DUAL BANACH SPACES WHICH CONTAIN AN ISOMETRIC COPY OF L_1

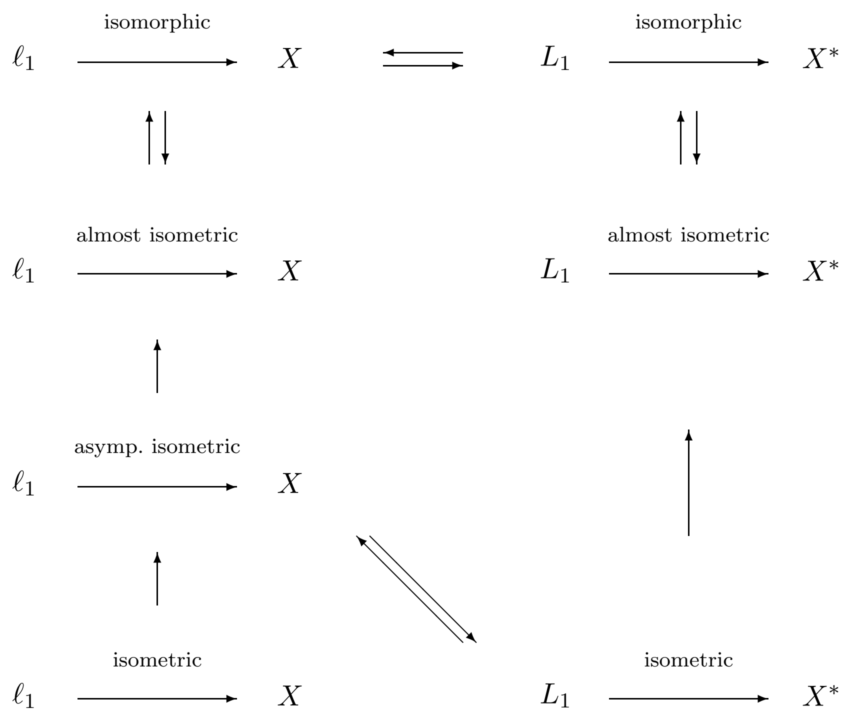
S. J. DILWORTH, MARIA GIRARDI AND J. HAGLER

Bull. Polish Acad. Sci. Math. 48 *(2000), no. 1, 1–12*

Abstract. A Banach space contains asymptotically isometric copies of ℓ_1 if and only if its dual space contains an isometric copy of L_1.

1. Introduction

The duality between a Banach space containing a 'nice' copy of ℓ_1 and its dual space containing a 'nice' copy of L_1 is summarized in the diagram below. Each upward implication follows straight from the definitions and the absence of a downward arrow indicates that the corresponding implication does not hold in general.



Date: July 4, 1999.

1991 *Mathematics Subject Classification.* Primary 46B04. Secondary 46B20.

Key words and phrases. Banach spaces, asymptotically isometric copies of ℓ_1, isometric copy of L_1.

Maria Girardi is supported by NSF grant DMS–9622841.

The investigation of this duality began when Pełczyński [P] proved that if X contains an isomorphic copy of ℓ_1 then X^* contains an isomorphic copy of L_1. He also proved the converse result under a technical assumption which was later removed by Hagler [H2]. Earlier, James [J] had shown that if X contains ℓ_1 isomorphically then X contains ℓ_1 almost isometrically. Recently, Dowling, N. Randrianantoanina and Turett [DRT] proved that a dual Banach space contains almost isometric copies of L_1 whenever it contains isomorphic copies of L_1 (see also [H1, Corollary 2.32] for this result). The main result of this paper, Theorem 2, shows that X contains asymptotically isometric copies of ℓ_1 if and only if X^* contains L_1 isometrically. In the real case, this is a hitherto unpublished result of Hagler [H1, Theorem 2.2].

2. Notation and Terminology

Henceforth, all Banach spaces are either real or complex. X, Y, and Z will denote arbitrary (infinite-dimensional) Banach spaces. Let $C(K)$ be the space of continuous functions on some compact Hausdorff space K, let L_1 be the space of Lebesgue–integrable functions on [0,1], and let $\ell_p(\Gamma)$ be the space of scalar-valued functions on the set Γ with finite $\|\cdot\|_p$-norm where $1 \leq p \leq \infty$, all with their usual norms. Let Δ be the Cantor set, ℓ_p be $\ell_p(\mathbb{N})$, and C be $C([0,1])$.

The concept of *asymptotically isometric copies of* ℓ_1 was introduced by Hagler [H1, pg. 14]. It was revitalized recently by Dowling and Lennard in fixed point theory [DL]. A Banach space contains *asymptotically isometric copies of* ℓ_1 provided it satisfies one (hence all) of the conditions in the lemma below.

