VECTOR-VALUED EXTENSIONS OF SOME CLASSICAL THEOREMS IN HARMONIC ANALYSIS *

Maria Girardi †
Department of Mathematics
University of South Carolina
Columbia, SC 29208
U.S.A.
girardi@math.sc.edu

Lutz Weis ‡
Mathematisches Institut I
Universität Karlsruhe
Englerstraße 2
76128 Karlsruhe
Germany
Lutz.Weis@math.uni-karlsruhe.de

Abstract This paper surveys some recent results on vector-valued Fourier multiplier theorems and pseudo differential operators, which have found important application in the theory of evolution equations. The approach used combines methods from Fourier analysis and the geometry of Banach spaces, such as R-boundedness.
Keywords R-boundedness, Mihlin multiplier theorem, pseudo differential operators, Fourier type, Littlewood-Paley decomposition, vector-valued Besov spaces
Subject Classification Primary 42B15, 46E40, 46B09. Secondary 46B20, 46E35.

*To appear: Analysis and Applications - ISAAC 2001 (H. G. W. Begehr, R. P. Gilbert, and M. W. Wong, eds.), Kluwer, Dordrecht.
†Girardi is supported in part by the Alexander von Humboldt Foundation.
‡Weis is supported in part by *Landesforschungsschwerpunkt Evolutionsgleichungen des Landes Baden-Württenberg*.

1. Introduction

Boundedness theorems for Fourier multiplier operators, singular integral operators, and pseudo differential operators play an important role in analysis. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow one to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For example, such theorems are used: in studying maximal regularity of parabolic equations (see, e.g., [1, 4, 5, 9, 17, 18, 19, 27, 28]), in stability theory (see, e.g., [21, 26]), in the theory of pseudo differential operators on manifolds with singularities (see, e.g., [23]),and for elliptic operators on infinite dimensional state spaces (see, e.g., [3, 10]).

This paper surveys some recent results in harmonic analysis of Banach space valued functions and tries to elucidate the interesting interplay with the geometry of the underlying Banach space, which in the end leads to significant applications to evolution equations.

The first Fourier multiplier theorem for *operator-valued* multiplier functions was J. Schwartz's version of Mihlin's theorem.

Theorem 1 (J. Schwartz) *Let H be a Hilbert space and*

$$m\colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(H)$$

be such that the sets

$$\left\{ |t|^{|\alpha|} D^\alpha m(t) : \ t \in \mathbb{R}^N \setminus \{0\} \right\} \tag{1}$$

are norm bounded for each multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \le [N/2] + 1$. Then

$$T_m(f) := \mathcal{F}^{-1}[m(\mathcal{F}f)] \qquad \text{for} \qquad f \in \mathcal{S}\left(\mathbb{R}^N, H\right)$$

defines a bounded operator on $L_q\left(\mathbb{R}^N, H\right)$ for each $q \in (1, \infty)$.

Here $\mathcal{F}$ is the Fourier transform and $\mathcal{S}\left(\mathbb{R}^N, H\right)$ is the Schwartz class of rapidly decreasing functions from $\mathbb{R}^N$ to H. Does Theorem 1 formally generalize by replacing the Hilbert space H by an arbitrary Banach space X? No; G. Pisier showed that (isomorphic images of) Hilbert spaces are the **only** Banach spaces for which Theorem 1 holds in the above form. In recent years, two approaches were found to circumvent this difficulty.

(i) Replace Bochner spaces $L_q\left(\mathbb{R}^N, X\right)$ by Besov spaces $B^s_{q,r}\left(\mathbb{R}^N, X\right)$. Using the characterization of Besov spaces in terms of the Paley-Littlewood decomposition, one can prove a Mihlin-type theorem as well as boundedness results for pseudo differential operators for *any* Banach space X. Section 4 elaborates on this approach. Note that in this setting one needs only norm boundedness, and not R-boundedness, of the sets in (1).

(ii) In the case of the Bochner spaces $L_q\left(\mathbb{R}^N, X\right)$, it makes sense to consider only those Banach spaces X for which the simplest multiplier function, i.e. $M(\cdot) = \operatorname{sign}(\cdot)\chi_X$, is a Fourier multiplier (or equivalently, for which the Hilbert transform is bounded) on $L_q\left(\mathbb{R}^N, X\right)$. Such Banach spaces are called **UMD spaces**. Subspaces of $L_q(\Omega, \mathbb{C})$-spaces, for $1 < q < \infty$, are examples of UMD spaces. There are many results showing the UMD spaces form the proper class of Banach spaces for vector-valued harmonic analysis (see, e.g., [6, 7, 29]). For starters, if X is a UMD space, then there is a Paley-Littlewood decompositition for $L_q\left(\mathbb{R}^N, X\right)$. But this decomposition is more delicate than the corresponding decomposition for Besov spaces. Therefore one has to replace the norm bounded condition in (1) by an R-bounded condition. This leads to boundedness results for Fourier multipliers and pseudo differential operators. Section 5 elaborates on this approach. Since large classes of classical operators are R-bounded (cf. [12, and references therein]), the assumptions in this approach are not too restrictive for applications.

