\right) \ .$$

Also, K and K^* are strongly integrable by Corollary 3.9 and $\operatorname{supp}\widehat{K} \subset \operatorname{supp}\phi_0$. Thus Remark 3.8b can be applied to $K \in L_\infty\left(\mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$. Hence for $K_j(\cdot) := 2^{Nj}K\left(2^j\ \cdot\right)$, condition (3.13) gives that for each $F \in \mathcal{S}\left(\mathbb{R}^N,\widetilde{X}\right)$

$$\mathbb{E}\left\|\sum_{j\in\mathbb{Z}}\varepsilon_j K_j * F\right\|_{L_q\left(\widetilde{Y}\right)} \leq C_2\, C_1\, D\, \|F\|_{L_q\left(\widetilde{X}\right)} . \tag{4.1}$$

To see that inequality (4.1) implies the boundedness of T_m, fix $f \in \mathcal{S}_o(X)$ and let $h_k := \phi_{k-1} + \phi_k + \phi_{k+1}$. Then $F := \left\{\check{h}_k * f\right\}_{k\in\mathbb{Z}} \in \mathcal{S}\left(\widetilde{X}\right)$ and by Corollary 3.3

$$\|F\|_{L_q\left(\widetilde{X}\right)} \leq \sum_{j=-1}^{1}\left\|\left\{\check{\phi}_{k+j} * f\right\}_{k\in\mathbb{Z}}\right\|_{L_q\left(\widetilde{X}\right)} = 3\left\|\left\{\check{\phi}_k * f\right\}_{k\in\mathbb{Z}}\right\|_{L_q\left(\widetilde{X}\right)} \leq C_3\ \|f\|_{L_q(X)} . \tag{4.2}$$

Note that $M \in L_\infty$ implies that $m \in L_\infty$; thus it follows from Remark 3.8b that

$$K_j * F = \left\{\check{\phi}_j * \check{h}_k * T_{m(2^{k-j}\,\cdot)}f\right\}_{k\in\mathbb{Z}} \in L_q\left(\widetilde{Y}\right) = \operatorname{Rad}_q\left(L_q(Y)\right)$$

since, for a.e. s,

$$\begin{aligned}(K_j * F)\widehat{\ }(s) &= M\left(2^{-j}s\right)\widehat{F}(s) = \left\{\phi_0\left(2^{-j}s\right)m\left(2^{k-j}s\right)h_k(s)\,\widehat{f}(s)\right\}_{k\in\mathbb{Z}}\\ &= \left\{\phi_j(s)\,h_k(s)\left(T_{m\left(2^{k-j}\,\cdot\right)}f\right)\widehat{\ }(s)\right\}_{k\in\mathbb{Z}} .\end{aligned}$$

Since h_k is 1 on the support of ϕ_k, if $j = k$ then $\check{\phi}_j * \check{h}_k * T_{m\left(2^{k-j}\,\cdot\right)}(f) = \check{\phi}_j * (T_m f)$. Hence, by Corollary 3.3 and Lemma 3.12,

$$\begin{aligned}\|T_m f\|_{L_q(Y)} &\leq C_4\ \mathbb{E}_{\varepsilon}\left\|\sum_{j\in\mathbb{Z}}\varepsilon_j\,\check{\phi}_j * T_m f\right\|_{L_q(Y)}\\ &\leq C_4\,\mathbb{E}_{\varepsilon,\varepsilon'}\left\|\sum_{k\in\mathbb{Z}}\varepsilon'_k\left[\sum_{j\in\mathbb{Z}}\varepsilon_j\check{\phi}_j * \check{h}_k * T_{m\left(2^{k-j}\,\cdot\right)}f\right]\right\|_{L_q(Y)}\\ &= C_4\,\mathbb{E}_{\varepsilon}\left\|\left\{\sum_{j\in\mathbb{Z}}\varepsilon_j\check{\phi}_j * \check{h}_k * T_{m\left(2^{k-j}\,\cdot\right)}f\right\}_{k\in\mathbb{Z}}\right\|_{\operatorname{Rad}_1(L_q(Y))}\\ &\leq C_5\ \mathbb{E}_{\varepsilon}\left\|\sum_{j\in\mathbb{Z}}\varepsilon_j\left\{\check{\phi}_j * \check{h}_k * T_{m\left(2^{k-j}\,\cdot\right)}f\right\}_{k\in\mathbb{Z}}\right\|_{L_q\left(\widetilde{Y}\right)} = C_5\ \mathbb{E}_{\varepsilon}\left\|\sum_{j\in\mathbb{Z}}\varepsilon_j K_j * F\right\|_{L_q\left(\widetilde{Y}\right)} .\end{aligned} \tag{4.3}$$

Combining (4.1), (4.2), and (4.3) gives that $\|T_m f\|_{L_q(Y)} \leq C_6 D\,\|f\|_{L_q(X)}$. ■

The following notation simplifies the statements of corollaries to come.