Lemma 1. *For a Banach space X, the following are equivalent.*

(A1) *There exist a null sequence (ε_n) of positive numbers less than one and a sequence (x_n) in X such that*

$$\sum_{n=1}^{m}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{m} a_n x_n\right\| \leq \sum_{n=1}^{m}|a_n|$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars.

(A2) *There exist a null sequence (ε_n) of positive numbers and a sequence (x_n) in X such that*

$$\sum_{n=1}^{m}|a_n| \leq \left\|\sum_{n=1}^{m} a_n x_n\right\| \leq \sum_{n=1}^{m}(1+\varepsilon_n)|a_n|$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars.

(A3) *There exist a null sequence (ε_n) of positive numbers and a sequence (x_n) in X such that*

$$\sum_{n=k}^{m}|a_n| \leq \left\|\sum_{n=k}^{m} a_n x_n\right\| \leq (1+\varepsilon_k)\sum_{n=k}^{m}|a_n|$$

for each finite sequence $(a_n)_{n=k}^m$ of scalars and $k \in \mathbb{N}$.

The proof of this lemma is elementary (cf. [DLT, Theorem 1.7] for further equivalent formulations). Note that each condition is equivalent to the variant obtained by replacing *'There exist a'* by *'For each'* and *'and'* by *'there exists.'* A sequence (x_n) satisfying one of the conditions in the lemma is called *an asymptotically isometric copy of* ℓ_1. See [DLT] for a splendid survey of this topic and its applications to fixed point theory.

The proof of James's theorem [J] for ℓ_1 shows that if X contains ℓ_1 almost isometrically, then for each null sequence (ε_n) of positive numbers there exists a sequence (x_n) in X such that

$$(1-\varepsilon_k)\sum_{n=k}^{m}|a_n| \le \left\|\sum_{n=k}^{m} a_n x_n\right\| \le \sum_{n=k}^{m}|a_n|$$

for each finite sequence $(a_n)_{n=k}^m$ of scalars and $k \in \mathbb{N}$. Indeed, the line between containing ℓ_1 almost isometrically and asymptotically isometrically is very fine.

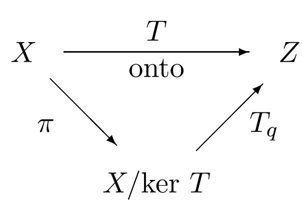
A sequence (x_n) in a Banach space X is a $(1+\varepsilon)$-*perturbation of an isometric copy of* ℓ_1 (for short, a $(1+\varepsilon)$-*p.i.* ℓ_1-*sequence*) provided that there exist a Banach space Y, a linear isometric embedding $T\colon X \to Y$, and a sequence (y_n) in Y such that (y_n) is isometrically equivalent to the unit vector basis of ℓ_1 and $\|y_n - Tx_n\| \le \varepsilon$ for each $n \in \mathbb{N}$. If furthermore

$$\|y_n - Tx_n\| \overset{\text{def}}{=} \varepsilon_n \xrightarrow{n\to\infty} 0\ ,$$

then (x_n) is a *perturbation of an isometric copy of* ℓ_1 (for short, a *p.i.* ℓ_1-*sequence*) with respect to (ε_n). Note that if X is separable then Y may be taken to be separable.

If X is a Banach space, then X^* is its dual space, $B(X)$ is its closed unit ball, and $S(X)$ is its unit sphere. The closed linear span of a subset A of X is $[A]$. If Y is a subspace of X then $\pi\colon X \to X/Y$ is the natural quotient mapping.

For a surjective bounded linear operator $T\colon X \to Z$, the corresponding bounded linear operator T_q is defined by the following (commutative) diagram.



The operator T is called an *isometric quotient mapping* provided T_q is an isometry, which is the case if and only if T^* is an isometric embedding. If $S\colon X \to Z$ is an isomorphic embedding, then S^* is an isometric quotient mapping if and only if S is an isometric embedding.

All notation and terminology, not otherwise explained, are as in [LT].

3. Main Result

Theorem 2, the main result of this paper, may be viewed as the isometric version of the theorems of Pełczyński and Hagler.

Theorem 2. *For a Banach space X, the following are equivalent.*

(a) *X contains asymptotically isometric copies of ℓ_1.*
(b) *X contains a perturbation of an isometric copy of ℓ_1.*
(c) *ℓ_1 is linearly isometric to a quotient space of a subspace X.*
(d) *L_1 is linearly isometric to a subspace of X^*.*
(e) *C^* is linearly isometric to a subspace of X^*.*
(f) *X^* contains an infinite set Γ which is isometrically equivalent to the usual basis of $\ell_1(\Gamma)$ and which is dense-in-itself in the weak-star topology on X^*.*

And if X is separable, then the following is equivalent to each of the above conditions.

(g) *$C(\Delta)$ is isometric to a quotient space of X.*

Recall that a subset K of a topological space is dense-in-itself if K has no isolated points in the relative topology. Our proof of Theorem 2 uses the following results.