2. Definitions and Notation

Notation is standard; consult [14, 15] for the needed definitions and notations. Here some basics are recalled.

Schwartz used Plancherel's identity for $L_2\left(\mathbb{R}^N, H\right)$ in his proof of Theorem 1. Since Plancherel's identity holds only for Hilbert space valued Bochner spaces $L_2\left(\mathbb{R}^N, X\right)$, the following concept from Banach space theory is needed.

Definition 2 ([22]) *Let $1 \leq p \leq 2$. A Banach space X has* ***Fourier type*** *p provided the Fourier transform $\mathcal{F}$ defines a bounded linear operator from $L_p\left(\mathbb{R}^N, X\right)$ to $L_{p'}\left(\mathbb{R}^N, X\right)$ for some (and thus then for each) $N \in \mathbb{N}$.*

The simple estimate $\|\mathcal{F}f(t)\|_X \leq \|f\|_{L_1(X)}$ shows that each Banach space X has Fourier type 1. The notion becomes more restrictive as p increases to 2. A Banach space has Fourier type 2 if and only if X is isomorphic to a Hilbert space [20]. A space $L_q(\Omega, \mathbb{R})$ has Fourier type $p = \min(q, q')$ [22]. Each closed subspace, the dual, and quotient space of a Banach space X has the same Fourier type as X; the first fact holds by definition, the last by duality.

The Fourier type is connected with the minimal smoothness assumptions on the multiplier function. For example, the condition (1) holding for $|\alpha| \leq [N/2]+1$, instead of $|\alpha| \leq N$ for the Bochner case and $|\alpha| \leq N+1$ for the Besov case, expresses the fact that a Hilbert space has Fourier type 2.

To define Besov spaces, first consider a partition of unity $\{\varphi_k\}_{k\in\mathbb{N}_0}$ of functions from $\mathcal{S}\left(\mathbb{R}^N,\mathbb{R}\right)$ as follows. Take a nonnegative function ψ in $\mathcal{S}\left(\mathbb{R},\mathbb{R}\right)$ with support in $\left[2^{-1},2\right]$ that satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \ \text{ for } \ s \in \mathbb{R}\setminus\{0\}$$

and let, for $t \in \mathbb{R}^N$,

$$\varphi_k(t) = \psi\left(2^{-k}|t|\right) \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \ .$$

To simplify notation, let $\varphi_k \equiv 0$ if $k < 0$. Note that φ_k and φ_j have overlapping support if and only if $|k-j| \leq 1$.

Among the many equivalent descriptions of Besov spaces, the most useful one in this context is given in terms of the so-called *Littlewood-Paley decomposition*. Roughly speaking this means that one considers $f \in \mathcal{S}'(X)$ as a distributional sum

$$f \ = \ \sum_{k=0}^{\infty} \left[\varphi_k\, \widehat{f}\,\right]^{\vee} \ = \ \sum_{k=0}^{\infty} \check{\varphi}_k * f \ = \ \sum_{k=0}^{\infty} f_k \ , \quad \text{where} \quad f_k := \check{\varphi}_k * f \ ,$$

of analytic functions f_k whose Fourier transforms have support in the (slightly overlapping dyadic-like) intervals $\{\operatorname{supp}\varphi_k\}_{k\in\mathbb{N}_0}$ and then one defines the Besov norm in terms of the *blocks* $\{f_k\}_{k\in\mathbb{N}_0}$ of the Littlewood-Paley decomposition of f.

Definition 3 *The* ***Besov space*** $B^s_{q,r}(\mathbb{R}^N, X)$*, where* $1 \leq q, r \leq \infty$ *and* $s \in \mathbb{R}$*, is the space of all* $f \in \mathcal{S}'(\mathbb{R}^N, X)$ *for which*

$$\|f\|_{B^s_{q,r}(\mathbb{R}^N,X)} := \left\| \left\{ 2^{ks} \|\check{\varphi}_k * f\|_{L_q(X)} \right\}_{k=0}^{\infty} \right\|_{\ell_r} \equiv \begin{cases} \left[\sum_{k=0}^{\infty} 2^{ksr} \|\check{\varphi}_k * f\|^r_{L_q(X)} \right]^{1/r} & \text{if } r \neq \infty \\ \sup_{k \in \mathbb{N}_0} \left[2^{ks} \|\check{\varphi}_k * f\|_{L_q(X)} \right] & \text{if } r = \infty \end{cases} \tag{2}$$

is finite; q is the ***main index*** *while s is the* ***smoothness index****. The space* $B^s_{q,r}(\mathbb{R}^N, X)$*, together with the norm in* (2)*, is a Banach space.*

3. A weak Fourier multiplier theorem for Bochner spaces

The following theorem is a *weak* Fourier multiplier theorem in the sense that its assumption (3) is quite *strong*; indeed, (3) implies that the Fourier multiplier function m is in $L_p(\mathbb{R}^N, \mathcal{B}(X,Y))$.

