GEOMETRY OF BANACH SPACES AND BIORTHOGONAL SYSTEMS

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ABSTRACT. A separable Banach space $\mathfrak{X}$ contains ℓ_1 isomorphically if and only if $\mathfrak{X}$ has a bounded fundamental total wc_0^*-stable biorthogonal system. The dual of a separable Banach space $\mathfrak{X}$ fails the Schur property if and only if $\mathfrak{X}$ has a bounded fundamental total wc_0^*-biorthogonal system.

1. INTRODUCTION

Generally it is easier to deal with Banach spaces that have some sort of basis structure, the most useful and commonly used structures being Schauder bases and finite-dimensional Schauder decompositions (FDD). Much research in Banach space theory has gone into proving that if a Banach space which has a Schauder basis or FDD possesses a certain property, then the space has a basis or FDD which reflects the property. While such theorems often give information (for example, by passing to suitable subspaces) about general spaces which do not have a basis or an FDD, they cannot give a classification of all separable spaces which have a certain property in terms of bases for the entire space unless the property itself implies the existence of a basis or FDD in a space which has the property. For that reason it is interesting to consider weaker structures than FDD's and Schauder bases which exist in every separable Banach space and try to prove that a separable Banach space has a certain property if and only if there is structure in the space which reflects the property.

One useful basis-like structure that has been considered for a long time is that of fundamental total biorthogonal system. Markushevich [M] showed in 1943 that each separable Banach space contains a fundamental total biorthogonal system. The main theorems of this paper characterize certain

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geometric properties of a Banach space by which types of bounded fundamental total biorthogoal systems exist in the space. Theorem 1 shows that the dual of a separable Banach space $\mathfrak{X}$ fails the Schur property if and only if $\mathfrak{X}$ contains a bounded fundamental total wc_0^*-biorthogonal system. Recall that the dual of a Banach space $\mathfrak{X}$ fails the Schur property if and only if $\mathfrak{X}$ fails the Dunford-Pettis property or ℓ_1 embeds into $\mathfrak{X}$. Theorem 2 shows that ℓ_1 embeds in a separable Banach space $\mathfrak{X}$ if and only if $\mathfrak{X}$ contains a bounded fundamental total wc_0^*-stable biorthogonal system.

Thirty-two years after Markushevich's result [M, 1943], Ovsepian and Pełczyński showed [OP] that for each positive ε, each separable Banach space contains a $[(1+\sqrt{2})^2+\varepsilon]$-bounded fundamental total biorthogonal system; the following year Pełczyński [P] improved the bound to $(1+\varepsilon)$. The proofs of Theorems 1 and 2 use a combination of the methods in [OP] and [P]. Theorem 15 shows that if $\mathfrak{X}$ is a separable Banach space containing ℓ_1, then there is a $[1+\sqrt{2}+\varepsilon]$-bounded fundamental biorthogonal system $\{x_n,\ x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ with the x_n^*'s arbitrarily close to an isomorphic copy of ℓ_2 sitting in $\mathfrak{X}^*$. Section 5 shows that, in the statement of Theorem 15, the $(1+\sqrt{2}+\varepsilon)$ can **not** be replaced with $(1.02+\varepsilon)$. To the best of our knowledge, this is the first result in the literature which provides the existence of a bounded fundamental biorthogonal system in all spaces which have a certain property, and yet the bound for the systems cannot be arbitrarily close to one.

2. NOTATION AND TERMINOLOGY

Throughout this paper, $\mathfrak{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ denote arbitrary (infinite-dimensional real) Banach spaces. If $\mathfrak{X}$ is a Banach space, then $\mathfrak{X}^*$ is its dual space, $B(\mathfrak{X})$ is its (closed) unit ball, $S(\mathfrak{X})$ is its unit sphere, $\delta\colon \mathfrak{X} \to \mathfrak{X}^{**}$ is the natural point-evaluation isometric embedding, and $\widehat{x} = \delta(x)$. If Y is a subset of $\mathfrak{X}$, then $\mathrm{sp}\{Y\}$ is the linear span of Y while $[Y]$ is the closed linear span of Y. Often used are the unit vector basis $\{\delta_n\}$ of ℓ_1, the Kronecker delta δ_{nm}, and the space $C(K)$ of continuous functions on a compact Hausdorff space K.

If $a > 0$, then $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$ is an *ab–isomorphic embedding* provided

$$a^{-1}\,\|x\| \;\leqslant\; \|Tx\| \;\leqslant\; b\,\|x\|$$

for each $x \in \mathfrak{X}$; in this case, $T_o \in \mathcal{L}(\mathfrak{X}, T\mathfrak{X})$ denotes the bijective operator that agrees with T on $\mathfrak{X}$. A surjective τ–isomorphic embedding $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$ is a *τ–isomorphism*; in this case, $\mathfrak{X}$ and $\mathcal{Y}$ are *τ–isomorphic.*

Recall that for a subset X of $\mathfrak{X}$ and a subset Z of $\mathfrak{X}^*$

(1) X is *fundamental* if $[X] = \mathfrak{X}$, or, equivalently, the annihlator $X^{\perp}$ of X in $\mathfrak{X}^*$ is $\{0\}$,
(2) Z is *total* if the weak*-closure of $\text{sp}\{Z\}$ is $\mathfrak{X}^*$, or, equivalently, the preannihilator $Z^{\top}$ of Z in $\mathfrak{X}$ is $\{0\}$,
(3) for a fixed $\tau \geqslant 1$, Z *τ-norms* X (or X is τ-normed by Z) if

$$\|x\| \ \leqslant \ \tau \sup_{z \in Z \setminus \{0\}} \frac{z(x)}{\|z\|}$$

for each $x \in X$,
(4) Z *norms* X if Z 1-norms X.

If Z τ-norms $\mathfrak{X}$ for a $\tau \geqslant 1$ then Z is total. Also, $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $X \times Z$ is

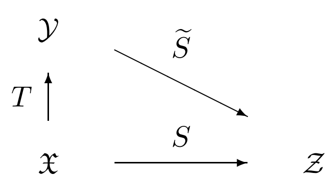
(1) a *biorthogonal system* if $x_n^*(x_m) = \delta_{nm}$,
(2) *M-bounded* if $\{x_n\}$ and $\{x_n^*\}$ are bounded and $\sup_n \|x_n\| \, \|x_n^*\| \leqslant M$,
(3) *bounded* if it is M-bounded for some (finite) M,
(4) *fundamental* if $\{x_n\}$ is fundamental,
(5) *total* if $\{x_n^*\}$ is total.

A biorthogonal system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ is:

(1) a *wc_0^*-biorthogonal system* if $\{x_n^*\}$ is a semi-normalized (i.e., bounded and bounded away from zero) weakly-null sequence,
(2) a *wc_0^*-stable biorthogonal system* if, for each isomorphic embedding T of $\mathfrak{X}$ into some $\mathcal{Y}$, there exists a lifting $\{y_n^*\}$ of $\{x_n^*\}$ (i.e., $T^* y_n^* = x_n^*$ for each n) such that $\{y_n^*\}$ is a semi-normalized weakly-null sequence in $\mathcal{Y}^*$ (or equivalently, such that $\{Tx_n,\ y_n^*\}$ in $\mathcal{Y} \times \mathcal{Y}^*$ is a wc_0^*-biorthogonal system).

Bases of type wc_0^* were introduced in [FS] (cf. [S1, II.7 and pg. 625–626]).

Recall that $\mathcal{Z}$ is *injective* if for each pair $\mathfrak{X}$ and $\mathcal{Y}$, each isomorphic embedding $T \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$, and each $S \in \mathcal{L}(\mathfrak{X}, \mathcal{Z})$, there exists $\widetilde{S} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that the following diagram commutes.



If $\mathcal{Z}$ is injective, then there exists $\lambda \geqslant 1$ so that $\mathcal{Z}$ is *λ-injective*, i.e. $\widetilde{S}$ can be chosen so that $\|\widetilde{S}\| \leqslant \lambda \left\| S T_o^{-1} \right\|$. Recall $\mathcal{Z}$ is a *Grothendieck space* if weak* and weak sequential convergence in $\mathcal{Z}^*$ coincide; an injective space is a Grothendieck space (cf. [LT3, p. 188]). $\mathcal{Z}$ has the *Schur property* if weak and strong sequential convergence in $\mathcal{Z}$ coincide.

All notation and terminology, not otherwise explained, are as in [DU] or [LT1].

3. THE FINE LINE BETWEEN WC_0^* AND WC_0^*-STABLE

The unit vectors $\{e_n^p,\ e_n^q\}$ in $\ell_p \times \ell_q$, where $1 \leqslant p < \infty$ and q is the conjugate exponent of p, form a 1-bounded fundamental total wc_0^*-biorthogonal system. For $p = 1$, they are even a wc_0^*-stable biorthogonal system, as the proof of (a) implies (b) in Theorem 2 shows. The next two theorems clarify the fine line between the existence of nice wc_0^*-biorthogonal and wc_0^*-stable biorthogonal systems.

Theorem 1. *The following statements are equivalent.*

(a) $\mathfrak{X}^*$ *fails the Schur property.*

(b) *There is a bounded* wc_0^**-biorthogonal system in* $\mathfrak{X} \times \mathfrak{X}^*$.

And in the case that $\mathfrak{X}$ *is separable:*

(c) *There is a bounded fundamental total* wc_0^**-biorthogonal system* $\{x_n,\ x_n^*\}$ *in* $\mathfrak{X} \times \mathfrak{X}^*$.

Furthermore for each $\varepsilon > 0$*: if* (b) *holds then the system can be taken to be* $(1+\varepsilon)$*-bounded; if* (c) *holds then the system can be taken to be* $[2(1+\sqrt{2})^2+\varepsilon]$*-bounded and so that* $[x_n^*]$ *norms* $\mathfrak{X}$.

Recall (cf. [D2, p. 23]) that $\mathfrak{X}^*$ fails the Schur property if and only if $\mathfrak{X}$ fails the Dunford-Pettis property or $\ell_1 \hookrightarrow \mathfrak{X}$.

Theorem 2. *The following statements are equivalent.*

(a) $\ell_1 \hookrightarrow \mathfrak{X}$.

(b) *There is a bounded* wc_0^**-stable biorthogonal system in* $\mathfrak{X} \times \mathfrak{X}^*$.

And in the case that $\mathfrak{X}$ *is separable:*

(c) *There is a bounded fundamental total* wc_0^**-stable biorthogonal system* $\{x_n,\ x_n^*\}$ *in* $\mathfrak{X} \times \mathfrak{X}^*$.

Furthermore for each $\varepsilon > 0$*: if* (b) *holds then the system can be taken to be* $(1+\varepsilon)$*-bounded; if* (c) *holds then the system can be taken to be* $[(1+\sqrt{2})+\varepsilon]$*-bounded and so that* $[x_n^*]$ $(2+\varepsilon)$*-norms* $\mathfrak{X}$.

In this section are the proofs of the easier implications in the above theorems. The other implications follow from the results of the next section.

Proof of (b) *implies* (a) *in Theorem 1.* A wc_0^*-biorthogonal system in $\mathfrak{X}\times\mathfrak{X}^*$ is enough to force $\mathfrak{X}^*$ to fail the Schur property. ■

Proof of (b) *implies* (a) *in Theorem 2.* Find an (isometric) embedding T of $\mathfrak{X}$ into a $C(K)$-space. Assume that there is a wc_0^*-biorthogonal system $\{Tx_n,\ y_n^*\}$ in $C(K)\times C^*(K)$ with $\{x_n\}$ bounded, which would be the case if (b) held. If $\{x_n\}$ had a weakly Cauchy subsequence $\{x_{n_k}\}$, then $\{Tx_{n_k}\}$ would be weakly Cauchy and $\{y_{n_k}^*\}$ would be weakly null, which cannot be since a $C(K)$ space has the Dunford-Pettis property (cf. [D2, p. 20]). So, by Rosenthal's ℓ_1 theorem, $\{x_n\}$ admits a subsequence that is equivalent to the unit vector basis of ℓ_1. ■

The above proof reveals somewhat more.

Remark 3. In the definition of wc_0^*-stable biorthogonal system, if the word *isomorphic* is replaced with *isometric* then the statement of Theorem 2 remains true. ■

Remark 4. If $\{x_n,\ x_n^*\}$ in $\mathfrak{X}\ \times\ \mathfrak{X}^*$ is either:

1. a bounded wc_0^*-biorthogonal system and $\mathfrak{X}$ has the Dunford-Pettis Property

or

2. a bounded wc_0^*-stable biorthogonal system,

then each subsequence of $\{x_n\}$ contains a further subsequence that is equivalent to the unit vector basis of ℓ_1. ■

That (a) implies (b) in Theorem 1 (with the $(1+\varepsilon)$ bound) follows from Facts 5–7.