Lemma 3. *If (x_n) is a p.i. ℓ_1-sequence, then $(\lambda_n x_n)$ is an asymptotically isometric copies of ℓ_1 satisfying (A2) for some suitable choice of scalars (λ_n). Conversely, an asymptotically isometric copies of ℓ_1 satisfying (A2) is a p.i. ℓ_1-sequence.*

Proof. Let $(\tilde{x}_n)$ be a p.i. ℓ_1-sequence with respect to $(\tilde{\varepsilon}_n)$. Then

$$\sum_{n=1}^{m}(1-\tilde{\varepsilon}_n)|a_n| \le \left\| \sum_{n=1}^{m} a_n \tilde{x}_n \right\| \le \sum_{n=1}^{m}(1+\tilde{\varepsilon}_n)|a_n| \ .$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. Define

$$\varepsilon_n \ \stackrel{\text{def}}{=} \ \frac{1+\tilde{\varepsilon}_n}{1-\tilde{\varepsilon}_n} \ - \ 1 \qquad \text{and} \qquad x_n \ \stackrel{\text{def}}{=} \ \frac{\tilde{x}_n}{1-\tilde{\varepsilon}_n} \quad .$$

Then (ε_n) and (x_n) satisfy (A2); thus, (x_n) is an asymptotically isometric copy of ℓ_1.

Conversely, let (ε_n) and (x_n) satisfy (A2). Then (x_n) is a p.i. sequence. To see this, let $X_0 \ = \ [x_n]$ and

$$W = \{(w_n)_{n=1}^{\infty} : w_n \in \mathbb{C} \ \text{ and } \ |w_n| = 1 \ \text{ for each } \ n \in \mathbb{N}\} \, .$$

For each $\omega = (w_n) \in W$, define $f_\omega \in B(X_0^*)$ by $f_\omega(x_n) = w_n$; for indeed,

$$\left| f_\omega \left(\sum_{n=1}^{m} a_n x_n \right) \right| \ = \ \left| \sum_{n=1}^{m} a_n w_n \right| \ \le \ \sum_{n=1}^{m} |a_n| \ \le \ \left\| \sum_{n=1}^{m} a_n x_n \right\|_{X_0}$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. For each $\omega \in W$, let $\tilde{f}_\omega \in B(X^*)$ be a norm-preserving Hahn-Banach extension of f_ω.

Let

$$Y \stackrel{\text{def}}{=} C\left(B\left(X^*\right), \sigma\left(X^*, X\right)\right) ,$$

endowed with the usual sup norm, and consider the isometric embedding

$$T : X \to Y$$

given by

$$(Tx)(x^*) \stackrel{\text{def}}{=} x^*(x) .$$

Let $y_n \in B(Y)$ be the 'truncation' of Tx_n; specifically,

$$y_n(x^*) = \begin{cases} (Tx_n)(x^*) & \text{if } \quad |(Tx_n)(x^*)| \le 1 \\ \dfrac{(Tx_n)(x^*)}{|(Tx_n)(x^*)|} & \text{if } \quad |(Tx_n)(x^*)| > 1 . \end{cases} \tag{1}$$

For each $n \in \mathbb{N}$, condition (A2) gives that $\|x_n\| \le 1 + \varepsilon_n$, and so by (1)

$$\|y_n - Tx_n\|_Y \le \varepsilon_n .$$

Since for each $n \in \mathbb{N}$ and $\omega = (w_j) \in W$

$$(Tx_n)(\tilde{f}_\omega) = \tilde{f}_\omega(x_n) = f_\omega(x_n) = w_n = y_n(\tilde{f}_\omega) ,$$

it follows that

$$\left\| \sum_{n=1}^{m} a_n y_n \right\|_Y \ge \sup_{\omega \in W} \left| \sum_{n=1}^{m} a_n y_n(\tilde{f}_\omega) \right| = \sup_{(w_n) \in W} \left| \sum_{n=1}^{m} a_n w_n \right| = \sum_{n=1}^{m} |a_n| .$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. Also, $\|y_n\| \le 1$ for each $n \in \mathbb{N}$. Thus (y_n) is isometrically equivalent to the unit vector basis of ℓ_1. □

Remark 4. Minor modifications to the above proof give an isomorphic version of Lemma 3. Indeed, if $(\tilde{x}_n)$ be a $(1+\tilde{\varepsilon})$-p.i. ℓ_1-sequence with $\tilde{\varepsilon} < 1$ and

$$\varepsilon \stackrel{\text{def}}{=} \frac{1+\tilde{\varepsilon}}{1-\tilde{\varepsilon}} - 1 \qquad \text{and} \qquad x_n \stackrel{\text{def}}{=} \frac{\tilde{x}_n}{1-\tilde{\varepsilon}} ,$$

then

$$\sum_{n=1}^{m} |a_n| \le \left\| \sum_{n=1}^{m} a_n x_n \right\| \le (1+\varepsilon) \sum_{n=1}^{m} |a_n| \tag{2}$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. Conversely, if (x_n) satisfies (2) for each finite sequence $(a_n)_{n=1}^m$ of scalars, then (x_n) is a $(1+\varepsilon)$-p.i. ℓ_1-sequence.