Notation 4.3. Let $\mathcal{RM}_l^N\left(\mathcal{B}(X,Y)\right)$ be the set of all measurable functions $m\colon \mathbb{R}^N\setminus\{0\} \to \mathcal{B}(X,Y)$ whose distributional derivatives $D^\alpha m$ are represented by measurable functions for each $\alpha \in \mathbb{N}_0^N$

with $|\alpha| \le l$ and

$$\|m\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))} \;:=\; \max_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le l}} \; R_2\left(\left\{|t|^{|\alpha|}\, D^\alpha m\,(t) : t\in\mathbb{R}^N\setminus\{0\}\right\}\right) \;<\; \infty \tag{4.4}$$

where $l \in \mathbb{N}_0$.

Note that if X and Y are Hilbert spaces, then $\|m\|_{\mathcal{M}_l^N(\mathcal{B}(X,Y))} = \|m\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))}$.

The following Mihlin-type multiplier theorem is an immediate consequence of Theorem 4.1.

Corollary 4.4. *Let X and Y be UMD spaces with Fourier type $p \in (1,2]$ and $l := \left[\frac{N}{p}\right] + 1$. If*

$$m \in \mathcal{RM}_l^N\left(\mathcal{B}\left(X,Y\right)\right) \qquad \textit{with} \qquad \|m\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))} := A$$

then m is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, Y)$ for each $q \in (1,\infty)$. Furthermore, $\|T_m\|_{L_q\to L_q} \;\le\; C\,A$ for some constant C independent of m.

Remark 4.5. a) Corollary 4.4 applies to Hilbert spaces X and Y with $l = [N/2]+1$, which recovers Schwartz's classical result. For arbitrary UMD spaces, Corollary 4.4, but with $l = N$, was first shown in [35], for a second proof see [16]. Furthermore, [36, Remark 3.7] shows that the exponent $l := [N/p] + 1$ is best possible for L_p-spaces.
b) The proof will show that the assumption (4.4) on m can be replaced by the following formally weaker assumption:

$$\max_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le l}} \; \sup_{\substack{t\in\mathbb{R}^N \\ 2^{-1}\le|t|\le 2^1}} \; R_2\left(\left\{\left|2^k t\right|^{|\alpha|} D^\alpha m(2^k t)\colon k\in\mathbb{Z}\right\}\right) \;:=\; A < \infty\ . \tag{4.5}$$

Proof of Corollary 4.4. Let $B := \mathcal{B}\left(\mathrm{Rad}_q\left(X\right), \mathrm{Rad}_q\left(Y\right)\right)$. Without loss of generality, via an approximation argument, $\operatorname{supp} m \subset \left\{t\in\mathbb{R}^N\colon 2^{-n} \le |t| \le 2^n\right\}$ for some $n \in \mathbb{N}$. Thus, if $s \in \mathbb{R}^N$ then $M\left(s\right) := \left\{\phi_0\left(s\right) m(2^k s)\right\}_{k\in\mathbb{Z}} \in B$ with the k^{th} coordinate of $M\left(s\right)$ being zero for $|k| > n$. By Leibniz's rule and Remark 2.10a,

$$\begin{aligned}
\|M\|_{W_p^l(\mathbb{R}^N,B)} \;&\le\; \sum_{|\alpha|\le l} 2^{|\alpha|} \left\| |t|^{|\alpha|}\, D^\alpha M\left(t\right)\, \chi_{\operatorname{supp}\phi_0}\left(t\right) \right\|_{L_p(dt,B)} \\
&\le\; \sum_{|\alpha|\le l}\sum_{\beta\le\alpha} 2^{|\alpha|}\binom{\alpha}{\beta} \left\| |t|^{|\alpha-\beta|}\, D^{\alpha-\beta}\phi_0\left(t\right) \left\{ \left|2^k t\right|^{|\beta|} D^\beta m\left(2^k t\right)\right\}_{k\in\mathbb{Z}} \right\|_{L_p(dt,B)} \\
&=\; \sum_{|\alpha|\le l}\sum_{\beta\le\alpha} 2^{|\alpha|}\binom{\alpha}{\beta} \left\| |t|^{|\alpha-\beta|}\, D^{\alpha-\beta}\phi_0\left(t\right)\, R_q\left(\left\{ \left|2^k t\right|^{|\beta|} D^\beta m\left(2^k t\right) : k\in\mathbb{Z}\right\}\right) \chi_{[2^{-1}\le|t|\le 2]} \right\|_{L_p(dt,\mathbb{R})} .
\end{aligned}$$