Fact 5. *Let $\{x_n\}_{n=1}^{\infty}$ be a weakly null sequence in $\mathfrak{X}$ and $\{g_n\}$ be a bounded sequence in $\mathfrak{X}^*$ and $\varepsilon>0$. Then there exists $m\in\mathbb{N}$ satisfying*

$$|\langle x_m, g_n\rangle| < \varepsilon$$

for infinitely many $n\in\mathbb{N}$.

This follows directly from the fact that, since $\{x_n\}_{n=1}^{\infty}$ is weakly null, there exists a finite sequence $\{\lambda_m\}_{m=1}^{N}$ of positive numbers satisfying

$$\max_{\pm}\left\|\sum_{m=1}^{N}\pm\lambda_m x_m\right\| \ < \ \frac{\varepsilon}{\sup_j\|g_j\|}$$

and $\sum_{m=1}^{N}\lambda_m = 1$ (cf. [W, p. 48, Exercise 13]).

Fact 6. *Let $\{x_n\}_{n=1}^{\infty}$ be a normalized weakly null sequence in $\mathfrak{X}$ and $\varepsilon > 0$. Then there are a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and functionals $\{x_{n_k}^*\}_{k=1}^{\infty}$ biorthogonal to $\{x_{n_k}\}_{k=1}^{\infty}$ so that $\sup_{k\in\mathbb{N}} \left\|x_{n_k}^*\right\| < 1+\varepsilon$.*

Proof. Fix a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive numbers satisfying

$$\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon/6 \ .$$

Without loss of generality (pass to a subsequence), $\{x_n\}_{n=1}^{\infty}$ is a basic sequence with biorthogonal functional $\{f_n\}_{n=1}^{\infty}$ satisfying $\|f_n\| < 3$. These f_n's will be used to perturb functionals as needed.

Without loss of generality (pass to a subsequence), there is a system $\{x_n,\ g_n\}_{n=1}^{\infty}$ in $\mathfrak{X} \ \times \ \mathfrak{X}^*$ satisfying $\|g_n\| < 1+\varepsilon/2$ and

$$\langle x_m, g_n\rangle \ = \ \delta_{mn} \qquad \text{when} \quad n \leqslant m \ .$$

To see how to find such a system by induction, consider a subsequence

$$\{n(j,k)\}_{k=1}^{\infty}$$

in $\mathbb{N}$ given at the beginning of the j^{th} step (for the base step, let $n(1,k) = k$). Let $x_{n_j} = x_{n(j,1)}$ and find $\widetilde{g}_{n_j}$ in $S(\mathfrak{X}^*)$ satisfying $\widetilde{g}_{n_j}(x_{n_j}) = 1$. Find a subsequence $\{n(j+1,k)\}_{k=1}^{\infty}$ of $\{n(j,k)\}_{k=2}^{\infty}$ satisfying

$$\left|\langle x_{n(j+1,k)}, \widetilde{g}_{n_j}\rangle\right| < \varepsilon_k$$

for each $k \in \mathbb{N}$ and let

$$g_{n_j} = \widetilde{g}_{n_j} - \sum_{k=1}^{\infty} \langle x_{n(j+1,k)}, \widetilde{g}_{n_j}\rangle \ f_{n(j+1,k)} \ .$$

Without loss of generality (pass to a subsequence),

$$|\langle x_m, g_n\rangle| < \varepsilon_m \qquad \text{when} \quad m < n \ .$$

To accomplish this, iterate Fact 5 to produce a sequence

$$\{\{n(j,k)\}_{k=1}^{\infty}\}_{j=1}^{\infty}$$

of sequences and a sequence $\{k_j\}_{j=1}^{\infty}$ so that $\{n(j+1,k)\}_{k=1}^{\infty}$ is a subsequence of $\{n(j,k)\}_{k=1}^{\infty}$ and

$$\left|\langle x_{n(j,k_j)}, g_{n(j+1,k)}\rangle\right| \ < \ \varepsilon_j \ .$$

Then the subsequence $n_j = n(j,k_j)$ works.

Clearly, the functionals

$$x_n^* \ := \ g_n \ - \sum_{\{m\in\mathbb{N}:\ m<n\}} \langle x_m, g_n\rangle \, f_m$$

are biorthogonal to $\{x_n\}_{n=1}^{\infty}$ and are of norm at most $1+\varepsilon$. ■

Fact 7. *Let $\{x_n^*,\ x_n^{**}\}_{n=1}^{\infty}$ be a biorthogonal system in $\mathfrak{X}^* \times \mathfrak{X}^{**}$ with $\sup_n \|x_n^{**}\| < 1+\varepsilon$ for some $\varepsilon > 0$ and $\{x_n^*\}$ normalized and weak-star null. Then there is a subsequence $\{n_k\}_{k=1}^{\infty}$ along with a biorthogonal system $\left\{x_{n_k},\ x_{n_k}^*\right\}_{k=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ with $\sup_k \|x_{n_k}\| < 1+\varepsilon$.*

Proof. Without loss of generality (pass to a subsequence), there is a biorthogonal system $\{y_n,\ x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ with

$$\sup_n \|y_n\| \leqslant M < \infty \ .$$

For just let $\mathfrak{X}_0$ be a separable subspace of $\mathfrak{X}$ that 1-norms $[x_n^*]_{n=1}^{\infty}$ and take a $\sigma(\mathfrak{X}_0^*, \mathfrak{X}_0)$-basic subsequence of $\{x_n^*|_{\mathfrak{X}_0}\}$ ([JR], cf. [D1, V.Exercise 7]).

For each $n \in \mathbb{N}$ let

$$E_n := [x_n^{**}]$$
$$F_n := [x_1^*, \ldots, x_n^*] \ .$$

Use the Principle of Local Reflexivity to find a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathfrak{X}$ satisfying

$$\langle z_n, x_k^* \rangle = \langle x_k^*, x_n^{**} \rangle = \delta_{nk} \qquad \text{when} \quad k \leqslant n$$

and

$$\sup_{n\in\mathbb{N}} \|z_n\| < 1+\varepsilon-\varepsilon_0$$

for some $\varepsilon_0 > 0$. Fix a sequence $\{\varepsilon_j\}_{j=2}^{\infty}$ of positive numbers satisfying

$$\sum_{j=2}^{\infty} \varepsilon_j < \frac{\varepsilon_0}{M} \ .$$

Without loss of generality (pass to a subsequence),

$$\left|\langle z_n, x_j^* \rangle\right| < \varepsilon_j \qquad \text{when} \quad n < j \ .$$

Clearly the vectors

$$x_n := z_n \ - \sum_{j=n+1}^{\infty} \langle z_n, x_j^* \rangle \ y_j$$

are biorthogonal to $\{x_n^*\}_{n=1}^{\infty}$ and are of norm at most $1+\varepsilon$. ■

The next lemma provides a means by which to determine whether a wc_0^*-biorthogonal system is wc_0^*-stable.

Lemma 8. *Let $\{x_n,\ x_n^*\}$ be a biorthogonal system such that $\{x_n^*\}$ is a semi-normalized weak*-null sequence in $\mathfrak{X}^*$. Then $\{x_n,\ x_n^*\}$ is a wc_0^*-stable biorthogonal system if and only if the operator $S\colon \mathfrak{X} \to c_0$ given by*

$$S(x) = (x_n^*(x))$$

factors through an injective space.

Proof. Let $\{x_n,\ x_n^*\}$ be a biorthogonal system such that $\{x_n^*\}$ is a semi-normalized weak*-null sequence in $\mathfrak{X}^*$.

First, assume that the above operator S factors through an injective space and let $T\colon \mathfrak{X} \to \mathcal{Y}$ be an isomorphic embedding. Consider the diagram

$$\begin{array}{ccccc} & \overset{T}{\swarrow} & \mathfrak{X} & \overset{S}{\searrow} & \\ & & \downarrow R & & \\ \mathcal{Y} & & \mathcal{Z} & \xrightarrow{\ L\ } & c_0 \end{array}$$

where $\mathcal{Z}$ is an injective space and $S = LR$. Since $\mathcal{Z}$ is injective, there exists $R_1 \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that the following diagram (totally) commutes.

$$\begin{array}{ccccc} & \overset{T}{\swarrow} & \mathfrak{X} & \overset{S}{\searrow} & \\ & & \downarrow R & & \\ \mathcal{Y} & \xrightarrow{\ R_1\ } & \mathcal{Z} & \xrightarrow{\ L\ } & c_0 \end{array}$$

Note that the operator $S \equiv LR$ is given by

$$(LR)(x) = (x_n^*(x)) \quad \text{and so} \quad x_n^* = R^*\, L^*(\delta_n)\ ;$$

similarly, the operator LR_1 has the form

$$(LR_1)(y) = (y_n^*(y)) \quad \text{where} \quad y_n^* = R_1^*\, L^*(\delta_n)\ .$$

It is easy to check that $\{x_n,\ x_n^*\}$ is indeed a wc_0^*-stable biorthogonal system: the commutativity of the diagram gives that $T^* y_n^* = x_n^*$, the weak-nullness of $\{y_n^*\}$ follows from the fact that $\mathcal{Z}$ is a Grothendieck space ($\{L^*\delta_n\}$ is weak*-null and thus weakly-null), and $\|y_n^*\| \geqslant \|T\|^{-1}\, \|x_n^*\|$.

Next assume that $\{x_n,\ x_n^*\}$ is a wc_0^*-stable biorthogonal system. Find an embedding R from $\mathfrak{X}$ into the injective space $\ell_\infty(\Gamma)$ for some index set Γ. By the stability of the system, there exists a weakly-null sequence $\{y_n^*\}$ in $\ell_\infty^*(\Gamma)$ such that $R^* y_n^* = x_n^*$. Define $L\colon \ell_\infty(\Gamma) \to c_0$ by $L(f) = (y_n^*(f))$, for then $S = LR$. ■

The commutative diagram in the next proof was inspired by the Hagler–Johnson proof [HJ] of the Josefson and Nissenzweig Theorem (cf. [D1, Chapter XII]).

Proof of (a) *implies* (b) *in Theorem 2, along with the* $(1+\varepsilon)$ *bound.* Consider the following commutative diagram

$$\begin{array}{ccccc} & \overset{j}{\swarrow} & \ell_1 & \overset{i}{\searrow} & \\ & & \downarrow R_0 & & \\ \mathfrak{X} & & L_\infty & \xrightarrow{\ L\ } & c_0 \end{array}$$

where j is an isomorphic embedding, i is the formal injection, and

$$R_0(\delta_n) = r_n \qquad \text{and} \qquad L(f) = \left(\int f r_n \, d\mu \right)_n$$

for the Rademacher functions $\{r_n\}$. Since L_∞ is 1-injective, there exists an operator R_2 such that the following diagram commutes

$$\begin{array}{ccccc} & \overset{j}{\swarrow} & \ell_1 & \overset{i}{\searrow} & \\ & & \downarrow R_0 & & \\ \mathfrak{X} & \xrightarrow{\ R_2\ } & L_\infty & \xrightarrow{\ L\ } & c_0 \end{array}$$

and $\|R_2\| \leqslant \left\|R_0 j_o^{-1}\right\|$.

The operator $S := L\, R_2$ takes the form

$$S(x) \;=\; (x_n^*\,(x)) \quad \text{where} \quad x_n^* = R_2^*\, L^*\,(\delta_n)\ .$$

It is easy to check that $\{j\delta_n,\ x_n^*\}$ is a wc_0^*-stable biorthogonal system. Biorthogonality follows from the commutativity of the diagram. Since $\{\delta_n\}$ is weak*-null, so is $\{x_n^*\}$. L_∞ is an injective space through which S factors. Furthermore, since R_0 and L both have norm one and $1 = x_n^*(j\delta_n)$,

$$\|j\|^{-1} \;\leqslant\; \|x_n^*\| \;=\; \|R_2^* L^* \delta_n\| \;\leqslant\; \|R_2\|\,\|L\| \;\leqslant\; \left\|R_0 j_o^{-1}\right\| \;\leqslant\; \left\|j_o^{-1}\right\|$$

and so $\{x_n^*\}$ is semi-normalized. If ℓ_1 embeds into $\mathfrak{X}$, then it $(1+\varepsilon)$-embeds into $\mathfrak{X}$, thus one can arrange that $\sup_n \|x_n\|\,\|x_n^*\| \leqslant \|j\|\,\left\|j_o^{-1}\right\| \leqslant 1+\varepsilon$. ■

It is not difficult to see that, if $\mathfrak{X}$ is any Banach space and $\varepsilon > 0$, then there is a $(2+\varepsilon)$-bounded biorthogonal system $\{x_n,\ x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ with $\{x_n^*\}$ weak*-null. The first step towards this is the lemma below.