Lemma 5. *If X satisfies (f) of Theorem 2, then there exists a separable subspace X_0 of X and a countable subset Γ' of X_0^* which satisfies (f) of Theorem 2.*

Proof. Let X be a Banach space satisfying (f) of Theorem 2. We shall inductively construct a sequence (Λ_i) of countably infinite subsets of Γ and a sequence (Z_i) of separable subspaces of X which satisfy, for each $n \in \mathbb{N}$,

(1) $Z_n \subset Z_{n+1}$,

(2) Z_n norms $[\,\cup_{i=1}^n \Lambda_i\,]$, i.e., if $x^* \in [\,\cup_{i=1}^n \Lambda_i\,]$ then
$$\|x^*\| = \sup_{z \in B(Z_n)} |x^*(z)| \ ,$$

(3) $\cup_{i=1}^n \Lambda_i$ is contained in the Z_n-cluster points of Λ_{n+1}, i.e., if $x^* \in \cup_{i=1}^n \Lambda_i$ and $(w_i)_{i=1}^k$ are from Z_n and $\varepsilon > 0$ then
$$\{y^* \in \Lambda_{n+1} \colon \ |(y^* - x^*)\,(w_i)| < \varepsilon \text{ for } 1 \le i \le k\} \setminus \{x^*\} \neq \emptyset \ .$$

For the first step of the induction choose a countably infinite subset Λ_1 of Γ and find a separable subspace Z_1 of X which satisfies (2). Suppose that we have chosen $(\Lambda_i)_{i=1}^n$ and $(Z_i)_{i=1}^n$ satisfying the three conditions. Since Z_n is separable and elements of Γ are of norm one, there is a countable subset Λ_{n+1} of Γ satisfying (3). Next we find a separable subspace Z_{n+1} of X which satisfies (1) and (2). This completes the inductive step.

Now let $X_0 = [\cup_{n=1}^\infty Z_n]$ and
$$\Gamma' \ \stackrel{\text{def}}{=} \ \{x^*|_{X_0} \colon x^* \in \cup_{n=1}^\infty \Lambda_n\} \ .$$
Condition (2) gives that Γ' is isometrically equivalent to the usual basis of $\ell_1(\Gamma')$. Conditions (1) and (3), along with the fact that $\Gamma \subset S(X^*)$, give that Γ' is dense-in-itself in the weak-star topology on X_0^*. □

Fact 6. (cf. [HS, Lemma 4]) *Let N and M be compact Hausdorff spaces with M perfect and suppose that $\phi \colon N \to M$ is continuous and onto. Then there exists a subset Q of N such that Q is dense-in-itself and $\phi|_Q \colon Q \to M$ is a bijection.*

Fact 7. (Haskell P. Rosenthal [R, Proposition 3 and its Remark 2]) *Let X_0 be a separable Banach space satisfying (f) of Theorem 2. Then there exists*
$$K \subset B(X_0^*) \ ,$$
which is homeomorphic to Δ, such that the restriction operator
$$R : X_0 \ \to \ C(K)$$
given by $(Rx_0)(x_0^*) \ = \ x_0^*(x_0)$ is an isometric quotient mapping.

Proof of Theorem 2. We shall assume that X is a complex Banach space as the proof in the real case is easier.

The equivalence of (a) and (b) follows directly from Lemma 3.

To see that (a) implies (c), let (ε_n) and (x_n) be sequences satisfying condition (A3). Partition $\mathbb{N}$ into infinite sets $\{J_n\}_{n\in\mathbb{N}}$ and let $T \colon [x_n] \to \ell_1$ be the bounded linear operator that maps x_j to the n^{th} unit vector of ℓ_1 when $j \in J_n$. Then T is an isometric quotient mapping.

To see that (c) implies (a), let
$$T \colon X_0/X_1 \to \ell_1$$
be an isometry from a quotient space of a subspace X_0 of X onto ℓ_1. Fix a null sequence (ε_n) of positive numbers. Find a sequence (x_n) in X_0 such that $T(x_n + X_1)$ is the n^{th} unit vector of ℓ_1 and
$$1 \le \|x_n\|_X \le 1 + \varepsilon_n \ .$$

Then (ε_n) and (x_n) satisfy (A2).