By (4.5) and Hölder's inequality, $M \in W_p^l\left(\mathbb{R}^N, B\right)$. By Remark 4.2c, Theorem 4.1 applies. ■

Corollary 4.6. *Let X and Y be UMD spaces with Fourier type $p \in (1,2]$ and $l := \left[\frac{N}{p}\right] + 1$. Let $m \in \mathcal{RM}_l^N\left(\mathcal{B}\left(X,Y\right)\right)$ and set $\|m\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))} := A$.*
(a) *Then m is a Fourier multiplier from $H_1\left(\mathbb{R}^N, X\right)$ to $L_1\left(\mathbb{R}^N, Y\right)$ and $\|T_m\|_{H_1\to L_1} \;\le\; C\,A$ for*

some constant C independent of m.

(b) *Then m is a Fourier multiplier from $L_1\left(\mathbb{R}^N, X\right)$ to $L_1^{wk}\left(\mathbb{R}^N, Y\right)$ and $\|T_m\|_{L_1 \to L_1^{wk}} \leq C\,A$ for some constant C independent of m.*

(c) *Then m is a Fourier multiplier from $L_\infty\left(\mathbb{R}^N, X\right)$ to $BMO\left(\mathbb{R}^N, Y\right)$ and $\|T_m\|_{L_\infty \to BMO} \leq C\,A$ for some constant C independent of m.*

Remark 4.7. A related result, which also covers the spaces H_p for $p < 1$ was found independently by T. Hytönen [18].

A word about what is meant by a Fourier multiplier for UMD spaces X and Y for the function spaces in Corollary 4.6 is in order. Since $\mathcal{S}_H\left(\mathbb{R}^N, X\right) := \left\{ f \in \mathcal{S}\left(\mathbb{R}^N, X\right) : \int_{\mathbb{R}^N} f(t)\, dt = 0 \right\}$ is norm-dense in $H_1\left(\mathbb{R}^N, X\right)$, it is clear what is meant by a Fourier multiplier from H_1 to L_1: in Definition 2.4, just replace $\mathcal{S}$ by $\mathcal{S}_H$ and L_q by either H_1 or L_1, accordingly. A linear mapping $T\colon L_1\left(\mathbb{R}^N, X\right) \to L_1^{\mathrm{wk}}\left(\mathbb{R}^N, Y\right)$ is (uniformly) continuous if and only if

$$\|T\|_{L_1 \to L_1^{\mathrm{wk}}} := \sup \left\{ \|Tf\|_{L_1^{\mathrm{wk}}} : \|f\|_{L_1} \leq 1 \right\} < \infty .$$

If a linear mapping $T\colon \mathcal{S}\left(\mathbb{R}^N, X\right) \to L_1^{\mathrm{wk}}\left(\mathbb{R}^N, Y\right)$ satisfies

$$\|Tf\|_{L_1^{\mathrm{wk}}(\mathbb{R}^N, Y)} \leq C\, \|f\|_{L_1(\mathbb{R}^N, X)} \qquad \text{for each} \quad f \in \mathcal{S}\left(\mathbb{R}^N, X\right),$$

then, since $\mathcal{S}\left(\mathbb{R}^N, X\right)$ is a norm-dense subspace of $L_1\left(\mathbb{R}^N, X\right)$, there is a unique continuous linear extension $\widetilde{T}\colon L_1\left(\mathbb{R}^N, X\right) \to L_1^{\mathrm{wk}}\left(\mathbb{R}^N, Y\right)$ of T and $\|\widetilde{T}\|_{L_1 \to L_1^{\mathrm{wk}}} \leq 2C$. Thus it is clear what is meant by a Fourier multiplier from L_1 to L_1^{wk}: in Definition 2.4, just replace L_q by either L_1 or L_1^{wk}, accordingly. The Schwartz class is not norm-dense in L_∞; however, it is weak*-dense in L_∞. Since X and Y are UMD spaces (in particular, reflexive)

$$\left[H_1\left(\mathbb{R}^N, Y^*\right)\right]^* = \mathrm{BMO}\left(\mathbb{R}^N, Y\right) \qquad \text{and} \qquad \left[L_1\left(\mathbb{R}^N, Y^*\right)\right]^* = L_\infty\left(\mathbb{R}^N, Y\right) ,$$