Theorem 4 ([14]) *Let X and Y have Fourier type $p \in [1,2]$ and*

$$m \in B^{N/p}_{p,1}(\mathbb{R}^N, \mathcal{B}(X,Y)) \ . \tag{3}$$

Then m is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, Y)$ for each $q \in [1, \infty]$; furthermore,

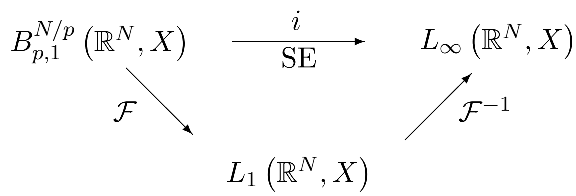
$$\|T_m\|_{L_q(X) \to L_q(Y)} \leq C \ \mathcal{M}_p(m) \ , \tag{4}$$

where C is a constant independent of m and

$$\mathcal{M}_p(m) := \inf \left\{ \|m(a\cdot)\|_{B^{N/p}_{p,1}(\mathbb{R}^N, \mathcal{B}(X,Y))} \ : \ a > 0 \right\} .$$

Theorem 4 leads to vector-valued extentions of classical multiplier theorems (such as Mihlin-, Hörmanders-, and Lipschitz-type theorems) for Besov spaces (see Section 4) and Bochner spaces (see Section 5) by considering a Littlewood-Paley decomposition of these spaces and then applying Theorem 4 to the blocks of the decomposition with the multiplier function m *restricted* to the support of the blocks. This is the charm behind using Littlewood-Paley decompositions: the classical (weak) assumptions on m imply the (strong) assumption of Theorem 4 *when* restricted to the support of the blocks.

The first step in proving Theorem 4 is to extend two other classical results to the vector-valued setting. The first extension shows that, for spaces with Fourier type p, the Sobolev Embedding (SE) factors through L_1 via the Fourier transform $\mathcal{F}$.



Lemma 5 ([14]) *Let X have Fourier type $p \in [1,2]$. Then the Fourier transform defines bounded operator from $B_{p,1}^{N/p}\left(\mathbb{R}^N, X\right)$ to $L_1\left(\mathbb{R}^N, X\right)$.*

The next lemma extends a well-known boundedness result for classical integral operators.

Lemma 6 ([13]) *Let $F \subseteq Y^*$ be a subspace that norms Y. Let*

$$k\colon \mathbb{R}^N \to \mathcal{B}\left(X, Y\right)$$

be such that k and k^* are srongly measurable and satisfy

$$\begin{aligned} \int_{\mathbb{R}^N} \left\|k\left(s\right)\, x\right\|_Y \, ds \;&\leq\; C_0 \left\|x\right\|_X \qquad \text{for each} \quad x \in X \\ \int_{\mathbb{R}^N} \left\|k\left(s\right)^* \, y^*\right\|_{X^*} \, ds \;&\leq\; C_1 \left\|y^*\right\|_{Y^*} \qquad \text{for each} \quad y^* \in F \end{aligned} \tag{5}$$

for some constants C_i. Then the convolution operator K, defined for finitely-valued functions $f\colon \mathbb{R}^N \to X$ with finite support by

$$\left(Kf\right)(t) \;=\; \int_{\mathbb{R}^N} k(t-s) f(s)\, ds \qquad \text{for} \;\; t \in \mathbb{R}^N \;,$$

extends to a bounded operator $K\colon L_q\left(\mathbb{R}^N, X\right) \;\to\; L_q\left(\mathbb{R}^N, Y\right)$ for each $q \in [1, \infty)$; furthermore, $\left\|K\right\|_{L_q \to L_q} \;\leq\; C_0^{\frac{1}{q}}\; C_1^{1-\frac{1}{q}}$. If, in addition, Y does not contain c_0, then the same holds true for $q = \infty$.

To prove Theorem 4, first consider a function m in the Schwartz class. Applying Lemma 5 to the functions

$$t \to m\left(t\right) x \quad \text{for } x \in X \qquad \text{and} \qquad t \to m^*\left(t\right) y^* \quad \text{for } y^* \in Y^*$$

gives that $k := \check{m}$ satisfies the assumptions in (5) and so one has that the corresponding operator T_m satisfies (4) (even if Y contains c_0). Now, thanks to the bound in (4) on the norm of the T_m's, a density argument finishes the job.