To see that (a) implies (e), let (ε_n) and (x_n) satisfy (A2). We shall define the bounded linear operators in the (commutative) diagram below

$$\begin{array}{cccccc} & X & & & & \\ & \uparrow{\scriptstyle j} & & & \searrow{\scriptstyle \widetilde{T}} & \\ [x_n] \stackrel{\mathrm{def}}{=} & Y & \xrightarrow{\ T\ } & C & \xrightarrow{\ \widehat{\imath}\ } & C^{**} \end{array} \tag{3}$$

as follows. Let (z_n) be dense in the unit sphere of C. Define T by $Tx_n = z_n$. Condition (A2) gives that T is a surjective norm-one bounded linear operator. Furthermore, T is an isometric quotient mapping for if $f \in S(C)$ then there is a subsequence (z_{k_n}) converging in norm to f and so

$$1 \;\le\; \left\| T_q^{-1} f \right\|_{Y/\ker T} \;\le\; \varliminf_{n\to\infty} \|x_{k_n}\|_X \;=\; 1\ .$$

Let j be the natural embedding and let $\widehat{\imath}$ be the canonical isometric embedding given by point evaluation. Since C^{**} has the Hahn-Banach Extension Property, T admits a norm-preserving extension $\widetilde{T}$. Dualizing gives the commutative diagram

$$\begin{array}{ccccccc} C^{*} & \xrightarrow{\ h\ } & C^{***} & \xrightarrow{\ \widehat{\imath}^{*}\ } & C^{*} & \xrightarrow{\ T^{*}\ } & Y^{*} \\ & & & \searrow{\scriptstyle \widetilde{T}^{*}} & & & \uparrow{\scriptstyle j^{*}} \\ & & & & & & X^{*} \end{array}$$

where h is the canonical isometric embedding given by point evaluation. To see that $R \stackrel{\mathrm{def}}{=} \widetilde{T}^{*}\, h$ is the desired isometric embedding, let $\mu \in C^{*}$. Then, since T^{*} is an isometric embedding and $\widehat{\imath}^{*}\, h$ is the identity mapping,

$$\|\mu\|_{C^{*}} \;=\; \|T^{*}\mu\|_{Y^{*}} \;=\; \|j^{*}R\mu\|_{Y^{*}} \;\le\; \|R\mu\|_{X^{*}} \;\le\; \|\mu\|_{C^{*}}\ .$$

Clearly, (e) implies (d).

To see that (d) implies (a), let $T\colon L_1 \to X^{*}$ be an isometric embedding and let (ε_n) be a null sequence of positive numbers. Then $T^{*}\colon X^{**} \to L_\infty$ is a weak-star continuous isometric quotient mapping. By Goldstine's Theorem,

$$W \stackrel{\mathrm{def}}{=} T^{*}(B(X))$$

is weak-star dense in $B(L_\infty)$. For each $n \in \mathbb{N}$, let

$$F_n \stackrel{\mathrm{def}}{=} \{z_j^n \colon 1 \le j \le M(n)\}$$

be an $(\varepsilon_n/2)$–net for $\{z \in \mathbb{C}\colon |z| = 1\}$.

Let $\mathcal{T}$ be the tree

$$\mathcal{T} = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$$

where $\mathcal{T}_n$, the n^{th}-level of $\mathcal{T}$, is

$$\mathcal{T}_n \stackrel{\text{def}}{=} \{(m_0, m_1, m_2, \ \ldots \ m_{n-1}) \in \mathbb{N}^n \colon \\ m_0 = 1 \text{ and } 1 \leq m_j \leq M(j) \text{ for each } j \in \mathbb{N}\} \ .$$

If $\alpha = (m_0, m_1, m_2, \ \ldots \ m_{n-1}) \in \mathcal{T}$ then

$$(\alpha, j) \stackrel{\text{def}}{=} (m_0, m_1, m_2, \ \ldots \ m_{n-1}, j) \ ;$$

thus, for each $n \in \mathbb{N}$

$$\mathcal{T}_{n+1} = \{(a, j) \colon \alpha \in \mathcal{T}_n \text{ and } 1 \leq j \leq M(n)\} \ .$$

We will define inductively, for each $n \in \mathbb{N}$, a collection $\{A_\alpha\}_{\alpha \in \mathcal{T}_n}$ of disjoint sets of positive (Lebesgue) measure and a function $f_n \in W$ such that, for each $n \in \mathbb{N}$ and $\alpha \in \mathcal{T}_n$,

$$\bigcup_{j=1}^{M(n)} A_{(\alpha,j)} \ \subset \ A_\alpha \ \subset \ [0,1] \tag{4}$$

and, for each $1 \leq j \leq M(n)$,

$$|f_n - z_j^n| < \frac{\varepsilon_n}{2} \text{ on } A_{(\alpha,j)} \ . \tag{5}$$