Lemma 9. *If $\mathfrak{X}_0$ is a finite co-dimensional subspace of $\mathfrak{X}$ and $\varepsilon > 0$, then there is a weak*-closed finite co-dimensional subspace $\mathcal{Y}$ of $\mathfrak{X}^*$ such that $\mathcal{Y}$ is $(2+\varepsilon)$-normed by $\mathfrak{X}_0$.*

To see how to use Lemma 9 to produce the desired biorthogonal system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$, start with a normalized weak*-null sequence $\{y_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X}^*$ (guaranteed to exist by the Josefson-Nissenzweig Theorem) and fix a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers tending to zero. Assume that

$$\left\{x_j,\ x_j^*\right\}_{j<n}$$

have been found. Let

$$\mathfrak{X}_n = \left[x_j^*\right]_{j<n}^{\top} \quad \text{and} \quad \mathcal{Z}_n = \left[x_j\right]_{j<n}^{\perp} \ .$$

By Lemma 9, there is a weak*-closed finite co-dimensional subspace $\mathcal{Y}_n$ of $\mathfrak{X}^*$ that is $(2+\varepsilon/2)$-normed by $\mathfrak{X}_n$. Since $\mathcal{Y}_n \cap \mathcal{Z}_n$ is finite co-dimensional and weak*-closed and $\{y_n^*\}_{n=1}^{\infty}$ is weak*-null, there exists $x_n^* \in S\left(\mathcal{Y}_n \cap \mathcal{Z}_n\right)$ with $\left\|x_n^* - y_{k_n}^*\right\| < \varepsilon_n$ for some large k_n. Next find $\widetilde{x}_n \in S\left(\mathfrak{X}_n\right)$ with $1 \leqslant (2+\varepsilon)x_n^*\left(\widetilde{x}_n\right)$ and let $x_n := \widetilde{x}_n / x_n^*\left(\widetilde{x}_n\right)$.

Proof of Lemma 9. Let $\mathcal{Y}$ be the annihilator of any finite dimensional subspace of $\mathfrak{X}$ that $(1+\varepsilon)$-norms the annihilator of $\mathfrak{X}_0$. For then if $f \in S(\mathcal{Y})$ then

$$\begin{aligned}
\sup_{x_0 \in S(\mathfrak{X}_0)} |f(x_0)| &= \inf_{y^* \in \mathfrak{X}_0^{\perp}} \|f - y^*\| \\
&\geqslant \inf_{y^* \in \mathfrak{X}_0^{\perp}} \max \left[\, \|f\| - \|y^*\| \ ,\ \sup_{x \in S(\mathcal{Y}^{\top})} \left|\left(f - y^*\right)(x)\right| \right] \\
&\geqslant \inf_{y^* \in \mathfrak{X}_0^{\perp}} \max \left[1 - \|y^*\| \ ,\ \tfrac{1}{1+\varepsilon} \|y^*\| \right] \\
&\geqslant \inf_{0 \leqslant t < \infty} \max \left[1 - t \ ,\ \tfrac{t}{1+\varepsilon}\right] \\
&= (2+\varepsilon)^{-1} \ .
\end{aligned}$$

■

Lemma 9 is nearly best possible since, for each $\varepsilon > 0$, the one co-dimensional subspace $\mathfrak{X}_0$ of mean zero functions in L_1 does not $(2-\varepsilon)$-norm any finite co-dimensional subspace of L_∞. Indeed, any finite co-dimensional subspace of L_∞ contains a norm one functional y^* that is bounded below by $-\varepsilon$ (just perturb a disjointly supported sequence of nonnegative norm one functions in L_∞ that are close to $\mathcal{Y}$) and so $y^*(x) \leqslant \frac{1}{2}(1+\varepsilon)$ for each $x \in S\left(\mathfrak{X}_0\right)$. However, any one co-dimensional subspace $\mathfrak{X}_0$ of a Banach space $\mathfrak{X}$ does 2-norm a one co-dimensional subspace $\mathcal{Y}$, namely $\mathcal{Y} := \ker P$ where $P\colon \mathfrak{X}^* \to \mathfrak{X}_0^\perp$ is a norm one projection. Indeed, if $f \in S(\mathcal{Y})$ then

$$\begin{aligned} \|f - y^*\| \ &\geqslant \ \frac{1}{2} \ [\ \|f - y^*\| \ + \ \|P(f - y^*)\|\ ] \\ &= \ \frac{1}{2} \ [\ \|f - y^*\| \ + \ \|y^*\|\ ] \ \geqslant \ \frac{1}{2} \end{aligned}$$

for each $y^* \in \mathfrak{X}_0^\perp$.

4. CONSTRUCTING FUNDAMENTAL TOTAL wc_0^*-BIORTHOGONAL SYSTEMS

The constructions of fundamental total biorthogonal systems in the proofs of (a) implies (c) in Theorems 1 and 2 use the Haar matrices, which are summarized below.

Remark 10. Fix $m \geqslant 0$ and consider the 2^m-dimensional Hilbert space $\ell_2^{2^m}$, along with its unit vector basis $\{e_j^2\}_{j=1}^{2m}$.

The Haar basis $\{h_j^m\}_{j=1}^{2^m}$ of $\ell_2^{2^m}$ can be described as follows. For $0 \leqslant n \leqslant m$ and $1 \leqslant k \leqslant 2^n$ let

$$I_k^n \ = \ \left\{ j \in \mathbb{N} : \ 2^{m-n}\,(k-1) \ < \ j \ \leqslant \ 2^{m-n}\,k \right\} \ .$$

Thus

$$I_1^0 = \{1, 2\,, \ \ldots\ , \ 2^m\}$$
$$I_1^1 = \left\{1, 2\,, \ \ldots\ , \ 2^{m-1}\right\} \quad \text{and} \quad I_2^1 = \left\{1 + 2^{m-1}\,, \ \ldots\ , \ 2^m\right\} \ .$$

In general, the collection $\{I_k^n\}_{k=1}^{2^n}$ of sets along the n^{th}-level (disjointly) partitions $\{1, 2, \ldots, 2^m\}$ into 2^n sets, each containing 2^{m-n} consecutive integers, and I_k^n is the disjoint union $I_k^n = I_{2k-1}^{n+1} \cup I_{2k}^{n+1}$. Now let

$$h_1^m \ = \ 2^{\frac{-m}{2}} \sum_{j \in I_1^0} e_j^2$$

and, for $0 \leqslant n < m$ and $1 \leqslant k \leqslant 2^n$, let $h_{2^n+k}^m$ be supported on I_k^n as

$$h_{2^n+k}^m \ = \ 2^{\frac{n-m}{2}} \left[\sum_{j \in I_{2k-1}^{n+1}} e_j^2 \ - \ \sum_{j \in I_{2k}^{n+1}} e_j^2 \right] \ .$$

Note that $\{h_j^m\}_{j=1}^{2^m}$ forms an orthonormal basis for $\ell_2^{2^m}$.

Let $H_m = \left(a_{ij}^m\right)$ be the $2^m \times 2^m$ Haar matrix that transforms the unit vector basis of $\ell_2^{2^m}$ onto the Haar basis; thus, the j^{th} column vector of H_m

is just h_j^m and so H_m is a unitary matrix. For example,

$$H_2 = \begin{bmatrix} 2^{-1} & +2^{-1} & +2^{-1/2} & 0 \\ 2^{-1} & +2^{-1} & -2^{-1/2} & 0 \\ 2^{-1} & -2^{-1} & 0 & +2^{-1/2} \\ 2^{-1} & -2^{-1} & 0 & -2^{-1/2} \end{bmatrix} .$$

Let $\left\{z_j,\ z_j^*\right\}_{j=1}^{2^m}$ be a biorthogonal sequence in $S(\mathfrak{X}) \times \mathfrak{X}^*$. Consider $\{x_i,\ x_i^*\}_{i=1}^{2^m}$ where

$$H_m \begin{bmatrix} z_1 \\ \vdots \\ z_{2^m} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{2^m} \end{bmatrix} \qquad \text{and} \qquad H_m \begin{bmatrix} z_1^* \\ \vdots \\ z_{2^m}^* \end{bmatrix} = \begin{bmatrix} x_1^* \\ \vdots \\ x_{2^m}^* \end{bmatrix} ,$$

thus

$$x_i := \sum_{j=1}^{2^m} a_{ij}^m z_j \qquad \text{and} \qquad x_i^* := \sum_{j=1}^{2^m} a_{ij}^m z_j^* .$$

Since H_m is a unitary matrix

(H1) $x_i^*(x_j) = \delta_{ij}$

(H2) $[x_i]_{i=1}^{2^m} = [z_j]_{j=1}^{2^m}$

(H3) $[x_i^*]_{i=1}^{2^m} = \left[z_j^*\right]_{j=1}^{2^m}$.

Note that, for each $1 \leqslant i \leqslant 2^m$,

(H4) $a_{i1}^m = 2^{-m/2}$

and the ℓ_1-norm of the i^{th} row of H_m is bounded

(H5) $\sum_{j=1}^{2^m} \left|a_{ij}^m\right| = 1 + \sqrt{2} - 2^{\frac{1-m}{2}} \quad \overset{m\to\infty}{\nearrow} \; 1 + \sqrt{2}$

and so

(H6) $\|x_i\| \leqslant 2^{-m/2} \|z_1\| + \left(1+\sqrt{2}\right) \max_{1<j\leqslant 2^m} \|z_j\|$

(H7) $\|x_i^*\| \leqslant 2^{-m/2} \|z_1^*\| + \left(1+\sqrt{2}\right) \max_{1<j\leqslant 2^m} \left\|z_j^*\right\|$

(H8) for each $x^{**} \in \mathfrak{X}^{**}$

$|x^{**}(x_i^*)| \leqslant \|x^{**}\| \, 2^{-m/2} \|z_1^*\| + \left(1+\sqrt{2}\right) \max_{1<j\leqslant 2^m} \left|x^{**}\left(z_j^*\right)\right|$.

Definition 11. A sequence $\{J_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ is a *blocking* of $\mathbb{N}$ if $\mathbb{N}$ is the disjoint union $\cup_{k=1}^{\infty} J_k$ and

$$\max J_k < \min J_{k+1}$$

for each $k \in \mathbb{N}$. Given a blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$, let $J_0 = \{0\}$ and

$$\begin{aligned} J_k^p &:= \bigcup_{0 \leqslant j < k} J_j \\ J_k^o &:= J_k \setminus \{\text{the first element in } J_k\} \\ J_k^{po} &:= \bigcup_{0 \leqslant j < k} J_k^o \\ \mathbb{N}^o &:= \bigcup_{k=1}^{\infty} J_k^o \end{aligned}$$

for each $k \in \mathbb{N}$.

From the next theorem it easily follows, when $\mathfrak{X}$ is separable, that (a) implies (c) in Theorem 1.

Theorem 12. *Let $\mathfrak{X}^*$ fail the Schur property. Fix $\varepsilon > 0$ along with $\{a_n,\ b_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$. Then there exists a $[2(1+\sqrt{2})^2 + \varepsilon]$-bounded wc_0^*-biorthogonal system $\{x_n,\ x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$ such that $[a_n] \subset [x_n]$ and $[b_n^*] \subset [x_n^*]$.*

Proof. Without loss of generality, $[a_n]_{n\in\mathbb{N}}$ and $[b_n^*]_{n\in\mathbb{N}}$ are each infinite dimensional. Fix a sequence $\{\delta_k\}_{k=1}^{\infty}$ of positive numbers decreasing to zero. Since $\mathfrak{X}^*$ fails the Schur property, there is a weakly-null sequence $\{w_i^*\}_{i=1}^{\infty}$ in $S(\mathfrak{X}^*)$.

It suffices to find a system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ along with a blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$, a sequence $\{\beta_n\}_{n\in\mathbb{N}^o}$ from $(0, 2+\varepsilon]$, and an increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$, satisfying

(1) $x_m^*(x_n) = \delta_{mn}$
(2) $\|x_n\| \leqslant \left(1+\sqrt{2}\right) + \varepsilon$
(3) $\|x_n^*\| \leqslant (2+\varepsilon)\left(1+\sqrt{2}\right) + \varepsilon$
(4) for each $x^{**} \in S\left(X^{**}\right)$, if $n \in J_k$ then
$|x^{**}\left(x_n^*\right)| \leqslant \delta_k + \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \left(\left|x^{**}\left(\beta_j w_{i_j}^*\right)\right| + \beta_j \delta_k \right)$
(5) $[a_n]_{n=1}^{\infty} \subset [x_n]_{n=1}^{\infty}$
(6) $[b_n^*]_{n=1}^{\infty} \subset [x_n^*]_{n=1}^{\infty}$.