(cf. [9] for H_1–BMO duality). Thus m is a Fourier multiplier from $L_\infty\left(\mathbb{R}^N, X\right)$ to $BMO\left(\mathbb{R}^N, Y\right)$ provided there is an operator $T_m \in \mathcal{B}\left(L_\infty\left(\mathbb{R}^N, X\right), BMO\left(\mathbb{R}^N, Y\right)\right)$ satisfying:

$$\begin{gathered} T_m f := \left[m\,\widehat{f}\,\right]^\vee \qquad \text{for each} \qquad f \in \mathcal{S}\left(\mathbb{R}^N, X\right) \\ T_m \ \text{ is weak}^*\text{-to-weak}^* \text{ continuous} . \end{gathered} \tag{4.6}$$

Note that (4.6) guarantees the uniqueness of a norm-to-norm continuous operator, *if* it exists.

Proof of Corollary 4.6. Throughout this proof, the C_i's are constants that are independent of m and *the* fixed $n \in \mathbb{N}$.

For each $j \in \mathbb{Z}$ and a fixed $n \in \mathbb{N}$, let

$$m_j \ := \ m\,\phi_j \qquad\qquad M_n \ := \ \sum_{j=-\infty}^{n} m_j$$

$$k_j \ := \ \check{m}_j \qquad\qquad K_n \ := \ \check{M}_n \ = \ \sum_{j=-\infty}^{n} k_j \ .$$

Note that $M_n \in \mathcal{RM}_l^N\left(\mathcal{B}\left(X,Y\right)\right)$ with

$$\|M_n\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))} \ \leq \ C_0\,A \ .$$

So $T_{M_n} \in \mathcal{B}\left(L_2\left(\mathbb{R}^N,X\right),L_2\left(\mathbb{R}^N,Y\right)\right)$ by Corollary 4.4. That T_{M_n} satisfies parts (a) and (b), for some constant C independent of m and the fixed $n \in \mathbb{N}$, follows from the Benedek-Calderon-Panzone theorem for convolutions on vector-valued spaces (see [13, Ch. V, Thm 3.4 and Remark (3.3′) on p. 494]) if one can show that

$$\int_{|t|>2|s|} \left\|K_n\left(t-s\right)x - K_n\left(t\right)x\right\|_Y \,dt \ \leq \ A\,C_1\left\|x\right\|_X \tag{4.7}$$

for each $s \in \mathbb{R}^N$ and $x \in X$. Inequality (4.7) is now shown via an adaption of the classical argument in [32, Ch. VI, Sect. 4.4].

Fix $x \in X$. For $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq l$, by Leibniz's rule,

$$\left\|D^\alpha m_j\left(t\right)\right\|_{\mathcal{B}(X,Y)} \ \leq \ A\,C_3 2^{-j|\alpha|} \ ;$$

thus, since $\left(D^\alpha m_j\right)^\vee\left(t\right) = \left(-it\right)^\alpha\, k_j\left(t\right)$ and Y has Fourier type p,

$$\left[\int_{\mathbb{R}^N} \left\|\left(-it\right)^\alpha k_j\left(t\right)x\right\|_Y^{p'}\,dt\right]^{1/p'} \ \leq \ C_4\ \left[\int_{\operatorname{supp}\phi_j} \left\|D^\alpha m_j\left(t\right)x\right\|_Y^{p}\,dt\right]^{1/p} \ \leq \ A\,C_5\,2^{-j|\alpha|}\,2^{\frac{jN}{p}}\ \left\|x\right\|_X \ .$$

Thus for each $l_0 \in \{0,1,\ldots,l\}$

$$\left[\int_{\mathbb{R}^N} \left\|\left|t\right|^{l_0} k_j\left(t\right)x\right\|_Y^{p'}\,dt\right]^{1/p'} \ \leq \ A\,C_6\ 2^{\frac{jN}{p}} 2^{-jl_0}\left\|x\right\|_X \ . \tag{4.8}$$

Using (4.8) with $l = l_0$, Hölder's inequality gives, for any $a > 0$, since $N/p < l$,

$$\begin{aligned}\int_{|t|\geq a} \left\|k_j\left(t\right)x\right\|_Y\,dt \ &\leq \ \left[\int_{\mathbb{R}^N} \left\|\left|t\right|^{l} k_j\left(t\right)x\right\|_Y^{p'}\,dt\right]^{1/p'}\ \left[\int_{|t|\geq a} \left|t\right|^{-lp}\,dt\right]^{1/p} \\ &\leq \ A\,C_6\ 2^{\frac{jN}{p}}\ 2^{-jl}\ \left\|x\right\|_X\ a^{\frac{N}{p}}\ a^{-l}\left[\int_{|t|\geq 1} \left|t\right|^{-lp}\,dt\right]^{1/p} \\ &\leq \ A\,C_7\ \left(2^j a\right)^{\frac{N}{p}-l}\ \left\|x\right\|_X \ .\end{aligned} \tag{4.9}$$