To start the induction, let

$$A_{(m_0)} = [0,1] \ .$$

For the inductive step, let $n \in \mathbb{N}$ and suppose that we have constructed disjoint sets

$$\{A_\alpha \colon \alpha \in \mathcal{T}_n\}$$

of positive measure. For each $\alpha \in \mathcal{T}_n$, partition A_α into sets $\{D_{(\alpha,j)}\}_{j=1}^{M(n)}$ of positive measure. Consider the function $g_n \in B(L_\infty)$ defined by

$$g_n(t) = \begin{cases} z_j^n & \text{if } t \in D_{(\alpha,j)} \text{ and } \alpha \in \mathcal{T}_n \\ 0 & \text{otherwise} \ . \end{cases}$$

Since W is weak-star dense in $B(L_\infty)$ there exists $f_n \in W$ approximating g_n closely enough to ensure that the sets

$$A_{(\alpha,j)} \stackrel{\text{def}}{=} \{|f_n - z_j^n| < \varepsilon_n/2\} \cap D_{(\alpha,j)}$$

all have positive measure. This completes the proof of the inductive step.

For each $n \in \mathbb{N}$, select $x_n \in B(X)$ such that $T^*(x_n) = f_n$. To see that (x_n) is an asymptotically isometric copy of ℓ_1, let $(a_n)_{n=1}^m$ be a finite complex sequence. Define $(\widetilde{a}_n)_{n=1}^m$ from $\{z \in \mathbb{C} \colon |z| = 1\}$ by

$$\widetilde{a}_n = \begin{cases} \dfrac{\bar{a}_n}{|a_n|} & \text{if } a_n \neq 0 \\ 1 & \text{if } a_n = 0 \ ; \end{cases}$$

thus, $a_n\widetilde{a}_n = |a_n|$. For each $1 \le n \le m$, find $1 \le j_n \le M(n)$ so that

$$|\widetilde{a}_n - z^n_{j_n}| \;<\; \frac{\varepsilon_n}{2} \;.$$

Then $\alpha \;\stackrel{\text{def}}{=}\; (1, j_1, \dots, j_m) \in \mathcal{T}_{m+1}$ and so by (4) and (5)

$$|f_n - z^n_{j_n}| \;<\; \frac{\varepsilon_n}{2} \;\text{ on } A_\alpha$$

for each $1 \le n \le m$. Thus

$$\begin{aligned}\left\|\sum_{n=1}^m a_n x_n\right\| \;&\ge\; \left\|T^*\left(\sum_{n=1}^m a_n x_n\right)\right\|_{L_\infty} \;=\; \left\|\sum_{n=1}^m a_n f_n\right\|_{L_\infty}\\ &\ge\; \left\|\sum_{n=1}^m a_n\widetilde{a}_n 1_{A_\alpha}\right\|_{L_\infty} \;-\; \left\|\sum_{n=1}^m a_n\left(\widetilde{a}_n - f_n\right)1_{A_\alpha}\right\|_{L_\infty}\\ &\ge\; \sum_{n=1}^m |a_n| \;-\; \sum_{n=1}^m \varepsilon_n|a_n| \;=\; \sum_{n=1}^m (1-\varepsilon_n)|a_n| \;.\end{aligned}$$

So (ε_n) and (x_n) do indeed satisfy (A1).

To show that (a) implies (f), let (a) hold. Then we have the situation depicted in (3). For $t \in [0,1]$, let $\delta_t \in C^*$ denote the point mass measure at t. Since T^* is a weak-star continuous isometric embedding

$$M \;\stackrel{\text{def}}{=}\; \{T^*(\delta_t) \colon t \in [0,1]\} \;\subset\; B(Y^*) \;,$$

equipped with the weak-star topology of Y^*, is homeomorphic to $[0,1]$ and is isometrically equivalent to the usual basis of $\ell_1([0,1])$. By the Hahn-Banach Theorem $j^*(B(X^*)) = B(Y^*)$ and so

$$N \;\stackrel{\text{def}}{=}\; j^{*-1}(M) \cap B(X^*)$$

is weak-star compact and satisfies $j^*(N) = M$. By Fact 6 there exists

$$\Gamma = \{n_t \colon t \in [0,1]\} \subset N$$

such that $j^*(n_t) = T^*(\delta_t)$ and such that Γ is dense-in-itself in the weak-star topology on X^*. Since for any finite set $\{a_t\}_{t\in F}$ of scalars

$$\begin{aligned}\sum_{t\in F}|a_t| \;&=\; \left\|\sum_{t\in F} a_t T^*\delta_t\right\|_{Y^*} \;=\; \left\|\sum_{t\in F} a_t j^* n_t\right\|_{Y^*}\\ &\le\; \left\|\sum_{t\in F} a_t n_t\right\|_{X^*} \;\le\; \sum_{t\in F}|a_t| \;,\end{aligned}$$

the set Γ is isometrically equivalent to the usual basis of $\ell_1([0,1])$.