The construction will inductively produce blocks $\{x_n,\ x_n^*\}_{n\in J_k}$. Let x_0 and x_0^* be the zero vectors and $j_0 = 0$. Fix $k \geqslant 1$. Assume that $\{J_j\}_{0\leqslant j<k}$ along with $\{x_n,\ x_n^*\}_{n\in J_k^p}$ and $\{i_n\}_{n\in J_k^{po}}$ and $\{\beta_n\}_{n\in J_k^{po}}$ have been constructed to satisfy conditions (1) through (4). Now to construct J_k along with $\{x_n,\ x_n^*\}_{n\in J_k}$ and $\{i_n\}_{n\in J_k^o}$ and $\{\beta_n\}_{n\in J_k^o}$.

Let

$$\mathcal{P}_k := [x_n^*]_{n\in J_k^p}^{\top} \qquad \text{and} \qquad \mathcal{Q}_k := [x_n]_{n\in J_k^p}^{\perp}$$

and

$$n_k = \max J_k^p \ .$$

The idea is to find a biorthogonal system $\{z_n,\ z_n^*\}_{n\in J_k}$ in $\mathcal{P}_k \times \mathcal{Q}_k$ by first finding $\{z_{1+n_k}, z_{1+n_k}^*\}$ which helps guarantee condition (5) if k is odd and condition (6) if k even; however, $\{z_{1+n_k}, z_{1+n_k}^*\}$ would not necessarily satisfy conditions (2) through (4) and so J_k^o and

$$\{z_n,\ z_n^*\}_{n\in J_k^o} \ ,$$

along with $\{i_n\}_{n\in J_k^o}$ and $\{\beta_n\}_{n\in J_k^o}$ are constructed and then the Haar matrix is applied to $\{z_n,\ z_n^*\}_{n\in J_k}$ to produce $\{x_n,\ x_n^*\}_{n\in J_k}$ so that

$$\{x_n,\ x_n^*\}_{n\in J_k^p\cup J_k}$$

with $\{i_n\}_{n\in J_k^{po}\cup J_k^o}$ and $\{\beta_n\}_{n\in J_k^{po}\cup J_k^o}$ satisfy conditions (1) through (4).

$\left\{z_{1+n_k},\ z_{1+n_k}^*\right\}$ is constructed by a standard Gram-Schmidt biorthogonal procedure. If k is odd, start in $\mathfrak{X}$. Let

$$h_k \ = \ \min\left\{h \colon a_h \notin [x_n]_{n\leqslant n_k}\right\} \ .$$

Set

$$z_{1+n_k} \ = \ a_{h_k} \ - \sum_{n\leqslant n_k} x_n^*(a_{h_k})x_n \ ,$$

and for any $y_{1+n_k}^*$ in $\mathfrak{X}^*$ such that $y_{1+n_k}^*(z_{1+n_k}) \neq 0$,

$$z_{1+n_k}^* \ = \ \frac{y_{1+n_k}^* \ - \ \sum_{n\leqslant n_k} y_{1+n_k}^*(x_n)x_n^*}{y_{1+n_k}^*(z_{1+n_k})} \ .$$

If k is even, start in $\mathfrak{X}^*$. Let

$$h_k \ = \ \min\left\{h \colon b_h^* \notin [x_n^*]_{n\leqslant n_k}\right\} \ .$$

Set

$$z_{1+n_k}^* \ = \ b_{h_k}^* \ - \sum_{n\leqslant n_k} b_{h_k}^*(x_n)x_n^* \ ,$$

and, for any y_{1+n_k} in $\mathfrak{X}$ such that $z_{1+n_k}^*(y_{1+n_k}) \neq 0$,

$$z_{1+n_k} \ = \ \frac{y_{1+n_k} \ - \ \sum_{n\leqslant n_k} x_n^*(y_{1+n_k})x_n}{z_{1+n_k}^*(y_{1+n_k})} \ .$$

Clearly $z_{1+n_k}^*(z_{1+n_k}) = 1$ and

$$z_{1+n_k} \ \in \ \mathcal{P}_k \qquad \text{and} \qquad z_{1+n_k}^* \ \in \ \mathcal{Q}_k \ .$$

Find a natural number m_k larger than one so that

$$2^{-m_k/2} \ \max\left(\ \|z_{1+n_k}\| \ , \ \left\|z^*_{1+n_k}\right\| \ \right) \ < \ \min\left(\ \varepsilon \ , \ \delta_k \ \right)$$

and let

$$J_k := \{1+n_k, \ldots, 2^{m_k}+n_k\} \quad \text{and so} \quad J_k^o := \{2+n_k, \ldots, 2^{m_k}+n_k\} \ .$$

Let

$$\widetilde{\mathcal{P}}_k \ := \ \mathcal{P}_k \cap \left[z^*_{1+n_k}\right]^{\top} \qquad \text{and} \qquad \widetilde{\mathcal{Q}}_k \ := \ \mathcal{Q}_k \cap \left[z_{1+n_k}\right]^{\perp} \ .$$

The next step is to find a biorthogonal system $\{z_n, \ z^*_n\}_{n\in J_k^o}$ along with $\{i_n\}_{n\in J_k^o}$ and $\{\beta_n\}_{n\in J_k^o}$ satisfying

$$\{z_n, \ z^*_n\} \in S\left(\widetilde{\mathcal{P}}_k\right) \ \times \ (2+\varepsilon)\, B\left(\widetilde{\mathcal{Q}}_k\right) \tag{1}$$

and

$$\left\| \frac{z^*_n}{\beta_n} - w^*_{i_n} \right\| \ < \ \delta_k \tag{2}$$

for each $n \in J_k^o$. Towards this, fix $j \in J_k^o$ and assume that a biorthogonal system $\{z_n, \ z^*_n\}_{2+n_k\leqslant n<j}$ along with $\{i_n\}_{2+n_k\leqslant n<j}$ and $\{\beta_n\}_{2+n_k\leqslant n<j}$ have been constructed so that conditions (1) and (2) hold for $2+n_k \leqslant n < j$. Let

$$\mathfrak{X}_j \ := \ \widetilde{\mathcal{P}}_k \cap [z^*_n]^{\top}_{2+n_k\leqslant n<j} \qquad \text{and} \qquad \mathcal{Y}_j \ := \ \widetilde{\mathcal{Q}}_k \cap [z_n]^{\perp}_{2+n_k\leqslant n<j} \ .$$

By Lemma 9, there is a weak*-closed finite co-dimensional subspace $\widetilde{\mathcal{Y}}_j$ of $\mathfrak{X}^*$ such that $\widetilde{\mathcal{Y}}_j$ is $(2+\varepsilon/2)$-normed by $\mathfrak{X}_j$. Find $i_j > i_{j-1}$ and $y^*_j \in S\left(\mathcal{Y}_j \cap \widetilde{\mathcal{Y}}_j\right)$ such that

$$\left\| y^*_j - w^*_{i_j} \right\| < \delta_k \ .$$

Find $z_j \in S(\mathfrak{X}_j)$ such that

$$\frac{1}{2+\varepsilon} \ \leqslant \ y^*_j\,(z_j) \ := \ \frac{1}{\beta_j}$$

and normalize

$$z^*_j \ := \ \beta_j \ y^*_j \ .$$

This completes the inductive construction of $\{z_n, \ z^*_n\}_{n\in J_k^o}$ along with the sets $\{i_n\}_{n\in J_k^o}$ and $\{\beta_n\}_{n\in J_k^o}$.

Now apply the Haar matrix to $\{z_n, \ z^*_n\}_{n\in J_k}$ to produce $\{x_n, \ x^*_n\}_{n\in J_k}$. With help from the observations in Remark 10, note that $\{x_n, \ x^*_n\}_{n\in J_k}$ is biorthogonal and is in $\mathcal{P}_k \ \times \ \mathcal{Q}_k$. Furthermore, for each n in J_k,

$$\begin{aligned} \|x_n\| \ &\leqslant \ 2^{-m_k/2} \ \|z_{1+n_k}\| \ + \ \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \|z_j\| \\ &\leqslant \ \varepsilon \ + \ \left(1+\sqrt{2}\right) \end{aligned}$$

and

$$\begin{aligned}\|x_n^*\| &\leqslant 2^{-m_k/2} \left\|z_{1+n_k}^*\right\| + \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \left\|z_j^*\right\| \\ &\leqslant \varepsilon + (2+\varepsilon) \left(1+\sqrt{2}\right)\end{aligned}$$

and for each $x^{**} \in S\left(\mathfrak{X}^{**}\right)$

$$\begin{aligned}\left|x^{**}\left(x_n^*\right)\right| &\leqslant 2^{-m_k/2} \left\|z_{1+n_k}^*\right\| + \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \left|x^{**}\left(z_j^*\right)\right| \\ &\leqslant \delta_k + \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \left(\left|x^{**}\left(\beta_j w_{i_j}^*\right)\right| + \left|\beta_j \delta_k\right| \right) .\end{aligned}$$

Thus

$$\{x_n,\ x_n^*\}_{n\in J_k^p \cup J_k}$$

with $\{i_n\}_{n\in J_k^{po}\cup J_k^o}$ and $\{\beta_n\}_{n\in J_k^{po}\cup J_k^o}$ satisfy conditions (1) through (4). If k is odd, then

$$[a_h]_{h\leqslant h_k} \in [x_n, z_{1+n_k}]_{n\in J_k^p} \subset [x_n]_{n\in J_k^p\cup J_k} ,$$

while if k is even, then

$$[b_h^*]_{h\leqslant h_k} \in \left[x_n^*, z_{1+n_k}^*\right]_{n\in J_k^p} \subset [x_n^*]_{n\in J_k^p\cup J_k} .$$

Clearly the constructed system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$, with the blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ and the increasing sequence $\{i_n\}_{n\in\mathbb{N}^o}$ from $\mathbb{N}$, and the sequence $\{\beta_n\}_{\in\mathbb{N}^o}$ from $(0, 2+\varepsilon]$, satisfy conditions (1) through (6). ■

Some notation will be helpful in the next construction.

Remark 13. Let $\mathfrak{X}$ be a Banach space containing an isomorphic copy of ℓ_1.

Recall [P1, H2] that $\mathfrak{X}$ contains an isomorphic copy of ℓ_1 if and only if $\mathfrak{X}^*$ contains an isomorphic copy of L_1. Thus $\mathfrak{X}^*$ also contains an isomorphic copy of ℓ_2. An isomorphic copy of L_1 (resp. ℓ_2) in $\mathfrak{X}^*$ will be denoted by $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$).

There is a norm $|||\cdot|||$ on $\mathcal{Z}_2$ which is equivalent to the usual norm on $\mathfrak{X}^*$ and for which $(\mathcal{Z}_2, |||\cdot|||)$ is Hilbertian; $\widetilde{\mathcal{Z}}_2$ denotes $\mathcal{Z}_2$ equipped with the new $|||\cdot|||$-norm. Since $\widetilde{\mathcal{Z}}_2$ is isometric to a Hilbert space, there is a unique inner product that induces its $|||\cdot|||$-norm; in $\mathcal{Z}_2$, Hilbert space concepts are understood to be in $\widetilde{\mathcal{Z}}_2$. For example, a subset of $\mathcal{Z}_2$ is *orthonormal* if, when viewed as a subset of $\widetilde{\mathcal{Z}}_2$, it is orthonormal in $\widetilde{\mathcal{Z}}_2$. A sequence $\{\mathcal{Y}_i\}$ of finite-dimensional subspaces of $\mathcal{Z}_2$ is an *orthogonal finite-dimensional decomposition* ($\perp$-fdd) provided $\mathcal{Y}_i \perp \mathcal{Y}_j$ for $i \neq j$ and each $\mathcal{Y}_i$ is finite dimensional. $\mathcal{Z}_2 \ominus \mathcal{Y}$ denotes the orthogonal complement of a subspace $\mathcal{Y}$ in $\mathcal{Z}_2$. ■

Lemma 14. *Let $\mathfrak{X}$ be a separable Banach space containing an isomorphic copy of ℓ_1 and $\varepsilon > 0$. Then $\mathcal{Z}_1$ can be taken so that a countable subset of it $(2+\varepsilon)$-norms $\mathfrak{X}$.*

Proof. By [H1, DRT] there is a $(1+\varepsilon)$-isomorphic copy of L_1 in $\mathfrak{X}^*$ and so there is an embedding $T\colon \ell_1 \oplus_1 L_1 \hookrightarrow \mathfrak{X}^*$ satisfying, for each $z \in \ell_1 \oplus_1 L_1$,

$$\| \, z \, \|_{\ell_1 \oplus_1 L_1} \ \leqslant \ \| \, Tz \, \|_{\mathfrak{X}^*} \ \leqslant \ (1+\varepsilon) \, \| \, z \, \|_{\ell_1 \oplus_1 L_1} \ .$$

Moreover, the image $\{T\delta_n\}$ of the unit vector basis of ℓ_1 can be assumed to be weak*-null (since $\mathfrak{X}$ is separable, $\{T\delta_n\}$ has a weak*-convergent subsequence $\{T\delta_{k_n}\}$, so just replace δ_n by $\frac{1}{2}(\delta_{k_{2n}} - \delta_{k_{2n+1}})$). Find a sequence $\{x_n^*\}_{n=1}^{\infty}$ in $S(\mathfrak{X}^*)$ such that $\{x_n^*\}_{n=N}^{\infty}$ norms $\mathfrak{X}$ for each $N \in \mathbb{N}$.