Similarly, taking $l_0 = 0$ in (4.8) gives

$$\int_{|t|\leq a} \left\|k_j\left(t\right)x\right\|_Y\,dt \ \leq \ \left[\int_{\mathbb{R}^N} \left\|k_j\left(t\right)x\right\|_Y^{p'}\,dt\right]^{1/p'}\ \left[\int_{|t|\leq a} 1\,dt\right]^{1/p} \ \leq \ A\,C_8\ \left(2^j a\right)^{\frac{N}{p}}\ \left\|x\right\|_X \ . \tag{4.10}$$

Choosing $a = 2^{-j}$ in (4.9) and (4.10) gives

$$\int_{\mathbb{R}^N} \|k_j(t)\,x\|_Y \; dt \;\leq\; A\, C_9 \|x\|_X \;.$$

Let $n_j(t) := [D^\alpha k_j]^\wedge(t) = (i)^{|\alpha|}\, t^\alpha m_j(t)$ for each $j \in \mathbb{Z}$. For $\beta \in \mathbb{N}_0^N$ with $|\beta| \leq l$

$$\left\| D^\beta n_j(t) \right\|_{\mathcal{B}(X,Y)} \;\leq\; A\, C_{10}\, 2^{j|\alpha|}\, 2^{-j|\beta|}$$

by Leibniz's rule; thus, since $\left[D^\beta n_j\right]^\vee(t) = (-i)^{|\beta|}\; t^\beta\, D^\alpha k_j(t)$ and Y has Fourier type p,

$$\begin{aligned}\left[\int_{\mathbb{R}^N} \left\| t^\beta D^\alpha k_j(t)\,x \right\|_Y^{p'} dt\right]^{1/p'} &\leq\; C_{11} \left[\int_{\mathbb{R}^N} \left\| D^\beta n_j(t)\,x \right\|_Y^{p} dt\right]^{1/p} \\ &\leq\; A\, C_{12}\, 2^{j|\alpha|}\, 2^{-j|\beta|} 2^{jN/p} \;.\end{aligned}$$

Arguing as above gives that

$$\int_{\mathbb{R}^N} \|D^\alpha k_j(t)\,x\|_Y \; dt \;\leq\; A\, C_{13} 2^{j|\alpha|}\, \|x\|_X \;. \tag{4.11}$$

Thus, for each $h = (h_1, \ldots, h_N) \in \mathbb{R}^N$,

$$\begin{aligned}\int_{\mathbb{R}^N} \|k_j(t+h)\,x - k_j(t)\,x\|_Y \; dt &\leq\; \int_0^1 \left(\sum_{i=1}^N |h_i| \int_{\mathbb{R}^N} \left\| \frac{\partial}{\partial t_i} k_j(t+sh)\,x \right\|_Y dt \right) ds \\ &\leq\; A\, C_{14}\, 2^j\; |h|\; \|x\|_X \;;\end{aligned} \tag{4.12}$$

indeed, just apply the Fundamental Theorem of Calculus to $\mathbb{R}^n \ni s \to k_j(t+sh)\,x \in Y$ and then use (4.11). Fix $s \in \mathbb{R}^N$. Note that the inequality

$$\sum_{j\in\mathbb{Z}} \int_{|t|>2|s|} \|k_j(t-s)\,x - k_j(t)\,x\|_Y \; dt \;\leq\; A\, C_{15}\; \|x\|_X \tag{4.13}$$

implies (4.7). To see (4.13), let $\Gamma_1 := \left\{ j \in \mathbb{N} \colon 2^j \leq |s|^{-1} \right\}$ and $\Gamma_2 := \left\{ j \in \mathbb{N} \colon\; |s|^{-1} < 2^j \right\}$. By (4.12),

$$\sum_{j\in\Gamma_1} \int_{|t|>2|s|} \|k_j(t-s)\,x - k_j(t)\,x\|_Y \; dt \;\leq\; A\, C_{14}\; \|x\|_X \left[|s| \sum\nolimits_{j\in\Gamma_1} 2^j \right] \;\leq\; A\, C_{16}\; \|x\|_X \;.$$

By (4.9)

$$\begin{aligned}\sum_{j\in\Gamma_2} \int_{|t|>2|s|} \|k_j(t-s)\,x - k_j(t)\,x\|_Y \; dt &\leq\; 2 \sum_{j\in\Gamma_2} \int_{|t|>1\,|s|} \|k_j(t)\,x\|_Y \; dt \\ &\leq\; 2\; A\, C_7\; \|x\|_X \left[|s|^{\frac{N}{p}-l} \sum\nolimits_{j\in\Gamma_2} \left(2^{\frac{N}{p}-l} \right)^j \right] \;\leq\; A\, C_{17}\; \|x\|_X\end{aligned}$$

since $N/p < l$. This completes the proof of (4.7). Thus T_{M_n} satisfies parts (a) and (b) for some constant C_{18} independent of m and $n \in \mathbb{N}$.