4. Fourier multiplier theorems for Besov spaces

One can think of a Besov space as a direct sum

$$B^s_{q,r}\left(\mathbb{R}^N, X\right) = \sum_{k\in\mathbb{N}_0} Z_k \quad \text{where} \quad Z_k \;:=\; \left\{\check{\varphi}_k * f \colon f \in B^s_{q,r}\left(\mathbb{R}^N, X\right)\right\}.$$

To see how a Fourier multiplier operator T_m, for a multiplier function $m\colon \mathbb{R}^N \to \mathcal{B}(X,Y)$, formally behaves on a *blocks* Z_k of the above Littlewood-Paley decomposition, fix an $f \in \mathcal{S}\left(\mathbb{R}^N, X\right)$. Since the function $\psi_k := \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ is 1 on $\operatorname{supp}\varphi_k$,

$$T_m\left(\check{\varphi}_k * f\right) = \left[\psi_k\, m\, \varphi_k\, \widehat{f}\,\right]^{\vee} = \check{\varphi}_k * \left[m\, \psi_k\, \widehat{f}\,\right]^{\vee} = \check{\varphi}_k * T_m\left(\check{\psi}_k * f\right) \in Z_k.$$

Thus T_m leaves the blocks Z_k invariant. Furthermore,

$$\left[\check{\varphi}_k * T_m f\right]^{\widehat{}} = \psi_k\, \varphi_k\, m\, \widehat{f} = m\, \psi_k\, \left(\check{\varphi}_k * f\right)^{\widehat{}} = \left[T_{m\psi_k}\left(\check{\varphi}_k * f\right)\right]^{\widehat{}}, \qquad (6)$$

and so

$$f = \sum_{k\in\mathbb{N}_0} \check{\varphi}_k * f \quad \text{and} \quad T_m f = \sum_{k\in\mathbb{N}_0} \check{\varphi}_k * T_m f = \sum_{k\in\mathbb{N}_0} T_{m\psi_k}\left(\check{\varphi}_k * f\right) \;.$$

Thus T_m behaves as a Fourier multiplier operator $T_{\psi_k\, m}$ on *each* block Z_k of the Littlewood-Paley decomposition; furthermore, the operator $T_{\psi_k\, m}$ depends only on the values of m on the supports of ψ_k. This suggests the following approach to boundedness results for T_m.

(1^{st}) Estimate $\|T_{m\psi_k}\|_{L_q\to L_q}$ on each block of the Littlewood-Paley decomposition *separably*. For this, apply Theorem 4 to the multiplier function $m\,\psi_k$.

(2^{nd}) Sum over the blocks; with the help of (6):

$$\begin{aligned}\|T_m f\|_{B^s_{p,r}(X)} \;&=\; \left[\sum_{k=0}^{\infty} 2^{ksr}\, \left\|T_{m\psi_k}\left(\check{\varphi}_k * f\right)\right\|^r_{L_q(X)}\right]^{1/r} \\ &\le\; \sup_k \|T_{m\psi_k}\|_{L_q\to L_q}\, \|f\|_{B^s_{p,r}(X)} \;.\end{aligned}$$

This gives the heuristic idea behind the proof of the main result of this section (additional considerations are necessary if q or r is ∞):

Theorem 7 [14] *Let X and Y be Banach spaces with Fourier type p. Let $m : \mathbb{R}^N \to \mathcal{B}(X,Y)$ satisfy, for each $k \in \mathbb{N}_0$,*

$$\varphi_k \cdot m \in B^{N/p}_{p,1}\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right) \qquad \textit{and} \qquad \mathcal{M}_p(\varphi_k \cdot m) \;\le\; A\;.$$

Then m is a Fourier multiplier from $B^s_{q,r}(\mathbb{R}^N, X)$ to $B^s_{q,r}(\mathbb{R}^N, Y)$ for each $q, r \in [1, \infty]$ and $s \in \mathbb{R}$. Furthermore, $\|T_m\|_{B^s_{q,r} \to B^s_{q,r}} \leq CA$ for some constant C that is independent of m.

Note that each Banach space has Fourier type 1 and each uniformly convex Banach space has Fourier type p for some $p > 1$. Our result shows that the required smoothness of the multiplier function m depends not only on the dimension of $\mathbb{R}^N$ but also on the geometry of the Banach spaces X and Y. It follows from results in [26] that the smoothness N/p is sharp for the Besov scale.

An advantage of the rather general formulation of the assumptions in Theorem 7 is that one can deduce from them, by simple estimates, several multiplier theorems with *classical* assumptions. For example, the Mihlin-type multiplier theorem below follows easily; it was the first multiplier theorem of this kind and its parts i) and iii) are due independently to H. Amann [2] and L. Weis [26], respectively.

Corollaries of Theorem 7. [14] Let $q, r \in [1, \infty]$ and $s \in \mathbb{R}$.