To see that (f) implies (e), let X satisfy (f). Then by Lemma 5, there is a separable subspace X_0 of X which satisfies (f). From Fact 7 and the fact that $C^*(\Delta)$ is linearly isometric to C^*, it follows that X_0 satisfies (e). The equivalence of (a) and (e) gives that X also satisfies (e).

Thus (a) through (f) are equivalent. The fact that $C^*(\Delta)$ is linearly isometric to C^* gives that (g) implies (e). That (f) implies (g) when X is separable is due to Rosenthal: Fact 7. □

Remark 8. Without the added assumption of separability, (g) is not equivalent to the other conditions. Clearly, ℓ_∞ satisfies conditions (a) through (f). But by a result of Grothendieck [G], a separable quotient of ℓ_∞ is reflexive and so ℓ_∞ does not satisfy (g).

Remark 9. A complemented isomorphic version of Theorem 2 is due to Hagler and Stegall [HS, Theorem 1]. A K-complemented isometric version of Theorem 2 is due to Hagler ([H1, Theorem 2.13] or [H3]).

Remark 10. Many Banach spaces (and their subspaces) which arise naturally in analysis contain an abundance of asymptotically isometric copies of ℓ_1: for example, Carothers, Dilworth and Lennard [CDL] proved that every non-reflexive subspace of the Lorentz space $L_{w,1}(0,\infty)$ contains asymptotically isometric copies of ℓ_1 whenever the weight w satisfies very mild regularity conditions. On the other hand, $L_{w,1}$ does *not* contain an isometric copy of the 2-dimensional space ℓ_1^2 whenever w is strictly decreasing [CDT].

Remark 11. Theorem 2 improves a recent result of Shutao Chen and Bor-Luh Lin [CL] who proved that X contains an asymptotically isometric copy of ℓ_1 whenever X^* contains an isometric copy of ℓ_∞.

Dowling, Johnson, Lennard and Turett [DJLT] gave some concrete examples of equivalent norms on ℓ_1 such that the corresponding Banach spaces do not contain asymptotically isometric copies of ℓ_1. Theorem 2 easily yields other equivalent norms on ℓ_1 with this property.

Corollary 12. *Let (γ_n) be a sequence of non-zero scalars with $\|(\gamma_n)\|_2 < \varepsilon$. Then the Banach space $(\ell_1, \|\cdot\|_1')$, where*

$$\|(a_n)\|_1' \overset{\text{def}}{=} \inf\left\{ \left[\|(a_n+b_n)\|_1^2 \;+\; \left\|\left(\gamma_n^{-1}b_n\right)\right\|_2^2 \right]^{\frac{1}{2}} : \left(\gamma_n^{-1}b_n\right) \in \ell_2 \right\} ,$$

does not contain asymptotically isometric copies of ℓ_1 and

$$\left(1+\varepsilon^2\right)^{-\frac{1}{2}} \|(a_n)\|_1 \;\le\; \|(a_n)\|_1' \;\le\; \|(a_n)\|_1 \tag{7}$$

for each $(a_n) \in \ell_1$.

Proof. We exhibit $(\ell_1, \|\cdot\|_1')$ as a quotient space of $X = \ell_1 \oplus_2 \ell_2$ with its usual norm

$$\|((a_n),(b_n))\|_X = \left[\|(a_n)\|_1^2 \;+\; \|(b_n)\|_2^2 \right]^{1/2} .$$

Let

$$\begin{aligned} Y &\overset{\text{def}}{=} \{((a_n),(b_n)) \in X \,:\, b_n = -\gamma_n^{-1}a_n\} \\ &\overset{\text{note}}{=} \{((a_n),(-\gamma_n^{-1}a_n)) \,:\, (\gamma_n^{-1}a_n) \in \ell_2\} . \end{aligned}$$

Since each element of X/Y has a representative of the form $((a_n), 0)$,

$$\|((a_n), 0) + Y\|_{X/Y} = \inf \left\{ \left\|((a_n), 0) + \left((b_n), \left(-\gamma_n^{-1} b_n\right)\right)\right\|_X : \left(\gamma_n^{-1} b_n\right) \in \ell_2 \right\} = \|(a_n)\|_1' .$$

Thus X/Y is isometrically isomorphic to $(\ell_1, \|\cdot\|_1')$.