Fix $\beta \in (0,1)$ and let

$$y_n^* = T\delta_n + \beta x_n^*$$

and

$$\mathcal{Z}_1 := [\ \{y_n^* \ : \ n \in \mathbb{N}\} \ \cup \ TL_1 \ ] \ .$$

Note that for each $n \in \mathbb{N}$,

$$\|y_n^*\| \ \leqslant \ 1 + \varepsilon + \beta \ . \tag{3}$$

The operator $S\colon \ell_1 \oplus_1 L_1 \to \mathcal{Z}_1$ defined by

$$S\left(\sum \alpha_n \delta_n \oplus_1 f\right) \ = \ \sum \alpha_n y_n^* \ + \ Tf$$

illustrates that $\mathcal{Z}_1$ is isomorphic to $\ell_1 \oplus_1 L_1$. Indeed, fix

$$\sum \alpha_n \delta_n \oplus_1 f \ \in \ \mathrm{sp}\{\delta_n\}_{n\in\mathbb{N}} \oplus_1 L_1 \ .$$

Then

$$\begin{aligned} \left\| S\left(\sum \alpha_n \delta_n \oplus_1 f\right)\right\|_{\mathfrak{X}^*} \ &\leqslant \ (1+\varepsilon+\beta) \sum |\alpha_n| \ + \ (1+\varepsilon) \ \|f\|_{L_1} \\ &\leqslant \ (1+\varepsilon+\beta) \left\| \sum \alpha_n \delta_n \oplus_1 f \right\|_{\ell_1 \oplus_1 L_1} \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} \left\| S\left(\sum \alpha_n \delta_n \oplus_1 f\right)\right\|_{\mathfrak{X}^*} \ &= \ \left\| T\left(\sum \alpha_n \delta_n \oplus_1 f\right) + \beta \sum \alpha_n x_n^* \right\|_{\mathfrak{X}^*} \\ &\geqslant \ \left\| \sum \alpha_n \delta_n \oplus_1 f \right\|_{\ell_1 \oplus_1 L_1} \ - \ \beta \left\| \sum \alpha_n \delta_n \right\|_{\ell_1} \\ &\geqslant \ (1-\beta) \left\| \sum \alpha_n \delta_n \oplus_1 f \right\|_{\ell_1 \oplus_1 L_1} \ . \end{aligned}$$

Thus $\mathcal{Z}_1$ is $\left(\frac{1+\beta+\varepsilon}{1-\beta}\right)$-isomorphic to $\ell_1 \oplus_1 L_1$, which is 3-isomorphic to L_1.

To see that $\{y_n^*\}_{n\in\mathbb{N}}$ is $\left(1+\frac{1+\varepsilon}{\beta}\right)$-norming for $\mathfrak{X}$, fix $x \in S(\mathfrak{X})$. Let $\delta > 0$ be such that $\delta(1+\delta) < \beta$ and find $n \in \mathbb{N}$ such that

$$|(T\delta_n)(x)| \leqslant \delta \qquad \text{and} \qquad 1 \leqslant (1+\delta)x_n^*(x)\ , \tag{4}$$

for then by (3) and (4)

$$\frac{y_n^*(x)}{\|y_n^*\|} \geqslant \frac{1}{1+\varepsilon+\beta}\left(\frac{\beta}{1+\delta} - \delta\right) .$$

Thus

$$\sup_{n\in\mathbb{N}} \frac{y_n^*(x)}{\|y_n^*\|} \geqslant \frac{\beta}{\beta+1+\varepsilon} .$$

So, for β sufficiently close to one, $\{y_n^*\}_{n\in\mathbb{N}}$ is $(2+2\varepsilon)$-norming for $\mathfrak{X}$. ■

From Lemma 14 and Theorem 15 it easily follows, when $\mathfrak{X}$ is separable, that (a) implies (c) in Theorem 2.

Theorem 15. *Let $\mathfrak{X}$ be a separable Banach space containing ℓ_1. From Remark 13, let $\mathcal{Z}_1$ be total and $\mathcal{Z}_2 \subset \mathcal{Z}_1$. Let $\{a_n,\ b_n^*\}_{n=1}^{\infty}$ be in $\mathfrak{X} \times \mathcal{Z}_1$ and fix $\varepsilon, \eta > 0$. Then there exists a $[(1+\sqrt{2})+\varepsilon]$-bounded wc_0^*-stable biorthogonal system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ so that*

(15a) $[a_n]_{n=1}^{\infty} \subset [x_n]_{n=1}^{\infty}$
(15b) $[b_n^*]_{n=1}^{\infty} \subset [x_n^*]_{n=1}^{\infty} \subset \mathcal{Z}_1$
(15c) $\sup_{n\in\mathbb{N}} d(x_n^*, \mathcal{Z}_2) < \eta$.

In Section 5 it is shown that the $[(1+\sqrt{2})+\varepsilon]$ can not be replaced with $(1+\varepsilon)$ in Theorem 15. The following fact helps with the bound of the system in Theorem 15. It is due to Dvoretzky [Dv] and Milman [Mil]; a proof may be found in [P].

Fact 16. *Let n, m, N be positive integers and $\delta > 0$. Then there is a positive integer $K = K(n, m, N, \delta)$ so that if*

(1) *Y is a Banach space with $K \leqslant \dim Y \leqslant \infty$*
(2) *E is a n-dimensional subspace of Y*
(3) *H is a m-codimensional subspace of Y*

then there is a subspace F of H which is $(1+\delta)$-isomorphic to ℓ_2^N and a projection P from $E+F$ onto F with $\ker P = E$ and $\|P\| < 1+\delta$.

In fact, they showed that P can be taken so that $\|P - I|_{E+F}\| < 1+\delta$.

Proof of Theorem 15. The proof of Theorem 15 is similar to the proof of Theorem 12; thus, notation from the proof of Theorem 12 will be retained.

Fix a strictly decreasing sequence $\{\eta_k\}_{k=1}^{\infty}$ converging to zero with $\eta_1 < \eta$. It suffices to construct a system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$ along with a blocking $\{J_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ and a sequence $\{u_n^*\}_{n=1}^{\infty}$ from $\mathcal{Z}_2$ satisfying

(1) $x_m^*(x_n) = \delta_{mn}$
(2) $\|x_n\| \leqslant \left(1+\sqrt{2}\right) + \varepsilon$
(3) $\|x_n^*\| \leqslant 1 + \varepsilon$
(4) $\|x_n^* - u_n^*\| \leqslant \eta_k$ if $n \in J_k$
(5) $[u_n^*]_{n\in J_{k_1}}$ is orthogonal to $[u_n^*]_{n\in J_{k_2}}$ for $k_1 \neq k_2$
(6) $[a_n]_{n=1}^{\infty} \subset [x_n]_{n=1}^{\infty}$
(7) $[b_n^*]_{n=1}^{\infty} \subset [x_n^*]_{n=1}^{\infty} \subset \mathcal{Z}_1$.

Note that conditions (3) through (5) imply that $\{x_n^*\}_{n\in\mathbb{N}}$ is weakly-null in $\mathfrak{X}^*$. Clearly all that remains at this point is to show that the wc_0^*-biorthogonal system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ is indeed stable, which is done in the last step by using the condition that $[x_n^*]_{n=1}^{\infty}$ stays inside of $\mathcal{Z}_1$.

Let

$$\delta \ := \ \frac{\varepsilon}{2\ +\ \sqrt{2}}\ .$$

The construction will inductively produce blocks

$$\{x_n,\ x_n^*\}_{n\in J_k} \qquad \text{and} \qquad \{u_n^*\}_{n\in J_k}\ .$$

Fix $k \geqslant 1$. Assume that $\{J_j\}_{0\leqslant j<k}$ along with $\{x_n,\ x_n^*\}_{n\in J_k^p}$ and $\{u_n^*\}_{n\in J_k^p}$ have been constructed to satisfy conditions (1) through (5). Now to construct J_k along with $\{x_n,\ x_n^*\}_{n\in J_k}$ and $\{u_n^*\}_{n\in J_k}$.

The idea is to find a biorthogonal system $\{z_n,\ z_n^*\}_{n\in J_k}$ in $\mathcal{P}_k \times \mathcal{Q}_k$ by first finding $\{z_{1+n_k}, z_{1+n_k}^*\}$ that helps guarantee condition (6) if k is odd and condition (7) if k even; however, $\{z_{1+n_k}, z_{1+n_k}^*\}$ would not necessarily satisfy conditions (2) and (3) and $z_{1+n_k}^*$ may be far from $\mathcal{Z}_2$ and so J_k^o and

$$\{z_n,\ z_n^*\}_{n\in J_k^o}$$

are then constructed and the Haar matrix is applied to $\{z_n,\ z_n^*\}_{n\in J_k}$ to produce $\{x_n,\ x_n^*\}_{n\in J_k}$ and $\{u_n^*\}_{n\in J_k}$ so that

$$\{x_n,\ x_n^*\}_{n\in J_k^p\cup J_k} \qquad \text{and} \qquad \{u_n^*\}_{n\in J_k^p\cup J_k}$$

satisfy conditions (1) through (5).

Find $\{z_{1+n_k}, z_{1+n_k}^*\}$ just as in the proof of Theorem 12: in the case that k is odd, be sure to choose $y_{1+n_k}^*$ in $\mathcal{Z}_1$, which is possible since $\mathcal{Z}_1$ is total.

Find a natural number m_k larger than one so that

$$2^{-m_k/2}\ \max\left(\ \|z_{1+n_k}\|\ ,\ \left\|z_{1+n_k}^*\right\|\ \right)\ <\ \min\left(\ \delta\ ,\ \eta_k\ \right)\ .$$

Let

$$\begin{aligned} E_k &:= \left[\{x_n^*\}_{n\in J_k^p} \cup \{z_{1+n_k}^*\} \right] \\ H_k &:= \left(\mathcal{Z}_2 \ominus [u_n^*]_{n\in J_k^p} \right) \cap [x_n]_{n\in J_k^p}^{\perp} \cap [z_{1+n_k}]^{\perp} \\ Y_k &:= [\mathcal{Z}_2 \cup E_k] \\ N_k &:= 2^{m_k} - 1 = |J_k^o| \ . \end{aligned}$$

Use Fact 16 to find a subspace F_k of H_k, a projection P_k with kernel E_k, and a norm one isomorphism T_k so that

$$E_k + F_k \overset{P_k}{\twoheadrightarrow} F_k \overset{T_k}{\longrightarrow} \ell_2^{N_k}$$

and

$$\max\left(\|T_k^{-1}\| , \|P_k\|^2 \right) < 1 + \delta \ .$$

Let $\{e_n\}_{n\in J_k^o}$ be an orthonormal basis for $\ell_2^{N_k}$. For each $n \in J_k^o$, let

$$z_n^* = T_k^{-1} e_n$$

and, using Local Reflexivity, find $z_n \in \mathfrak{X}$ that agrees, on E_k and F_k, with a norm-preserving Hahn-Banach extension of $P_k^* T_k^* e_n \in (E_k + F_k)^*$ to $\mathfrak{X}^*$ and satisfies

$$\|z_n\| < \sqrt{1+\delta} \ \|P_k^* T_k^* e_n\| \ .$$

Then $\{z_n,\ z_n^*\}_{n\in J_k^o}$ is a biorthogonal system in $E_k^{\top} \times F_k$ and

$$\max \ \{ \ \|z_n\| \ : \ n \in J_k^o \ \} < 1 + \delta \ .$$

Now apply the Haar matrix to $\{z_n,\ z_n^*\}_{n\in J_k}$ to produce $\{x_n,\ x_n^*\}_{n\in J_k}$ and let

$$u_n^* := \sum_{j\in J_k^o} a_{nj}^{m_k} z_j^*$$

for each n in J_k.