Towards showing part (a), fix $f \in \mathcal{S}_H\left(\mathbb{R}^N, X\right)$ and let

$$G := \left\{ g \in \mathcal{S}\left(\mathbb{R}^N, Y^*\right) :\; \|g\|_{L_\infty(Y^*)} \leq 1 \right\} \;.$$

Then

$$\begin{aligned}\|T_m f\|_{L_1(\mathbb{R}^N,Y)} &= \sup_{g\in G} \langle g, T_m f\rangle \;=\; \sup_{g\in G} \lim_{n\to\infty} \langle g, T_{M_n} f\rangle \;\leq\; \sup_{g\in G} \varliminf_{n\to\infty} \|g\|_{L_\infty(Y^*)} \|T_{M_n} f\|_{L_1(Y)} \\ &\leq \varliminf_{n\to\infty} \|T_{M_n}\|_{H_1(X)\to L_1(Y)} \|f\|_{H_1(X)} \;\leq\; A\,C_{18}\, \|f\|_{H_1(X)} \;.\end{aligned}$$

Thus part (a) holds.

If $f \in \mathcal{S}_*\left(\mathbb{R}^N, X\right) := \left\{g \in \mathcal{S}\left(\mathbb{R}^N, X\right) : \operatorname{supp}\widehat{g} \text{ is compact}\right\}$, then $T_m f = T_{M_n} f$ for n sufficiently large. Since $\mathcal{S}_*\left(\mathbb{R}^N, X\right)$ is norm-dense in $L_1\left(\mathbb{R}^N, X\right)$, part (b) holds.

Part (c) follows from part (a) and a duality argument. Since X and Y have nontrivial (Fourier) type, $m^* \in \mathcal{RM}_l^N\left(\mathcal{B}\left(Y^*, X^*\right)\right)$ and $\|m^*\|_{\mathcal{RM}_l^N(\mathcal{B}(Y^*,X^*))} \leq C_{19}\, \|m\|_{\mathcal{RM}_l^N(\mathcal{B}(X,Y))}$ (cf., eg., [39, Thm. 2.2.14]). Thus $T_{m^*} \in \mathcal{B}\left(H_1\left(\mathbb{R}^N, Y^*\right), L_1\left(\mathbb{R}^N, X^*\right)\right)$ with $\|T_{m^*}\|_{H_1\to L_1} \leq A\,C_{20}$ by part (a). Hence $\left(T_{m^*}\right)^* \in \mathcal{B}\left(L_\infty\left(\mathbb{R}^N, X\right), BMO\left(\mathbb{R}^N, Y\right)\right)$ is weak*-to-weak* continuous and is of norm at most $A\,C_{20}$. Furthermore, $\left(T_{m^*}\right)^* |_{\mathcal{S}(\mathbb{R}^N,X)} = T_{m(-\cdot)} |_{\mathcal{S}(\mathbb{R}^N,X)}$. Thus part (c) holds. ■

Remark 4.8. One can use the estimates in the proof of Corollary 4.6 to give a more direct proof of Corollary 4.4 avoiding Besov spaces. We indicate this for $p = 1, l = [N] + 1$: By assumption $M(t) = (\phi_0(t)m(2^k t))$ and $M^{(\alpha)}$, where $|\alpha| \leq N{+}1$, have norm less than C in $\mathcal{B}\left(\operatorname{Rad}(X), \operatorname{Rad}(Y)\right)$. Estimates (4.9) and (4.10) applied to $K(t) = M(t)^\vee$ with $p = 1$ give $\int_{\mathbb{R}^N} \|k(t)\|\, dt < \infty$ for the operator norm on $\mathcal{B}\left(\operatorname{Rad}(X), \operatorname{Rad}(Y)\right)$. Now apply Remark 3.8a and finish the proof as in the last part of the proof of Theorem 4.1.

Our results are easily adapted to Sobolev spaces. Corollary 4.9 gives a flavor of this for fractional Sobolev spaces (see Definition 2.15).