Observe that $X^* = \ell_\infty \oplus_2 \ell_2$ with its usual norm

$$\|((c_n), (d_n))\|_{X^*} = \left[\|(c_n)\|_\infty^2 + \|(d_n)\|_2^2 \right]^{1/2}$$

and

$$Y^\perp = \{((c_n), (d_n)) \in X^* : d_n = \gamma_n c_n\} = \{((c_n), (\gamma_n c_n)) : (c_n) \in \ell_\infty\} .$$

It follows that $Y^\perp$ is isometrically isomorphic to $(\ell_\infty, \|\cdot\|_\infty')$ where

$$\|(c_n)\|_\infty' = \left[\|(c_n)\|_\infty^2 + \|(\gamma_n c_n)\|_2^2 \right]^{1/2} \tag{8}$$

and the mapping

$$i\colon \left(\ell_\infty, \|\cdot\|_\infty'\right) \to \left(\ell_1, \|\cdot\|_1'\right)^*$$

given by $(i(c_n))(a_n) = \sum_n a_n c_n$ is an isometry. Since the norm $\|\cdot\|_\infty'$ is strictly convex, the space $(\ell_\infty, \|\cdot\|_\infty')$ does not contain an isometric copy of L_1. Thus by Theorem 2 the space $(\ell_1, \|\cdot\|_1')$ does not contain asymptotically isometric copies of ℓ_1. From (8) it follows that

$$\|(c_n)\|_\infty \le \|(c_n)\|_\infty' \le \sqrt{1+\varepsilon^2}\, \|(c_n)\|_\infty$$

for each $(c_n) \in \ell_\infty$ and so (7) holds by duality. □

Finally, Alspach's [A] example of an isometry $T\colon K \to K$ without a fixed point, where K is a certain weakly compact convex subset of L_1, yields an obvious corollary.

Corollary 13. *If X contains asymptotically isometric copies of ℓ_1, then there exists a (nonempty) weakly compact convex subset K of X^* and an isometry $T\colon K \to K$ without a fixed point.*

Acknowledgement. The authors thank Patrick Dowling, William B. Johnson, and Haskell P. Rosenthal for their helpful comments. H. P. Rosenthal suggested the term p.i. ℓ_1-sequence and his helpful conversations [HPR] led us to the proof of Lemma 3.

References

[A] D. E. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* **82** (1981), 423–424.

[CDL] N. L. Carothers, S. J. Dilworth and C. J. Lennard, On a localization of the UKK property and the Fixed Point Property in $L_{w,1}$, *Interaction between Functional Analysis, Harmonic Analysis and Probability*, Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, 1996, pp. 111–124.

[CDT] N. L. Carothers, S. J. Dilworth and D. A. Trautman, On the geometry of the unit spheres of the Lorentz spaces $L_{w,1}$, *Glasgow Math. J.* **34** (1992), 21–25.

[CL] Shutao Chen and Bor-Luh Lin, Dual action of asymptotically isometric copies of ℓ_p $(1 \leq p < \infty)$ and c_0, *Collect. Math.* **48** (1997), 449–458.

[DJLT] P. N. Dowling, W. B. Johnson, C. J. Lennard and B. Turett, The optimality of James's distortion theorems, *Proc. Amer. Math. Soc.* **125** (1997), 167–174.

[DL] P. N. Dowling and C. J. Lennard, Every nonreflexive subspace of L_1 fails the fixed point property, *Proc. Amer. Math. Soc.* **125** (1997), 443–446.

[DLT] P. N. Dowling, C. J. Lennard, and B. Turett, Asymptotically perturbed norms of classical sequence spaces with applications to fixed point theory. Proceedings of Workshop on Fixed Point Theory (Kazimierz Dolny, 1997). *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **51** (1997), 67–98.

[DRT] Patrick N. Dowling, Narcisse Randrianantoanina and Barry Turett, Remarks on James's distortion theorems, *Bull. Austral. Math. Soc.* **57** (1998), 49–54.

[G] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, *Canadian J. Math.* **5** (1953), 129–173.

[H1] J. Hagler, Embeddings of L^1 into conjugate Banach spaces, Ph.D. Thesis, University of California, Berkeley, Calif., 1972.

[H2] J. Hagler, Some more Banach spaces which contain ℓ^1, *Studia Math.* **46** (1973), 35–42.

[H3] J. Hagler, in preparation.

[HS] J. Hagler and C. Stegall, Banach spaces whose duals contain complemented subspaces isomorphic to $C[0,1]^*$, *J. Funct. Anal.* **13** (1973), 233–251.

[J] Robert C. James, Uniformly non-square Banach spaces, *Ann. of Math. (2)* **80** (1964), 542–550.

[LT] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. *Lecture Notes in Math.*, Vol. 338. Springer-Verlag, Berlin, 1973.

[P] A. Pełczyński, On Banach spaces containing $L_1(\mu)$, *Studia Math.* **30** (1968), 231–246.

[R] H. P. Rosenthal, On factors of $C([0,1])$ with non-separable dual, *Israel J. Math.* **13** (1972), 361–378.

[HPR] H. P. Rosenthal, *Personal Communication*, SUMIRFAS '98, Texas A&M University, July 25, 1998.

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: dilworth@math.sc.edu and girardi@math.sc.edu

Department of Mathematics and Computer Science, University of Denver, Denver, CO 80208, U.S.A.

E-mail address: jhagler@cs.du.edu