Mihlin condition If $m \colon \mathbb{R}^N \to \mathcal{B}(X, Y)$ satisfies, for some constant A, the estimate

$$\sup_{t \in \mathbb{R}^N} \left\| (1 + |t|)^{|\alpha|} D^\alpha m(t) \right\|_{\mathcal{B}(X,Y)} \leq A$$

for each multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq l$, then m is a Fourier multiplier from $B^s_{q,r}(\mathbb{R}^N, X)$ to $B^s_{q,r}(\mathbb{R}^N, Y)$ provided one of the following conditions hold:

i) X and Y are arbitrary Banach spaces and $l = N + 1$

ii) X and Y are uniformly convex Banach spaces and $l = N$

iii) X and Y have Fourier type p and $l = [\frac{N}{p}] + 1$.

Hörmander condition Let X and Y have Fourier type p and $l = \left[\frac{N}{p}\right] + 1$. Let $m \colon \mathbb{R}^N \to \mathcal{B}(X, Y)$ satisfy, for some constant A and each $R \in [1, \infty)$, the estimates:

$$\left[\int_{|t| \leq 2} \|D^\alpha m(t)\|^p \, dt \right]^{1/p} \leq A$$

$$\left[R^{-N} \int_{R < |t| < 4R} \|D^\alpha m(t)\|^p \, dt \right]^{1/p} \leq A R^{-|\alpha|}$$

for each α with $|\alpha| \leq l$. Then m is a Fourier multiplier from $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ to $B^s_{q,r}\left(\mathbb{R}^N, Y\right)$.

Lipschitz condition Let X and Y have Fourier type p and $l \in (1/p, 1)$. Assume that $m\colon \mathbb{R} \to \mathcal{B}(X,Y)$ satisfies, for some constant A, the estimates:

$$\|m(t)\| \leq A \quad \text{for} \quad t \in \mathbb{R}$$
$$(1+|t|)^l \left\| \frac{m(t+u)-m(t)}{|u|^l} \right\| \leq A \quad \text{for} \quad u,t \in \mathbb{R}\ ,\ u \neq 0\ .$$

Then m is a Fourier multiplier from $B^s_{q,r}(X)$ to $B^s_{q,r}(Y)$.

It is also useful to consider pseudo differential operators with operator-valued symbols. A pseudo differential operators Ψ_a with symbol a is formally defined by

$$\Psi_a f(t) := (2\pi)^{-N} \int_{\mathbb{R}^N} e^{it\cdot s} a(t,s) \widehat{f}(s)\, ds \qquad , \qquad f \in \mathcal{S}\left(\mathbb{R}^N, X\right)\ .$$

In analogy to classical symbol classes, for $\delta \in [0,1)$ and an $r > 0$, let $S^0_{1,\delta}(r,X)$ be the the class of symbols $a\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{B}(X)$ so that for all multi-indices α there is a constant C_α with

$$\left\|(1+|s|)^{|\alpha|} \partial^\alpha_s a(t,s)\right\| \leq C_\alpha \quad \text{for each} \quad t,s \in \mathbb{R}^N \tag{7}$$
$$\|\partial^\alpha_s a(\cdot,s)\|_{B^r_{\infty,\infty}} \leq C_\alpha\ (1+|s|)^{\delta r - |\alpha|}\ .$$

By extending the Coifman-Meyer decomposition of symbols to the operator-valued case, Z. Štrkalj showed the following theorem.

Theorem 8 [24] *Let X be a separable Banach space. Let $q,r \in [1,\infty]$ and $-(1-\delta)r < s < r$. If $a \in S^0_{1,\delta}(r,X)$ then Ψ_a is bounded on $B^s_{q,r}\left(\mathbb{R}^N, X\right)$.*

5. Fourier multiplier theorems for Bochner spaces

This section presents Fourier multiplier theorems on Bochner spaces. The methods are similar to those in the Besov case: one uses Theorem 4 and a Littlewood-Paley decomposition for Bochner spaces. For this decomposition, one needs to decompose $\mathbb{R}^N$ not only for $|t| \to \infty$ but also for $|t| \to 0$.

So consider a *partition of unity* $\{\phi_k\}_{k\in\mathbb{Z}}$ of functions from $\mathcal{S}\left(\mathbb{R}^N, \mathbb{R}\right)$ defined as follows. Take a nonnegative function $\phi_0 \in C^\infty(\mathbb{R}^N, \mathbb{R})$ that has support in $\{t : 2^{-1} \leq |t| \leq 2\}$ and satisfies, for $\phi_k(t) := \phi_0(2^{-k}t)$ for each $k \in \mathbb{Z}$, that $\sum_{k\in\mathbb{Z}} \phi_k(t) = 1$ for each $t \neq 0$. Note that

$$\|\check{\varphi}_k\|_{L_1} = \|\check{\varphi}_0\|_{L_1} \quad \text{and} \quad \operatorname{supp} \phi_k \subset \left\{t : 2^{k-1} \leq |t| \leq 2^{k+1}\right\}$$

for each $k \in \mathbb{Z}$.