Corollary 4.9. *Let X and Y have UMD and Fourier type $p \in (1,2]$ and $l := \left[\frac{N}{p}\right] + 1$ and $s \in \mathbb{R}$. Assume that $m \in C^l\left(\mathbb{R}^N \setminus \{0\}, \mathcal{B}(X,Y)\right)$ satisfies that*

$$\max_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\,\leq\, l}} R_2\left(\left\{\, \left(1+|t|^2\right)^{s/2} |t|^{|\alpha|} \left(D^\alpha m\right)(t) : t \in \mathbb{R}^N \setminus \{0\}\right\}\,\right) \;:=\; A \;\leq\; \infty\;. \tag{4.14}$$

Then m is a Fourier multiplier from $H_q^r\left(\mathbb{R}^N, X\right)$ to $H_q^{r+s}\left(\mathbb{R}^N, Y\right)$ for each $r \in \mathbb{R}$ and $q \in (1,\infty)$. Also, $\|T_m\|_{H_q^r \to H_q^{r+s}} \leq C\,A$ for some constant C independent of m.

Proof. It suffices to show that $n(\cdot) := \left(1+|\cdot|^2\right)^{s/2} m(\cdot)$ is a Fourier multiplier from $L_q\left(\mathbb{R}^N, X\right)$ to $L_q\left(\mathbb{R}^N, Y\right)$; for then, the following diagram

$$\begin{array}{ccc} H_q^r\left(\mathbb{R}^N, X\right) & \xrightarrow{T_m} & H_q^{r+s}\left(\mathbb{R}^N, Y\right) \\ {\scriptstyle J^r}\big\downarrow & & \big\downarrow{\scriptstyle J^{r+s}} \\ L_q\left(\mathbb{R}^N, X\right) & \xrightarrow{T_n} & L_q\left(\mathbb{R}^N, Y\right) \end{array}$$

commutes, where $J^u \colon H^u_q\left(\mathbb{R}^N, Z\right) \to L_q\left(\mathbb{R}^N, Z\right)$ is the isometry defined by

$$J^u(f) \;:=\; \left[\left(1+|\cdot|^2\right)^{u/2}\, \widehat{f}(\cdot)\right]^{\vee} \qquad \text{for} \quad f \in \mathcal{S}\left(\mathbb{R}^N, Z\right) .$$

But

$$\begin{aligned} |t|^{|\alpha|}\, D^\alpha n(t) \;&=\; |t|^{|\alpha|}\, D^\alpha \left[\left(1+|\cdot|^2\right)^{s/2} m(\cdot)\right](t) \\ &=\; \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left\{ |t|^{|\beta|}\, \frac{D^\beta\left[\left(1+|\cdot|^2\right)^{s/2}\right](t)}{\left(1+|t|^2\right)^{s/2}} \right\} \left[\left(1+|t|^2\right)^{s/2} |t|^{|\alpha-\beta|}\, D^{\alpha-\beta} m(t)\right] \end{aligned} \tag{4.15}$$

and the terms $\{\cdots\}$ in (4.15) are uniformly bounded on $\mathbb{R}^N \setminus \{0\}$. Thus the assumption (4.14) gives that $n \in \mathcal{RM}^N_l\left(\mathcal{B}(X,Y)\right)$ and so Corollary 4.4 finishes off the proof. ■

Hörmander's condition takes here the following form.

Corollary 4.10. *Let X and Y be UMD spaces with Fourier type $p \in (1,2]$ and $l := \left[\frac{N}{p}\right] + 1$. Let $m \colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$ be a measurable function whose distributional derivatives $D^\alpha m$ are represented by measurable functions for each $\alpha \in \mathbb{N}^N_0$ with $|\alpha| \le l$ and*

$$\max_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \,\le\, l}} \int_{2^{-1}<|t|<2} \left[R_2\left(\left\{ \left|2^k t\right|^{|\alpha|} D^\alpha m\left(2^k t\right) : k \in \mathbb{Z} \right\} \right) \right]^p dt \;:=\; A \;<\; \infty \,. \tag{4.16}$$

Then m is a Fourier multiplier from $L_q(X)$ to $L_q(Y)$ for each $1<q<\infty$. Also, $\|T_m\|_{L_q \to L_q} \le CA$ for some constant C independent of m.