With help from the observations in Remark 10, note that for each n in J_k

$$\begin{aligned} \|x_n\| &\leqslant 2^{-m_k/2} \ \|z_{1+n_k}\| + \left(1+\sqrt{2}\right) \max_{j\in J_k^o} \|z_j\| \\ &\leqslant \delta + \left(1+\sqrt{2}\right)(1+\delta) \\ &\leqslant \left(1+\sqrt{2}\right) + \varepsilon \end{aligned}$$

and

$$\begin{aligned}\|x_n^*\| &\leqslant 2^{-m_k/2} \left\|z_{1+n_k}^*\right\| + \left\|\sum_{j\in J_k^o} a_{nj}^{m_k} z_j^*\right\| \\ &\leqslant \delta + (1+\delta) \left\|\sum_{j\in J_k^o} a_{nj}^{m_k} e_j\right\| \\ &\leqslant 1 + 2\delta \\ &\leqslant 1 + \varepsilon .\end{aligned}$$

and since $x_n^* - u_n^* = a_{n1}^{m_k} z_{n_k+1}^*$,

$$\|x_n^* - u_n^*\| = 2^{-m_k/2} \left\|z_{n_k+1}^*\right\| \leqslant \eta_k .$$

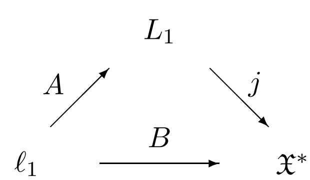
Thus,

$$\{x_n, x_n^*\}_{n\in J_k^p\cup J_k} \qquad \text{and} \qquad \{u_n^*\}_{n\in J_k^p\cup J_k}$$

clearly satisfy conditions (1) through (5).

This completes the inductive construction of the system $\{x_n, x_n^*\}_{n=1}^{\infty}$ in $\mathfrak{X} \times \mathfrak{X}^*$, along with the blocking $\{J_k\}$ of $\mathbb{N}$ and the sequence $\{u_n^*\}_{n=1}^{\infty}$ from $\mathcal{Z}_2$, that satisfy conditions (1) through (7).

The last step is to verify that $\{x_n, x_n^*\}_{n=1}^{\infty}$ is indeed stable, which, by Lemma 8, is equivalent to verifying that the operator $S\colon \mathfrak{X} \to c_0$ given by $S(x) = (x_n^*(x))$ factors through an injective space. Towards this, consider the following commutative diagram



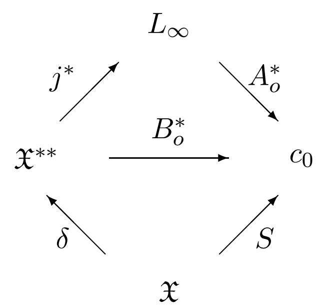
where j is an isomorphic embedding with range $\mathcal{Z}_1$ and A and B are given by

$$A(\delta_n) = j_o^{-1} x_n^* := \widetilde{r}_n \quad \text{and} \quad B(\delta_n) = x_n^* .$$

Since A^* and B^* are of the form

$$A^*(f) = (f(\widetilde{r}_n))_n \qquad \text{and} \qquad B^*(x^{**}) = (x^{**}(x_n^*))_n ,$$

their ranges are contained in c_0; let A_o^* and B_o^* be the corresponding maps with their ranges restricted to c_0. Thus the following diagram commutes.



An appeal to Lemma 8 finishes the proof. ■

Theorems 1 (c) is much easier to proof if one drops the total condition since then one can use the technique of Davis-Johnson-Singer ([DJ, Thm. 1] and [S2, Prop. 1]). As a partial illustration of this, we offer the following theorem, which gives a weaker result but a smaller constant than is provided by Theorems 1 (c).

Theorem 17. *Let $\mathfrak{X}$ be a separable Banach space not containing ℓ_1 such that $\mathfrak{X}^*$ fails the Schur property. Fix $\varepsilon > 0$. Then there is a $(2+\varepsilon)$-bounded fundamental wc_0^*-biorthogonal system $\{x_n,\ x_n^*\}$ in $\mathfrak{X} \times \mathfrak{X}^*$.*

The meat in the proof of Theorem 17 is the following lemma.

Lemma 18. *Let $\mathfrak{X}$ be a separable Banach space such that $\mathfrak{X}^*$ fails the Schur property. Fix $\varepsilon > 0$. Then there is a wc_0^*-biorthogonal sequence $\{x_n,\ x_n^*\}$ in $S(\mathfrak{X}) \times \mathfrak{X}^*$ satisfying*

(1) $\|x_n^*\| \leqslant 2+\varepsilon$
(2) $\{x_n\}$ *is basic*
(3) $[x_n^*]^{\top} + [x_n]$ *is dense in* $\mathfrak{X}$.

Proof of Lemma 18. Fix a normalized weakly-null sequence $\{w_n^*\}$ in $\mathfrak{X}^*$, a dense sequence $\{d_n\}$ in $\mathfrak{X}$, a sequence $\{\varepsilon_n\}$ decreasing to zero, and a sequence $\{\tau_n\}$ such that $\tau_n > 1$ and $\Pi\tau_n < \infty$. It is sufficient to construct

(a) a sequence $\{x_n\}_{n\geqslant 1}$ in $S(\mathfrak{X})$
(b) a sequence $\{\widetilde{x}_n^*\}_{n\geqslant 1}$ in $S(\mathfrak{X}^*)$
(c) finite sets $\{F_n\}_{n\geqslant 0}$ in $S(\mathfrak{X}^*)$ with $F_0 := \emptyset$
(d) an increasing sequence $\{k_n\}_{n=1}^{\infty}$ of integers

that satisfy

(4) $x_n \in F_{n-1}^{\top} \cap \{x_i^*\}_{i<n}^{\top} := \mathfrak{X}_n$
(5) $\widetilde{x}_n^* \in \{d_i\}_{i<n}^{\perp} \cap \{x_i\}_{i<n}^{\perp} := \mathcal{Y}_n$

(6) $\frac{1}{2+\varepsilon} \leqslant \widetilde{x}_n^*(x_n)$
(7) $\left\|\widetilde{x}_n^* - w_{k_n}^*\right\| \leqslant \varepsilon_n$
(8) if $x \in [x_j]_{j \leqslant n}$, then there is $f \in F_n$ with $\|x\| \leqslant \tau_n f(x)$.

For then just take $x_n^* = \widetilde{x}_n^* / \widetilde{x}_n^*(x_n)$. Note that (4) and (8) imply (2) while (5) and biorthogonality imply (3) since each d_i has the form

$$d_i = \left(d_i - \sum_{n=1}^{i} x_n^*(d_i) x_n \right) \; + \; \sum_{n=1}^{i} x_n^*(d_i) x_n \; .$$

The construction is by induction on n. To start, let $\widetilde{x}_1^* = w_1^*$. Find x_1 in $S(\mathfrak{X})$ that satisfies (6) and F_1 that satisfies (8).

Fix $n > 1$ and assume that the items in (a) through (d) have been constructed up through the $(n-1)^{\text{th}}$-level. From this it is possible to find $\mathfrak{X}_n$ and $\mathcal{Y}_n$.

By Lemma 9, there is a finite co-dimensional subspace $\mathcal{Y}$ of $\mathfrak{X}^*$ that is $(2 + \frac{\varepsilon}{2})$-normed by $\mathfrak{X}_n$. Find $\widetilde{x}_n^* \in S\left(\mathcal{Y} \cap \mathcal{Y}_n\right)$ along with $k_n > k_{n-1}$ such that (7) holds. Since $\mathcal{Y}$ is $(2 + \frac{\varepsilon}{2})$-normed by $\mathfrak{X}_n$, there is $x_n \in S(\mathfrak{X}_n)$ such that

$$1 \leqslant (2 + \varepsilon) \widetilde{x}_n^*(x_n) \; .$$

Now find F_n satisfying (8). ■

Proof of Theorem 17. First find the biorthogonal system $\{x_n,\ x_n^*\}$ given by Lemma 18. The next step is to perturb this system to produce the desired system.

Begin by finding a bijection $p \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ satisfying

(i) $\{p(n,i)\}_{i=1}^{\infty}$ is an increasing sequence

for each n in $\mathbb{N}$. Take a dense set $\{y_n\}$ in $B\left([x_n^*]^\top\right)$. The underlying idea is to use $\{x_{p(n,i)}\}_i$ to capture y_n, along with $\{x_{p(n,i)}\}_i$, in the span of a small perturbation of $\{x_{p(n,i)}\}_i$.

Towards this, with the help of (2) and the fact that $\{x_{p(n,i)}\}_i$ is not equivalent to the unit vector basis of ℓ_1, for each n find a sequence $\{a_{p(n,i)}\}_i$ such that

(ii) $\sum_i \left|a_{p(n,i)}\right| = \infty$
(iii) $\sum_i a_{p(n,i)} x_{p(n,i)} \in \mathfrak{X}$.

Let

$$w_{p(n,i)} := x_{p(n,i)} \; - \; \varepsilon \left(\text{sign } a_{p(n,i)}\right) y_n \; .$$

Clearly $\{w_n,\ x_n^*\}$ is a $[(1+\varepsilon)(2+\varepsilon)]$-bounded wc_0^*-biorthogonal system.

Fix $n_0 \in \mathbb{N}$ and consider $x^* \in \left[w_{p(n_0,i)}\right]_i^{\perp}$. For each $m \in \mathbb{N}$,

$$x^* \left(\sum_{i=1}^{m} a_{p(n_0,i)} x_{p(n_0,i)} \right) = \varepsilon x^*(y_{n_0}) \sum_{i=1}^{m} \left| a_{p(n_0,i)} \right| \ .$$

Combined with (ii) and (iii), this gives that $x^* \in [y_{n_0}]^{\perp}$, which in turn implies that $x^* \in \left[x_{p(n_0,i)}\right]_i^{\perp}$. Thus

$$\left[\ \left\{ x_{p(n_0,i)} \right\}_i \cup \{y_{n_0}\} \ \right] \ \subset \ \left[w_{p(n_0,i)} \right]_i \ .$$

Combined with (3), it follows that $\{w_n\}_{n \in \mathbb{N}}$ is fundamental. ■

5. BOUNDED FUNDAMENTAL BIORTHOGONAL SYSTEMS

The knowledgeable reader notices that, in our proof of Theorem 15, a combination of the [OP]–method (which produces $[(1+\sqrt{2})^2+\varepsilon]$-bounded systems) and the [P]–method (which produces $(1+\varepsilon)$-bounded systems) is used to produce a $(1+\sqrt{2}+\varepsilon)$-bounded system. Using just the [P]–method in our proof of Theorem 15 will not guarantee that the x_n^*'s are in $\mathcal{Z}_1$ nor close to $\mathcal{Z}_2$. This difficulty is not purely technical. For indeed, consider the below special case of Theorem 15.

Corollary 19. *Fix $\varepsilon, \eta > 0$. Let $\mathcal{Z}_1$ be a total subspace of $\ell_\infty = \ell_1^*$ that is isomorphic to L_1 and $\mathcal{Z}_2$ be a subspace of $\mathcal{Z}_1$ that is $(1+\eta)$-isomorphic to ℓ_2. Then there exists a $[(1+\sqrt{2})+\varepsilon]$-bounded fundamental biorthogonal system $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ in $S(\ell_1) \ \times \ \ell_\infty$ satisfying*

$$\sup_{n \in \mathbb{N}} \ d\left(x_n^*, \mathcal{Z}_2\right) < \eta \ .$$

Lemma 20 shows that such subspaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in Corollary 19 do exist. Corollary 24 shows that in Corollary 19, the $[(1+\sqrt{2})+\varepsilon]$ can not be replaced with $(1+\varepsilon)$. However, if the requirement (15a) in the statement of Theorem 15 is removed (which would basically remove the fundamental condition), then the [P]–method can be used to obtain this variant of Theorem 15 with $(1+\varepsilon)$ replacing $[(1+\sqrt{2})+\varepsilon]$.

The proof of Lemma 14 gives that for each $\varepsilon > 0$ there exists

a strictly increasing surjective function $\quad i\colon\ (0,1) \to (3(1+\varepsilon), \infty)$

a strictly decreasing surjective function $\quad n\colon\ (0,1) \to (2+\varepsilon, \infty)$

so that if $\mathfrak{X}$ is a separable Banach space whose dual contains an isomorphic copy of L_1, then for each $\beta \in (0,1)$, there is a subspace $\mathcal{Z}_1$ of $\mathfrak{X}^*$

which is $i(\beta)$–isomorphic to L_1 and which has a countable subset that $n(\beta)$–norms $\mathfrak{X}$. However, if the dual space contains an isometric copy $\mathcal{Z}$ of L_1 then the above isomorphism constant $i(\beta)$ can be improved.