Bourgain [7, $N = 1$] and Zimmermann [29, $N > 1$] proved that if a scalar-valued function $m\colon \mathbb{R}^N \setminus \{0\} \to \mathbb{C}$ satisfied a certain Mihlin-type smoothness condition and X is a UMD space, then $m(\cdot)\, I_X$ is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, X)$ for each $q \in (1, \infty)$. Their result leads to a Littlewood-Paley decomposition for Bochner spaces. Henceforth, $\{r_k\}_{k\in\mathbb{Z}}$ is just any enumeration of the Rademacher functions.

Corollary 9 ([15]) *Let X be a UMD space and $1 < q < \infty$. There is a constant C so that*

$$\frac{1}{C}\,\|f\|_{L_q(\mathbb{R}^N,X)} \le \int_{[0,1]} \left\| \sum_{k\in\mathbb{Z}} r_k(t) \left(\check{\phi}_k * f\right) \right\|_{L_q(\mathbb{R}^N,X)} dt \le C\,\|f\|_{L_q(\mathbb{R}^N,X)} \quad (8)$$

for each $f \in L_q\left(\mathbb{R}^N, X\right)$.

To see the fundamental difference between the Besov- and Bochner-space case (for $1 < q < \infty$), let's compare the norms. If $f \in B^0_{q,2}(X)$ (which is *closest* to $L_q(X)$), then

$$\|f\|_{B^0_{q,2}(X)} \;=\; \left[\sum_{k=0}^{\infty} \|\check{\varphi}_k * f\|^2_{L_q(X)}\right]^{1/2} ; \quad (9)$$

thus, $\{\check{\varphi}_k * f\}_{k\in\mathbb{N}_0}$ is <u>absolutely 2-summable</u> in $L_q(X)$. If $f \in L_q(X)$, then Corollary 9 gives not only that $\left\{\check{\phi}_k * f\right\}_{k\in\mathbb{Z}}$ is <u>almost unconditionally summable</u> in $L_q(X)$ but also (in the scalar case) that

$$\|f\|_{L_q(\mathbb{C})} \;\sim\; \left\| \left[\sum_{k\in\mathbb{Z}} \left|\check{\phi}_k * f\right|^2\right]^{1/2} \right\|_{L_q(\mathbb{C})} , \quad (10)$$

with the help of Kahane's and Khintchine's inequalities. Compare (9) and (10)! In the Besov case, one can estimate the Bochner norm of each block $\check{\varphi}_k * f$ of f *separately* (via Theorem 4) and then sum over the blocks. But this approach is not possible in the Bochner case since one sums over the blocks inside the Bochner norm. Therefore one needs tools to estimate the blocks simultaneously as an unconditionally summable sequence. Definition 10 is the first tool; Definition 12 is the second tool.

Definition 10 *Let X be a Banach space. Then the space **Rad**(X), or simply $\widetilde{X}$, is*

$$Rad(X) \overset{or}{=} \widetilde{X} := \Big\{ \{x_k\}_{k\in\mathbb{Z}} \in X^{\mathbb{Z}} \colon \sum_{k=-n}^{n} r_k(\cdot)\, x_k \colon [0,1] \to X \text{ is convergent in } L_1([0,1], X) \Big\}.$$

For $1 \leq p < \infty$, when equipped with one of the following norms, which are equivalent by Kahane's inequality,

$$\|\{x_k\}_{k\in\mathbb{Z}}\|_{\mathrm{Rad}_p(X)} := \left\|\sum_{k\in\mathbb{Z}} r_k(\cdot)\, x_k\right\|_{L_p([0,1],X)},$$

$\mathrm{Rad}_p(X)$ becomes a Banach space. Much can be found about $\mathrm{Rad}(X)$ in the literature (see, e.g. [11]).

Condition (8) can thus be reformulated as follows:

$$\|f\|_{L_q(\mathbb{R}^N,X)} \sim \left\|\left\{\check{\phi}_k * f\right\}_{k\in\mathbb{Z}}\right\|_{L_q(\mathbb{R}^N,\widetilde{X})}.$$

It is now possible to estimate the blocks simultaneously in the following way: for a given function $m\colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$, simultaneously *roll in* all the blocks to the 0^{th} block by defining the corresponding mapping $M\colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)$ by

$$M(s) := \left\{\phi_0(s) m(2^k s)\right\}_{k\in\mathbb{Z}}. \tag{11}$$

Next (if possible), apply Theorem 4 to the function M to get a Fourier multiplier operator $T_M\colon L_q\left(\widetilde{X}\right) \to L_q\left(\widetilde{Y}\right)$, which then can be *rolled back out* to a Fourier multiplier operator $T_m\colon L_q(X) \to L_q(Y)$. This approach leads to the following theorem (note that if X has Fourier type p (resp. UMD), then so does $\widetilde{X}$).