Proof. In the proof of Corollary 4.4, just replace (4.5) by (4.16) to see that Theorem 4.1 applies in this setting also. ■

The following corollary shows that our result improves on the multiplier theorem of Bourgain even in the case of scalar-valued multiplier functions m. Its proof uses the following equivalent norm [29, Prop. 3.1] on the Besov spaces $B^s_{p,1}(\mathbb{R}, Z)$ for $0<s<1$ and $1 \le p < \infty$:

$$\|f\|'_{B^s_{p,1}(\mathbb{R},Z)} \;:=\; \|f\|_{L_p(\mathbb{R},Z)} \;+\; B^s_{p,1}(f)$$

where $f_h(u) := f(u+h)$,

$$\begin{aligned} B^s_{p,1}(f) \;&:=\; \int_0^\infty t^{-s}\, w_p(f,t)\, \frac{dt}{t} \\ w_p(f,t) \;&:=\; \sup_{|h| \le t} \|f_h - f\|_{L_p(\mathbb{R},Z)} \;. \end{aligned}$$

Corollary 4.11. *Let X and Y be UMD spaces with Fourier type $p \in (1,2]$. Assume that for the function $m : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X,Y)$ there is some $l \in \left(\frac{1}{p}, 1\right)$ and $h_0 > 0$ so that, for some constant A,*

$$R_2\ (\{m(u)\ : u \in \mathbb{R} \setminus \{0\}\}) \ \leq\ A$$

$$R_2\left(\left\{|u|^l\, \frac{m(u+h) -\ m(u)}{|h|^l}\ : u, u+h \in \mathbb{R} \setminus \{0\}\ \text{ and } 0 < |h| \leq |u|\ h_0\right\}\right) \ \leq\ A\ . \tag{4.17}$$

Then m is a Fourier multiplier from $L_q(\mathbb{R}, X)$ to $L_q(\mathbb{R}, Y)$ for each $q \in (1, \infty)$. Furthermore, $\|T_m\|_{L_q \to L_q} \ \leq\ C\, A$ for some constant C independent of m.

Proof. Throughout this proof, the C_j's are constants independent of m. Without loss of generality, $h_0 < \frac{1}{4}$. Fix s so that $0 < \frac{1}{p} < s < l < 1$.

Set $m(0) := 0$. For $t \in \mathbb{R}$, let $m_k(t)\ :=\ \phi_0(t) m(2^k t)$ and $M(t)\ :=\ \{m_k(t)\}_{k \in \mathbb{Z}}$. Note that

$$\left|\frac{u}{h}\right|^l [m_k(u+h) - m_k(u)]\ =$$
$$\phi_0(u) \left|\frac{u}{h}\right|^l \left[m\left(2^k(u+h)\right) - m\left(2^k u\right)\right]\ +\ m\left(2^k(u+h)\right) \left|\frac{u}{h}\right|^l [\phi_0(u+h) - \phi_0(u)]\ .$$

So, since $\phi_0 \in \mathcal{S}$ and $l < 1$, the assumption (4.17) yields, for each $u, h \in \mathbb{R} \setminus \{0\}$ with $0 < |h| \leq |u|\, h_0$,

$$R_2\ (\{m_k(u+h) - m_k(u) : k \in \mathbb{Z}\})\ \equiv$$
$$\left|\frac{h}{u}\right|^l R_2\ \left(\left\{\left|\frac{u}{h}\right|^l [m_k(u+h) - m_k(u)] : k \in \mathbb{Z}\right\}\right)\ \leq\ \left|\frac{h}{u}\right|^l A\, C_1\ .$$

Thus, if $|h| \leq \frac{h_0}{4}$ then, with $B := \mathcal{B}(\mathrm{Rad}_q(X), \mathrm{Rad}_q(Y))$,

$$\|M_h - M\|_{L_p(B)}\ =\ \left[\int_{\frac{1}{4}}^{3} \|M(u+h) - M(u)\|_B^p\, du\right]^{\frac{1}{p}}\ \leq\ A\, C_2 \left[\int_{\frac{1}{4}}^{3} \left|\frac{h}{u}\right|^{lp} du\right]^{\frac{1}{p}}\ \leq\ A\, C_3\ |h|^l$$

since $\frac{1}{p} < l$. Also, $\|M_h - M\|_{L_p(B)}\ \leq\ 2\ \|M\|_{L_p(B)}\ \leq\ A\, C_4$ for any $h \in \mathbb{R}$. Thus

$$w_p(M, t)\ \leq\ \begin{cases} A\, C_5\, t^l & \text{if } t \leq \frac{h_0}{4} \\ A\, C_6 & \text{if } t > \frac{h_0}{4}\ . \end{cases}$$

Thus $B^s_{p,1}(M)\ \ \leq A\, C_8$ since $0 < s < l$ and so $M \in B^s_{p,1}(\mathbb{R}, B)$ with norm at most $A\, C_9$. By Remark 4.2c, since $\frac{1}{p} < s$, we can apply Theorem 4.1, one last time. ∎

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