Lemma 20. *There exists*

$$\text{a strictly increasing surjective function} \quad i\colon\ (0,1) \to (1,\infty)$$
$$\text{a strictly decreasing surjective function} \quad n\colon\ (0,1) \to (2,\infty)$$

so that if $\mathfrak{X}$ is a separable Banach space whose dual contains an isometric copy of ℓ_1 that is contractively complemented in some subspace $\mathcal{Z}$ of $\mathfrak{X}^*$, then for each $\beta \in (0,1)$, there is a subspace $\mathcal{Z}_1$ of $\mathfrak{X}^*$ which is $i(\beta)$–isomorphic to $\mathcal{Z}$ and which has a countable subset that $n(\beta)$–norms $\mathfrak{X}$.

Since ℓ_∞ contains an isometric copy $\mathcal{Z}$ of L_1, which in turn contains a contractively complemented subspace which is isometric to ℓ_1, for each positive η, applying Lemma 20 with β sufficiently close to zero gives that there is a total subspace $\mathcal{Z}_1$ of ℓ_∞ that is $(1+\eta)$–isomorphic to L_1, which in turn contains a subspace $\mathcal{Z}_2$ which is $(1+\eta)$–isomorphic to ℓ_2.

Proof of Lemma 20. Find $\{e_n^*\}_{n\in\mathbb{N}}$ in $\mathcal{Z}$ which is 1–equivalent to the standard unit vector basis of ℓ_1 and a surjective contractive projection

$$P\colon \mathcal{Z} \to [e_n^*]_{n\in\mathbb{N}} \ .$$

Without loss of generality, $\{e_n^*\}_{n\in\mathbb{N}}$ is weak*-null (similar to before, just replace $\{e_n^*\}$ with a weak*-convergent subsequence $\{\frac{1}{2}(e_{k_{2n}}^* - e_{k_{2n+1}}^*)\}$, which will be 1–equivalent to $\{e_n^*\}$ and contractively complemented in $[e_n^*]$). Find a sequence $\{x_n^*\}_{n=1}^\infty$ in $S(\mathfrak{X}^*)$ such that $\{x_n^*\}_{n=N}^\infty$ norms $\mathfrak{X}$ for each $N \in \mathbb{N}$.

Fix $\beta \in (0,1)$ and let

$$y_n^* = e_n^* + \beta x_n^*$$

and

$$\mathcal{Z}_1 := [\ \{y_n^*\ :\ n \in \mathbb{N}\}\ \cup\ \ker P\ ]\ .$$

Note that for each $n \in \mathbb{N}$,

$$\|y_n^*\| \ \leqslant\ 1+\beta\ . \tag{5}$$

Each element in $\mathcal{Z}$ has a unique expression as

$$z = \sum \alpha_n e_n^* + f$$

where $f \in \ker P$; the operator $S\colon \mathcal{Z} \to \mathcal{Z}_1$ defined by

$$S\left(\sum \alpha_n e_n^* + f\right) \;=\; \sum \alpha_n y_n^* \;+\; f$$

illustrates that $\mathcal{Z}_1$ is isomorphic to $\mathcal{Z}$. Indeed, fix

$$\sum \alpha_n e_n^* + f \;\in\; \mathrm{sp}\{e_n^*\}_{n\in\mathbb{N}} + \ker P \;.$$

Then

$$\begin{aligned} \|z - Sz\| \;&=\; \left\|\sum \alpha_n (e_n^* - y_n^*)\right\| \\ &\leqslant\; \beta \sum |\alpha_n| \;=\; \beta\,\|P(z)\| \\ &\leqslant\; \beta\,\|z\| \;. \end{aligned}$$

Thus

$$(1-\beta)\;\|z\| \;\leqslant\; \|Sz\| \;\leqslant\; (1+\beta)\;\|z\|$$

for each $z \in \mathcal{Z}$; thus, $\mathcal{Z}_1$ is $(\frac{1+\beta}{1-\beta})$-isomorphic to $\mathcal{Z}$.

To see that $\{y_n^*\}_{n\in\mathbb{N}}$ is $\left(1+\frac{1}{\beta}\right)$-norming for $\mathfrak{X}$, fix $x \in S(\mathfrak{X})$. Let $\delta > 0$ be such that $\delta(1+\delta) < \beta$ and find $n \in \mathbb{N}$ such that

$$|e_n^*(x)| \leqslant \delta \qquad \text{and} \qquad 1 \leqslant (1+\delta) x_n^*(x) \;, \tag{6}$$

for then by (5) and (6)

$$\frac{y_n^*(x)}{\|y_n^*\|} \;\geqslant\; \frac{1}{1+\beta}\;\left(\frac{\beta}{1+\delta} \;-\; \delta\right) \;.$$

Thus

$$\sup_{n\in\mathbb{N}} \frac{y_n^*(x)}{\|y_n^*\|} \;\geqslant\; \frac{\beta}{\beta+1} \;.$$

So $\{y_n^*\}_{n\in\mathbb{N}}$ is $(1+\frac{1}{\beta})$-norming for $\mathfrak{X}$. ■

Recall that the modulus of convexity $\delta_{\mathcal{W}}\colon [0,2] \to [0,1]$ of a Banach space $\mathcal{W}$ is

$$\delta_{\mathcal{W}}(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \;:\; x, y \in B(\mathcal{W}) \text{ and } \;\; \|x-y\| \geqslant \varepsilon\right\}$$

and $\mathcal{W}$ is *uniformly convex* if $\delta_{\mathcal{W}}(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$. If $\mathcal{W}$ is uniformly convex then $\delta_{\mathcal{W}}$ is a surjective continuous strictly increasing function (cf. [GK, pp. 53–55]).

In a uniformly convex space, the midpoint of points near to the sphere that are far apart is uniformly bounded away from the sphere. This can be extended to convex combinations of points near to the sphere.

Lemma 21. *Let $\mathcal{W}$ be a uniformly convex Banach space and ε and b be constants satisfying*

$$0 \ < \ \varepsilon \ \leqslant b \ \leqslant \ 1 \ .$$

If $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{R}$ and $\{y_j\}_{j=1}^{\infty} \subset B(\mathcal{W})$ satisfy

$$\sum_{j=1}^{\infty} |\alpha_j| = 1 \qquad \textit{and} \qquad \left\| \sum_{j=1}^{\infty} \alpha_j y_j \right\| \ > \ 1 \ - \ \varepsilon \ , \tag{7}$$

then there is a finite subset F of $\mathbb{N}$ so that

$$\sum_{j \in F} |\alpha_j| \ > \ 1 \ - \ \frac{\varepsilon}{2b - \varepsilon} \tag{8}$$

and for each $j \in F$,

$$\left\| (\mathit{sign}\ \alpha_j)\, y_j \ - \ \sum_{j=1}^{\infty} \alpha_j y_j \right\| \ < \ \delta_{\mathcal{W}}^{-1}(b) \tag{9}$$

and $\alpha_j \neq 0$.

Proof. Let $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{R}$ and $\{y_j\}_{j=1}^{\infty} \subset B(\mathcal{W})$ satisfy (7). Set

$$x_0 := \sum_{j=1}^{\infty} \alpha_j y_j \ .$$

Without loss of generality each $\alpha_j \geqslant 0$.

Find $x_0^* \in S(\mathcal{W}^*)$ so that $x_0^*(x_0) = \|x_0\|$ and let

$$F = \{ j \in \mathbb{N} : x_0^*(y_j) \ > \ 1 \ + \ \varepsilon \ - \ 2b \ \text{ and } \ \ \alpha_j \neq 0 \}.$$

The condition $\varepsilon \leqslant b$ guarantees that F is non-empty. Since

$$\begin{aligned} 1 - \varepsilon \ &< \ \sum_{j \notin F} \alpha_j x_0^*(y_j) \ + \ \sum_{j \in F} \alpha_j x_0^*(y_j) \\ &\leqslant \ (1 + \varepsilon - 2b) \left(1 \ - \ \sum_{j \in F} \alpha_j \right) \ + \ \left(\sum_{j \in F} \alpha_j \right) \\ &= \ (1 + \varepsilon - 2b) + (2b - \varepsilon) \ \left(\sum_{j \in F} \alpha_j \right) \ , \end{aligned}$$

condition (8) holds. For each $j \in F$,

$$\frac{1}{2} \|y_j + x_0\| \ > \ \frac{1}{2} \left(\ (1 + \varepsilon - 2b) \ + \ (1 - \varepsilon) \ \right) \ = \ 1 - b$$

and so

$$\|y_j \ - \ x_0\| \ < \ \delta_{\mathcal{W}}^{-1}(b)$$

by uniform convexity. ■

Proposition 22. *Let $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ be a biorthogonal system in $S(\ell_1) \times \ell_\infty$ and $\mathcal{W}$ be a uniformly convex Banach space and $Q \in \mathcal{L}(\ell_1, \mathcal{W})$ be of norm at most one, all of which satisfy*

$$d\left(x_n^*, Q^*\left(K\, B\left(\mathcal{W}^*\right)\right)\right) \ < \ 2\eta \tag{10}$$

for some constants $K \geq 1$ and $\eta > 0$. If

$$1 \ - \ \frac{1-2\eta}{K} \ < \ \frac{2}{3}\ \delta_{\mathcal{W}}\left(\frac{1-4\eta}{2K}\right) \tag{11}$$

then $\{x_n\}_{n=1}^{\infty}$ is equivalent to the standard unit vector basis of ℓ_1. More specifically if constants a and b satisfy

$$\frac{3}{2}\left(1 \ - \ \frac{1-2\eta}{K}\right) \ \leqslant \ a \ < \ b \ \leqslant \delta_{\mathcal{W}}\left(\frac{1-4\eta}{2K}\right) \tag{12}$$

then

$$\frac{3\,(b-a)}{3b-a}\ \sum_{n=1}^{\infty}|\beta_n| \ \leqslant \ \left\|\sum_{n=1}^{\infty}\beta_n x_n\right\| \ \leqslant \sum_{n=1}^{\infty}|\beta_n| \tag{13}$$

for each $\{\beta_n\}_{n=1}^{\infty}$ in ℓ_1.

Note that if $K = 1$ and $\eta = 0$ then (11) becomes

$$0 \ < \ \frac{2}{3}\ \delta_{\mathcal{W}}\left(\frac{1}{2}\right) \ ;$$

thus, if K is sufficiently close to 1 and η is sufficiently close to 0, as they often are in practice, then (11) does indeed hold.

Proof. The underlying idea behind the proof is to use Lemma 21 to find a small perturbation $\{\widetilde{x}_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ that are disjointly supported on the standard unit vector basis of ℓ_1. For then, $\{\widetilde{x}_n\}_{n=1}^{\infty}$ is equivalent to the standard unit vector basis of ℓ_1 and so, for a small enough perturbation, $\{x_n\}_{n=1}^{\infty}$ is also equivalent to the standard unit vector basis of ℓ_1.

Find $\{w_n^*\}_{n=1}^{\infty}$ in $K\, B\left(\mathcal{W}^*\right)$ so that

$$\|x_n^* \ - \ Q^* w_n^*\| \ < \ 2\eta \ . \tag{14}$$

Thus

$$\begin{aligned}\|Qx_n\| &\geqslant \left|\left\langle Qx_n, \frac{w_n^*}{\|w_n^*\|}\right\rangle\right| \\ &\geqslant \frac{1}{\|w_n^*\|}\ \left|\langle x_n, x_n^*\rangle \ - \ \langle x_n, x_n^* - Q^* w_n^*\rangle\right| \\ &> \frac{1-2\eta}{K}\end{aligned}$$

and so by the first inequality in (12)

$$\|Qx_n\| \ > \ 1 \ - \ 2a/3 \ .$$

Write

$$x_n = \sum_{j=1}^{\infty} \ \alpha_j^n \delta_j$$

where $\sum_{j=1}^{\infty} \left|\alpha_j^n\right| = 1$ and let $\varepsilon_j^n = \text{sign } \alpha_j^n$; thus,

$$Qx_n = \sum_{j=1}^{\infty} \ \alpha_j^n \ Q\delta_j \ .$$

So by Lemma 21 there is a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets in $\mathbb{N}$ so that

$$\sum_{j \in F_n} \left|\alpha_j^n\right| \ > \ 1 \ - \ \frac{a}{3b-a}$$

and for each $j \in F_n$

$$\left\|\varepsilon_j^n Q\delta_j \ - \ Qx_n\right\| \ < \ \delta_{\mathcal{W}}^{-1}(b)$$

and $\alpha_j^n \neq 0$. Let

$$\widetilde{x}_n \ = \ \sum_{j \in F_n} \alpha_j^n \ \delta_j \ .$$

To see that the F_n's are disjoint, suppose that there is $j_0 \in F_n \cap F_m$ for some distinct $n, m \in \mathbb{N}$. Pick $\tau \in \{-1, 1\}$ so that $\tau\varepsilon_{j_0}^m = \varepsilon_{j_0}^n$. Then

$$\begin{aligned} \|Qx_n \ - \ \tau Qx_m\| \ &\leqslant \ \left\|Qx_n \ - \ \varepsilon_{j_0}^n Q\delta_{j_0}\right\| \ + \ \left\|\tau\varepsilon_{j_0}^m Q\delta_{j_0} \ - \ \tau Qx_m\right\| \\ &< \ 2\delta_{\mathcal{W}}^{-1}(b) \ . \end{aligned}$$

On the other hand, from (14) and the third inequality of (12) it follows that

$$\begin{aligned} \|Qx_n \ \pm \ Qx_m\| \ &\geqslant \ \left|\left\langle Q(x_n \pm x_m) \ , \ \frac{w_n^*}{\|w_n^*\|}\right\rangle\right| \\ &\geqslant \ \frac{1}{\|w_n^*\|} \ \left|\langle x_n \pm x_m, x_n^*\rangle \ - \ \langle x_n \pm x_m, x_n^* - Q^* w_n^*\rangle\right| \\ &> \ \frac{1-4\eta}{K} \\ &\geqslant \ 2\delta_{\mathcal{W}}^{-1}(b) \ . \end{aligned}$$

A contradiction, thus the finite subsets $\{F_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ are indeed disjoint.