Theorem 11 [15] *Let X and Y be UMD Banach spaces with Fourier type $p \in (1,2]$ and $1 < q < \infty$. Let $m : \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$ be a measurable function so that the corresponding mapping M, as defined in (11), satisfies that $M \in B^s_{p,1}\left(\mathbb{R}^N, \mathcal{B}\left(\widetilde{X},\widetilde{Y}\right)\right)$ for some $s > N/p$. Then m is a Fourier multiplier from $L_q(\mathbb{R}^N,X)$ to $L_q(\mathbb{R}^N,Y)$.*

The assumption in Theorem 11 may look awkward; however, it is general enough to yield *classical* multiplier theorems, with the help of our second tool.

Definition 12 *A subset τ of $\mathcal{B}(X,Y)$ is **R-bounded** provided there is a constant C_p so that for each $n \in \mathbb{N}$ and subset $\{T_j\}_{j=1}^n$ of τ and subset $\{x_j\}_{j=1}^n$ of X*

$$\left\|\sum_{j=1}^n r_j(\cdot)\, T_j(x_j)\right\|_{L_p(\Omega,Y)} \leq C_p \left\|\sum_{j=1}^n r_j(\cdot)\, x_j\right\|_{L_p(\Omega,X)} \tag{12}$$

for some (and thus then, by Kahane's inequality, for each) $p \in [1, \infty)$. *The* ***R-bound*** *of* τ, $R(\tau)$, *is smallest constant* C_1 *for which* (12) *holds.*

Note the following connection between our two tools: Definition 10 and Definition 12.

Remark 13 *A sequence* $\{T_j\}_{j\in\mathbb{Z}}$ *from* $\mathcal{B}(X,Y)$ *is R-bounded if and only if the mapping*

$$\widetilde{X} \ni \{x_j\}_{j\in\mathbb{Z}} \quad \xrightarrow{\widetilde{T}} \quad \{T_j x_j\}_{j\in\mathbb{Z}} \in \widetilde{Y}$$

defines an element in $\mathcal{B}\left(\widetilde{X}, \widetilde{Y}\right)$ *for some (or equiv., for each)* $p \in [1, \infty)$.

The statements, and proofs, of the following corollaries to Theorem 11 are similar to the corresponding corollaries to Theorem 7.

Corollaries of Theorem 11. [15]

Let X and Y be UMD spaces with Fourier type p and $1 < q < \infty$.

Mihlin condition If for $m \colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$ the set

$$\left\{ |t|^{|\alpha|} D^\alpha m(t) : t \in \mathbb{R}^N \setminus \{0\} \right\}$$

is R-bounded for each multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq [\frac{N}{p}] + 1$, then m is a Fourier multiplier from $L_q\left(\mathbb{R}^N, X\right)$ to $L_q\left(\mathbb{R}^N, Y\right)$.

Hörmander condition If for $m \colon \mathbb{R}^N \setminus \{0\} \to \mathcal{B}(X,Y)$ the term

$$\left[\int_{\frac{1}{2}<|t|<2} R\left(\left\{ \left|2^k t\right|^{|\alpha|} D^\alpha m\left(2^k t\right) \right\}_{k\in\mathbb{Z}} \right)^p dt \right]^{1/p}$$

is finite for each multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq [\frac{N}{p}] + 1$, then m is a Fourier multiplier from $L_q\left(\mathbb{R}^N, X\right)$ to $L_q\left(\mathbb{R}^N, Y\right)$.

Lipschitz condition If for $m \colon \mathbb{R} \to \mathcal{B}(X,Y)$ the set

$$\left\{ m(t), |t|^l \frac{m(t+s) - m(t)}{|s|^l} : t, s \in \mathbb{R} \setminus \{0\} \right\}$$

is R-bounded for some $l \in (1/p, 1)$, then m is a Fourier multiplier from $L_q\left(\mathbb{R}^N, X\right)$ to $L_q\left(\mathbb{R}^N, Y\right)$.

For the connection between R-boundedness and multiplier theorems with scalar multiplier functions, see [8]. The operator-valued Mihlin multiplier theorem was first proved for $N = 1$ in [28] and for higher dimensions in [25]. For variants of the proof of the Mihlin-type result, see [4, 5, 9, 10, 16].

Now returning to pseudo differential operators, define the symbol class $\Re S^0_{1,\delta}(r, X)$ similarly to the symbol class $S^0_{1,\delta}(r, X)$: just replace (7) by the condition that

$$R\left(\left\{(1+|s|)^{|\alpha|}\,\partial_s^\alpha a(t,s) : s \in \mathbb{R}^N\right\}\right) \leq C_\alpha \quad \text{for each} \quad t \in \mathbb{R}^N .$$

In this context, using the same tools and the Coifman-Meyer decomposition of symbols, Z. Štrkalj has shown, among other things, the following boundedness result.

Theorem 14 [24] *Let X be a separable UMD Banach space, $\delta \in [0, 1)$, and $r > 0$. If $a \in \Re S^0_{1,\delta}(r, X)$ then Ψ_a is bounded on $L_q(\mathbb{R}^N, X)$ for each $q \in (1, \infty)$.*

References

[1] Amann, H.: Linear and quasilinear parabolic problems. Vol. I, Birkhäuser Boston Inc., Boston, MA, 1995, Abstract linear theory.

[2] Amann, H.: Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications. Math. Nachr. 186 (1997), 5–56.

[3] Amann, H.: Elliptic operators with infinite-dimensional state spaces. J. Evol. Equ. 1 (2001), no. 2, 143–188.

[4] Arendt, W., Bu, S.: The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. Math. Z., (to appear).

[5] Blunck, S.: Maximal regularity of discrete and continuous time evolution equations. Studia Math. 146 (2001), no. 2, 157–176.

[6] Bourgain, J.: Some remarks on Banach spaces in which martingale difference sequences are unconditional. Ark. Mat. 21 (1983), no. 2, 163–168.

[7] Bourgain, J.: Vector-valued singular integrals and the H^1-B M O duality. Probability theory and harmonic analysis (Cleveland, Ohio, 1983), Dekker, New York, 1986, pp. 1–19.

[8] Clément, P., de Pagter, B., Sukochev, F.A., Witvliet, H.: Schauder decomposition and multiplier theorems. Studia Math. 138 (2000), no. 2, 135–163.

[9] Clément, P., Prüss, J.: An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces. Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Dekker, New York, 2001, pp. 67–87.

[10] Denk,R., Hieber, M., Prüß, J.: R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Amer. Math. Soc., (to appear).

[11] Diestel, J., Jarchow, H., Tonge, A.: Absolutely summing operators. Cambridge University Press, Cambridge, 1995.

[12] Girardi, M., Weis, L.: Criteria for R-boundedness of operator families. Recent Contributions to Evolution Equations. Marcel Dekker Lecture Notes in Math., (to appear).

[13] Girardi, M., Weis, L.: Integral operators with operator-valued kernels. (in preparation).

[14] Girardi, M., Weis, L.: Operator-valued Fourier multiplier theorems on Besov spaces. Mathematische Nachrichten, (to appear).

[15] Girardi, M., Weis, L.: Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces. (submitted).

[16] Haller, R., Heck, H., Noll, A.: Mikhlin's theorem for operator-valued Fourier multipliers on n-dimensional domains. Math. Nachr., (to appear).

[17] Kalton, N.J., Lancien, G.: A solution to the problem of L^p-maximal regularity. Math. Z. 235 (2000), no. 3, 559–568.

[18] Kunstmann, P.C.: Maximal L_p-regularity for second order elliptic operators with uniformly continuous coefficients on domains. (submitted).

[19] Kunstmann, P.C., Weis, L.: Perturbation theorems for maximal L_p-regularity. Ann. Sc. Norm. Sup. Pisa, 30 (2001), 415–435.

[20] Kwapień, S.: Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. Studia Math. 44 (1972), 583–595, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI.

[21] Latushkin, Y., Räbiger, F.: Fourier multipliers in stability and control theory. (preprint).

[22] Peetre, J.: Sur la transformation de Fourier des fonctions à valeurs vectorielles. Rend. Sem. Mat. Univ. Padova 42 (1969), 15–26.

[23] Schulze, B.-W.: Boundary value problems and singular pseudo-differential operators. John Wiley & Sons Ltd., Chichester, 1998.

[24] Štrkalj, Ž.: $\mathcal{R}$-Beschränktheit, Summensätze abgeschlossener Operatoren und operatorwertige Pseudodifferentialoperatoren. Ph.D. thesis, Universität Karlsruhe, Karlsruhe, Germany, 2000.

[25] Štrkalj, Ž., Weis, L.: On operator-valued Fourier multiplier theorems. (submitted).

[26] Weis, L.: Stability theorems for semi-groups via multiplier theorems. Differential equations, asymptotic analysis, and mathematical physics (Potsdam, 1996), Akademie Verlag, Berlin, 1997, pp. 407–411.

[27] Weis, L.: A new approach to maximal L_p-regularity. Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Dekker, New York, 2001, pp. 195–214.

[28] Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity. Math. Ann. 319 (2001), no. 4, 735–758.

[29] Zimmermann, F.: On vector-valued Fourier multiplier theorems. Studia Math. 93 (1989), no. 3, 201–222.