For each $\{\beta_n\}_{n=1}^{\infty}$ in ℓ_1,

$$\begin{aligned}\sum_{n=1}^{\infty} |\beta_n| &\geqslant \left\| \sum_{n=1}^{\infty} \beta_n x_n \right\| \\ &\geqslant \left\| \sum_{n=1}^{\infty} \beta_n \widetilde{x}_n \right\| - \sum_{n=1}^{\infty} |\beta_n| \, \|\widetilde{x}_n - x_n\| \\ &\geqslant \left(1 - \frac{a}{3b-a}\right) \sum_{n=1}^{\infty} |\beta_n| - \left(\frac{a}{3b-a}\right) \sum_{n=1}^{\infty} |\beta_n| \\ &= \left(1 - \frac{2a}{3b-a}\right) \sum_{n=1}^{\infty} |\beta_n| \end{aligned}$$

and so (13) holds. ■

If $\mathcal{W}$ is uniformly convex then $\mathcal{W}^*$ is super-reflexive and so $\mathcal{W}^*$ has finite cotype; thus, there exists a cotype constant $C_q(\mathcal{W}^*) \geqslant 1$ for some $q \in [2, \infty)$ so that

$$\left(\sum_{i=1}^{n} \|w_i^*\|^q \right)^{1/q} \leqslant C_q(\mathcal{W}^*) \left[\operatorname{avg}_{\theta_i = \pm 1} \left\| \sum_{i=1}^{n} \theta_i w_i^* \right\|^2 \right]^{1/2} \tag{15}$$

for each finite sequence $\{w_i^*\}_{i=1}^{n}$ in $\mathcal{W}^*$.

Theorem 23. *Let a uniformly convex Banach space $\mathcal{W}$ and $\varepsilon_0 > 0$ satisfy*

$$1 - \frac{1}{1+\varepsilon_0} = \frac{2}{3} \delta_{\mathcal{W}} \left(\frac{1}{2(1+\varepsilon_0)} \right) . \tag{16}$$

Let $C_q(\mathcal{W}^*)$ be as in (15) and $0 < \varepsilon_1 < \varepsilon_0$. Then there exists a constant

$$\eta = \eta\left(C_q(\mathcal{W}^*), \delta_{\mathcal{W}}, \varepsilon_1\right) > 0 \tag{17}$$

so that if

(23a) $\{x_n, \ x_n^*\}_{n=1}^{\infty}$ *is a $(1+\varepsilon)$–bounded fundamental biorthogonal system in $S(\ell_1) \times \ell_\infty$*
(23b) $Q \in \mathcal{L}(\ell_1, \mathcal{W})$
(23c) Q^* *is a $(1+\eta)$–isomorphic embedding*
(23d) $d(x_n^*, Q^*\mathcal{W}^*) < \eta$

then

$$\varepsilon > \varepsilon_1 .$$

The following notation helps crystallize condition (16) and simplify some technical arguments in the proof of Theorem 23.

Notation. Consider the functions $l, u\colon\ [0,1/4]\times[0,\infty)\times[0,\infty)\to[0,1]$ given by

$$\begin{aligned} l(\eta_1,\eta_2,\varepsilon) &= 1-\frac{1-2\eta_1}{(1+\eta_2)(1+\varepsilon)} \\ u(\eta_1,\eta_2,\varepsilon) &= \frac{2}{3}\,\delta_{\mathcal{W}}\left(\frac{1-4\eta_1}{2(1+\eta_2)(1+\varepsilon)}\right)\ . \end{aligned} \tag{18}$$

Note that in each variable, l is a strictly increasing continuous function and u is a strictly decreasing continuous function. Condition (16) is equivalent to

$$l(0,0,\varepsilon_0) \;=\; u(0,0,\varepsilon_0)$$

and

$$\begin{aligned} l(0,0,\cdot)\ &:\ [0,\infty)\ \overset{\text{onto}}{\longrightarrow}\ [0,1) \\ u(0,0,\cdot)\ &:\ [0,\infty)\ \overset{\text{onto}}{\longrightarrow}\ \left(0,\frac{2}{3}\,\delta_{\mathcal{W}}\left(\frac{1}{2}\right)\right]\ ; \end{aligned}$$

thus, for a uniformly convex space $\mathcal{W}$, there is a unique $\varepsilon_0>0$ satisfying (16).

Proof. The underlying idea behind the proof is that for sufficiently small η Proposition 22 gives that $\{x_n\}$ is equivalent to the standard unit vector basis of ℓ_1: indeed, condition (10) will hold and condition (16) implies (11). Then $\{x_n^*\}$ is equivalent to the standard unit vector basis of c_0. But if ε is small enough, then condition (23d) cannot hold since $\mathcal{W}^*$ has finite cotype.

Let the hypotheses of Theorem 23 hold. Since

$$l(0,0,\varepsilon_1)<l(0,0,\varepsilon_0)=u(0,0,\varepsilon_0)<u(0,0,\varepsilon_1)$$

there are constants a and b so that

$$l(0,0,\varepsilon_1)<a<b<u(0,0,\varepsilon_1)\ .$$

Find $\eta=\eta\left(C_q\left(\mathcal{W}^*\right),\delta_{\mathcal{W}},\varepsilon_1\right)>0$ sufficiently small enough so that

$$l(\eta,\eta,\varepsilon_1)<a<b<u(\eta,\eta,\varepsilon_1)$$

and so that there exists $N\in\mathbb{N}$ satisfying

$$C\ (1+\eta)\left[1+\frac{2a}{3(b-a)}\ +\ 2N\eta\right]\ <\ N^{1/q} \tag{19}$$

where $C=C_q\left(\mathcal{W}^*\right)$. To see that condition (19) is easily accomplished, note that if

$$\left[2C\left(3+\frac{2a}{3(b-a)}\right)\right]^q<N\in\mathbb{N}$$

and $0<\eta\leqslant\frac{1}{N}$ then (19) holds.

Let conditions (23a) through (23d) hold. Assume that $\varepsilon\leqslant\varepsilon_1$.

By (23c), without loss of generality, for each $w^* \in \mathcal{W}^*$,

$$\frac{1}{1+\eta}\ \|w^*\| \ \leqslant\ \|Q^* w^*\| \ \leqslant\ \|w^*\| \ . \tag{20}$$

Keeping with the notation from Proposition 22, let

$$K = (1+\eta)\,(1+\varepsilon)\ .$$

From (23a), (23d), and (20) it follows that there is $\{w_n^*\}_{n=1}^{\infty} \subset KB(\mathcal{W}^*)$ such that

$$\begin{aligned} \|x_n^* - Q^* w_n^*\| &< 2\eta \\ \|x_n^*\| \ &= \ \|Q^* w_n^*\| \ . \end{aligned}$$

Thus (10) from Proposition 22 holds. Furthermore (11) also holds since

$$l(\eta,\eta,\varepsilon) \leqslant l(\eta,\eta,\varepsilon_1) < a < b < u(\eta,\eta,\varepsilon_1) \leqslant u(\eta,\eta,\varepsilon)\ .$$

So by Proposition 22, since $\{x_n\}_{n=1}^{\infty}$ is fundamental, $\{x_n^*\}_{n=1}^{\infty}$ is equivalent to the standard unit vector basis of c_0 with

$$\frac{3\,(b-a)}{3b-a}\left\|\sum_{n=1}^{\infty}\beta_n x_n^*\right\| \ \leqslant\ \sup_n|\beta_n| \ \leqslant\ \left\|\sum_{n=1}^{\infty}\beta_n x_n^*\right\|$$

for each $\{\beta_n\}_{n=1}^{\infty}$ in c_0.

For each finite subset F of $\mathbb{N}$

$$\left[\sum_{n\in F}\|w_n^*\|^q\right]^{\frac{1}{q}} \ \leqslant\ C\ \left[\ \mathrm{avg}_{\theta_i=\pm1}\ \left\|\sum_{i=1}^{n}\theta_i w_i^*\right\|^2\ \right]^{1/2} \tag{21}$$

The right-hand side of (21) mimics c_0-growth in that for each choice $\{\theta_n\}_{n\in F}$ of signs

$$\begin{aligned} \left\|\sum_{n\in F}\theta_n w_n^*\right\| \ &\leqslant\ (1+\eta)\ \left\|\sum_{n\in F}\theta_n Q^* w_n^*\right\| \\ &\leqslant\ (1+\eta)\ \left[\ \left\|\sum_{n\in F}\theta_n x_n^*\right\| + \left\|\sum_{n\in F}\theta_n\,(x_n^* - Q^* w_n^*)\right\|\ \right] \\ &\leqslant\ (1+\eta)\ \left[\frac{3b-a}{3(b-a)}\ +\ 2\,|F|\,\eta\right]\ . \end{aligned}$$

The left-hand side of (21) mimics l_q-growth since

$$1\ \leqslant\ \|x_n^*\|\ =\ \|Q^* w_n^*\|\ \leqslant\ \|w_n^*\|$$

for each $n \in \mathbb{N}$. Thus

$$|F|^{\frac{1}{q}}\ \leqslant\ C(1+\eta)\ \left[\frac{3b-a}{3(b-a)}\ +\ 2\,|F|\,\eta\right]\ .$$

This contradicts (19). Thus $\varepsilon_1 < \varepsilon$. ■

Corollary 24. *Let*

$$0 < \varepsilon_1 < -2 + \frac{\sqrt{147}}{6}$$

and, following the notation in (17),

$$\eta = \eta\left(1, \delta_{\ell_2}, \varepsilon_1\right) \ .$$

Let $\{x_n,\ x_n^*\}_{n=1}^{\infty}$ *be a* $(1+\varepsilon)$*–bounded fundamental biorthogonal system in* $S\left(\ell_1\right) \ \times \ \ell_\infty$ *satisfying*

$$\sup_{n\in\mathbb{N}} \ d\left(x_n^*, \mathcal{Z}_2\right) < \eta$$

for some subspace $\mathcal{Z}_2$ *of* ℓ_∞ *that is a* $(1+\eta)$*–isomorph of a Hilbert space. Then*

$$\varepsilon > \varepsilon_1 \ . \tag{22}$$

Proof. Let $\mathcal{W} = \ell_2$. Thus $C_2(\mathcal{W}^*) = 1$. It is straight forward to verify that

$$\varepsilon_0 := -2 + \frac{\sqrt{147}}{6}$$

satisfies condition (16) of Theorem 23. Note that $\varepsilon_0 \approx 0.0207$.

Let ${}^*\mathcal{Z}_2$ be the predual of $\mathcal{Z}_2$. There is an operator $T \ \in \ \mathcal{L}(\ell_1, {}^*\mathcal{Z}_2)$ such that $T^* \in \mathcal{L}(\mathcal{Z}_2, \ell_\infty)$ is the formal pointwise embedding; for indeed, since $\mathcal{Z}_2$ is reflexive, this formal pointwise embedding is weak*-to-weak* continuous. Similarly, by reflexivity, there is $S \in \mathcal{L}({}^*\mathcal{Z}_2, \ell_2)$ such that $S^* \in \mathcal{L}(\ell_2, \mathcal{Z}_2)$ is a $(1+\eta)$–isomorphism. Let $Q = S \circ T$. Thus

$$Q\colon \ell_1 \overset{T}{\to} {}^*\mathcal{Z}_2 \overset{S}{\to} \ell_2 \qquad \text{and} \qquad Q^*\colon \ell_2 \overset{S^*}{\to} \mathcal{Z}_2 \overset{T^*}{\to} \ell_\infty$$

and Q^* is a $(1+\eta)$–isomorphic embedding.

Thus condition (22) follows from Theorem 23. ■

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